Journal of Mathematical Inequalities
Volume 19, Number 2 (2025), 491–516
doi:10.7153/jmi-2025-19-31

THE HERMITE—HADAMARD'S INEQUALITY OVER p–SPHERES AND RELATED INEQUALITIES FOR THE INTEGRAL MEANS

MOHAMMAD W. ALOMARI

(*Communicated by M. Krnić*)

Abstract. The goal of this work is to extend the Hermite-Hadamard inequality, to the p-sphere. Additionally, we present Trapezoid and Midpoint type inequalities for the integral means of the generalized Hermite-Hadamard inequality. Specifically, for functions defined on p-spheres that meet certain Hölder-type conditions, we establish corresponding Trapezoid and Midpoint type inequalities.

1. Introduction

In 1879, Lundberg [16] introduced a remarkable generalization of the sine and cosine functions, which expanded the classical trigonometric framework into more generalized geometric contexts. Lundberg's functions are based on the inverses of certain integrals that depend on a parameter p, and are denoted by $S := S_{\frac{p-1}{p}}(\varphi)$ and $C := C_{\frac{p-1}{p}}(\varphi)$, representing generalized sine and cosine functions, respectively. These functions arise from the following integral expressions:

$$\varphi = \int_0^S \frac{dt}{(1-t^p)^{\frac{p-1}{p}}},$$
$$\varphi = \int_C^1 \frac{dt}{(1-t^p)^{\frac{p-1}{p}}}.$$

For $p = 2$, these generalized functions reduce to the classical trigonometric sine and cosine: $S_{\frac{1}{2}}(\varphi) = \sin(\varphi)$ and $C_{\frac{1}{2}}(\varphi) = \cos(\varphi)$. Consequently, this generalization not only extends our understanding of trigonometric functions but also introduces a natural extension of the concept of π as [10]:

$$\omega_p = \frac{2\Gamma^2\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{2}{p}\right)}, \quad p \geqslant 1,$$

Mathematics subject classification (2020): 26D15.
Keywords and phrases: Hermite-Hadamard inequality, p-sphere, trapezoid, midpoint.

where $\omega_2 = \pi$ in the classical case.

The properties of these generalized sine and cosine functions parallel those of their classical counterparts, including symmetry and periodicity, but with a more flexible dependence on the parameter p. These properties provide a foundation for extending trigonometric identities and functions into new geometric contexts, such as Minkowski space or other non-Euclidean geometry. For more information see [13]–[15].

1.1. Properties of $S_{\frac{p-1}{p}}(\varphi)$ and $C_{\frac{p-1}{p}}(\varphi)$

Some of the key properties of these generalized functions include:

1. $S_{\frac{p-1}{p}}(\varphi)$ is an odd function, and $C_{\frac{p-1}{p}}(\varphi)$ is an even function.
2. These functions exhibit a shift-symmetry, with $S_{\frac{p-1}{p}}(\varphi) = C_{\frac{p-1}{p}}\left(\frac{\omega_p}{2} - \varphi\right)$ and vice versa.
3. The functions are periodic with a period related to ω_p, and satisfy identities similar to the classical sine and cosine functions.
4. Generalized versions of trigonometric identities, such as the Pythagorean identity, take the form $S^p_{\frac{p-1}{p}}(\varphi) + C^p_{\frac{p-1}{p}}(\varphi) = 1$.
5. Derivatives of these functions extend the classical relationships, with $\frac{d}{d\varphi}S_{\frac{p-1}{p}}(\varphi) = C^{p-1}_{\frac{p-1}{p}}(\varphi)$, and $\frac{d}{d\varphi}C_{\frac{p-1}{p}}(\varphi) = -S^{p-1}_{\frac{p-1}{p}}(\varphi)$.
6. The generalized tangent function is defined as $\mathfrak{T}_{\frac{2}{p}}(\varphi) = \frac{S^{p-1}_{\frac{p-1}{p}}(\varphi)}{C^{p-1}_{\frac{p-1}{p}}(\varphi)}$, with $p > 1$.

   For $p = 2$, this reduces to $\tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)}$.

These generalizations lead to the natural definition of a generalized tangent function, as well as an extended geometric framework, where curves, surfaces, and volumes are measured using norms derived from the parameter p. In particular, Lindqvist and Peetre [13] expanded on these ideas to explore the p-circle and its arc lengths using the Minkowski metric.

1.2. The p-circle and Minkowski geometry

Lindqvist and Peetre [13] defined the concept of a p-circle, which generalizes the classical Euclidean circle to one governed by the Minkowski p-norm. The distance between two points (u_1, v_1) and (u_2, v_2) in Minkowski geometry is given by the p-metric:

$$d_p(u,v) = \begin{cases} (|u_2 - u_1|^p + |v_2 - v_1|^p)^{\frac{1}{p}}, & 1 \leqslant p < \infty, \\ \max\{|u_2 - u_1|, |v_2 - v_1|\}, & p = \infty, \end{cases}$$

where $p \geqslant 1$ is a parameter. The equation for a p-circle of radius R centered at (u_0, v_0) is:

$$|u-u_0|^p + |v-v_0|^p = R^p.$$

As p varies, this equation describes different geometric shapes: for $p = 2$, it reduces to the standard Euclidean circle; for $p = 1$, it describes a rhombus, and for $p = \infty$, it becomes a square.

Similarly, the p-sphere generalizes the classical notion of a sphere to accommodate various norms and geometries, extending our understanding of spherical shapes beyond Euclidean space. In three-dimensional space $\mathbb{R}^3$, the p-sphere offers a flexible framework for exploring geometric properties under different norms.

In Euclidean three-dimensional space, a classical sphere of radius R centered at the (u_0, v_0, w_0) is given by:

$$(u-u_0)^2 + (v-v_0)^2 + (w-w_0)^2 = R^2.$$

For the p-sphere, this definition is adapted to use the p-norm, resulting in:

$$\|\mathbf{u}\|_p^p = |u-u_0|^p + |v-v_0|^p + |w-w_0|^p = R^p,$$

where $\mathbf{u} = (u, v, w)$ and $\|\mathbf{u}\|_p$ denotes the p-norm defined by:

$$\|\mathbf{u}\|_p = \left(|u|^p + |v|^p + |w|^p\right)^{\frac{1}{p}}.$$

Here, $\|\mathbf{u}\|_p^p$ represents the generalized distance from the origin to the surface of the p-sphere in $\mathbb{R}^3$.

The p-sphere in three dimensions has several interesting properties that depend on the value of p. Among others, the "*Shape Variability*". i.e,

1. For $p = 2$, the p-sphere reduces to the classical Euclidean sphere.

2. For $p \neq 2$, the shape of the p-sphere varies: - As p approaches 1, the p-sphere approximates a polyhedral shape with flat faces, resembling an octahedron or rhombic dodecahedron.

3. As p approaches infinity, the p-sphere approximates a cube, with flat faces aligned with the coordinate axes.

Extending these ideas further into surface geometry, the classical notion of surface area can be generalized using a q-norm, leading to the definition of q-area. Consider a parametric surface S defined by a vector function $\mathbf{r}(x,y) = U(x,y)\mathbf{i} + V(x,y)\mathbf{j} + W(x,y)\mathbf{k}$ over a region T in the xy-plane. The classical surface area is proportional to the magnitude of the cross product of the partial derivatives $\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}$, representing the local scaling factor for areas.

To generalize this notion, we define the q-area $A_q(S)$ of the surface as:

$$A_q(S) = \iint_T \left(\left| \frac{\partial(V,W)}{\partial(x,y)} \right|^q + \left| \frac{\partial(U,W)}{\partial(x,y)} \right|^q + \left| \frac{\partial(U,V)}{\partial(x,y)} \right|^q \right)^{\frac{1}{q}} dx\,dy, \qquad q > 1.$$

For $q = 2$, this reduces to the classical surface area in Euclidean space. However, for other values of q, it represents surface area measurements in different geometric contexts. This generalization is useful in non-Euclidean or anisotropic geometries, where the geometry is influenced by direction or other factors.

1.3. The Hermite–Hadamard's inequality

Dragomir [8] was one of the earliest to extend the classical Hermite-Hadamard inequality to convex mappings defined on balls in higher-dimensional spaces. His work provided significant applications to the study of integral inequalities in multi-dimensional settings, offering valuable tools for analyzing convex functions over non-Euclidean domains. This set the stage for further exploration into generalizing these inequalities to more complex shapes and domains. Namely, he proved the following version of Hermite-Hadamard's inequality of a sphere in the space.

THEOREM 1. *If the mapping $f : B(A,R) \to \mathbb{R}$ is convex on the ball $B(A,R)$, then the inequality*

$$f(A) \leqslant \frac{1}{vol(B)} \iiint_{B(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \leqslant \frac{1}{\mathfrak{S}(A,R)} \iint_{S(A,R)} f(x_1,x_2,x_3)\,dS \qquad (1)$$

holds, where

$$S(A,R) := \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)^2 = R^2\right\},$$

is the sphere centered at the point A with radius R, and

$$vol(B) = \frac{4}{3}\pi R^3, \qquad \mathfrak{S}(A,R) := 4\pi R^2.$$

The above inequalities are sharp.

For more generalizations, counterparts, and new inequalities of Hadamard's type focusing on higher-dimensional geometric objects such as triangles, regular polygons, and hyperspheres the reader may find the references [1]–[9], [11], [12], and [17]–[22] very useful.

The aim of this work is to generalize the Hermite-Hadamard inequality given in (1) on p-spheres $(p > 1)$. Trapezoid and Midpoint type inequalities for the integral means of the established Hermite–Hadamard inequality are also presented. Namely, for functions defined on p-spheres that satisfies certain Hölder type conditions, Trapezoid and Midpoint type inequalities are proved.

2. The Hermite–Hadamard's inequality on the p-sphere

Consider $A = (a_1,a_2,a_3) \in \mathbb{R}^3$ and the p-ball $B_p(A,R)$ $(p > 1)$ centered at point A with radius $R > 0$. Consider

$$B_p(A,R) := \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1-a_1|^p + |x_2-a_2|^p + |x_3-a_3|^p \leqslant R^p, p \gtreqqless 1\right\}$$

be the p-ball centered at the point A with radius $R > 0$.

LEMMA 1. *Let* $p = \frac{q}{q-1}$, $q > 1$. *The surface area of the* p*-Sphere is* $\mathfrak{S}_p(A,R) := 4\omega_p R^2$ $(p \gneqq 1)$.

Proof. Let

$$S_p(A,R) := \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1 - a_1|^p + |x_2 - a_2|^p + |x_3 - a_3|^p = R^p, p > 1\right\}$$

be the p-sphere centered at the point A with radius R.

To determine the p-sphere volume described above we compute the fundamental vector product and then integrate its q-length over the region $T = S_p(A,R)$. For this purpose, consider the parameterized p-surface $\lambda : \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right] \times [0, 2\omega_p] \to \mathbb{R}^2$ defined by

$$S_p(A,R) : \begin{cases} x_1(\theta,\psi) = a_1 + RC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi) \\ x_2(\theta,\psi) = a_2 + RC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi) \\ x_3(\theta,\psi) = a_3 + RS_{\frac{p-1}{p}}(\theta) \end{cases};$$

where $p \gneqq 1$, $R > 0$, $\theta \in \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$, and $\psi \in [0, 2\omega_p]$. Therefore,

$$\frac{\partial(x_1,x_2)}{\partial(\theta,\psi)} = \begin{vmatrix} -RS^{p-1}_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi) & -RS^{p-1}_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi) \\ -RC_{\frac{p-1}{p}}(\theta) S^{p-1}_{\frac{p-1}{p}}(\psi) & RC_{\frac{p-1}{p}}(\theta) C^{p-1}_{\frac{p-1}{p}}(\psi) \end{vmatrix} = -R^2 S^{p-1}_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\theta)$$

$$\frac{\partial(x_1,x_3)}{\partial(\theta,\psi)} = \begin{vmatrix} -RS^{p-1}_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi) & RC^{p-1}_{\frac{p-1}{p}}(\theta) \\ -RC_{\frac{p-1}{p}}(\theta) S^{p-1}_{\frac{p-1}{p}}(\psi) & 0 \end{vmatrix} = R^2 C^{p}_{\frac{p-1}{p}}(\theta) S^{p-1}_{\frac{p-1}{p}}(\psi)$$

$$\frac{\partial(x_2,x_3)}{\partial(\theta,\psi)} = \begin{vmatrix} -RS^{p-1}_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi) & RC^{p-1}_{\frac{p-1}{p}}(\theta) \\ RC_{\frac{p-1}{p}}(\theta) C^{p-1}_{\frac{p-1}{p}}(\psi) & 0 \end{vmatrix} = -R^2 C^{p}_{\frac{p-1}{p}}(\theta) C^{p-1}_{\frac{p-1}{p}}(\psi).$$

Hence, for all conjugate exponents p and $q > 1$; i.e., $pq = p + q$, we have

$$\begin{aligned} &\left|\frac{\partial(\mathbf{x})}{\partial(\theta,\psi)}\right| \\ &= \left|\frac{\partial(x_1,x_2)}{\partial(\theta,\psi)}\right|^q + \left|\frac{\partial(x_1,x_3)}{\partial(\theta,\psi)}\right|^q + \left|\frac{\partial(x_2,x_3)}{\partial(\theta,\psi)}\right|^q \\ &= R^{2q}\left(S^{p-1}_{\frac{p-1}{p}}(\theta)\right)^q \left(C_{\frac{p-1}{p}}(\theta)\right)^q + R^{2q}\left(C^{p}_{\frac{p-1}{p}}(\theta)\right)^q \left(S^{p-1}_{\frac{p-1}{p}}(\psi)\right)^q \\ &\qquad + R^{2q}\left(C^{p}_{\frac{p-1}{p}}(\theta)\right)^q \left(C^{p-1}_{\frac{p-1}{p}}(\psi)\right)^q \end{aligned} \tag{2}$$

$$
\begin{aligned}
&= R^{2q}\left(S^{p-1}_{\frac{p-1}{p}}(\theta)\right)^{q}\left(C_{\frac{p-1}{p}}(\theta)\right)^{q} \\
&\qquad + R^{2q}\left(C^{p}_{\frac{p-1}{p}}(\theta)\right)^{q}\left[\left(S_{\frac{p-1}{p}}(\psi)\right)^{pq-q}+\left(C_{\frac{p-1}{p}}(\psi)\right)^{pq-q}\right] \\
&= R^{2q}\left(S^{p-1}_{\frac{p-1}{p}}(\theta)\right)^{q}\left(C_{\frac{p-1}{p}}(\theta)\right)^{q} \\
&\qquad + R^{2q}\left(C^{p}_{\frac{p-1}{p}}(\theta)\right)^{q}\left[\left(S_{\frac{p-1}{p}}(\psi)\right)^{p}+\left(C_{\frac{p-1}{p}}(\psi)\right)^{p}\right] \\
&= R^{2q}\left\{\left(S_{\frac{p-1}{p}}(\theta)\right)^{pq-q}\left(C_{\frac{p-1}{p}}(\theta)\right)^{q}+\left(C_{\frac{p-1}{p}}(\theta)\right)^{pq}\right\} \\
&= R^{2q}\left\{\left(S_{\frac{p-1}{p}}(\theta)\right)^{p}\left(C_{\frac{p-1}{p}}(\theta)\right)^{q}+\left(C_{\frac{p-1}{p}}(\theta)\right)^{p+q}\right\} \\
&= R^{2q}\left\{\left(C_{\frac{p-1}{p}}(\theta)\right)^{q}\left[\left(S_{\frac{p-1}{p}}(\theta)\right)^{p}+\left(C_{\frac{p-1}{p}}(\theta)\right)^{p}\right]\right\} \\
&= R^{2q}\left(C_{\frac{p-1}{p}}(\theta)\right)^{q},
\end{aligned}
$$

where we used the identity $S^{p}_{\frac{p-1}{p}}(\psi)+C^{p}_{\frac{p-1}{p}}(\psi)=1$.

Thus, changing variables yields that

$$
\begin{aligned}
\mathfrak{S}_p(A,R) = \iint\limits_{S_p(A,R)} dS_p &= \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}}\int_0^{2\omega_p}\left(\left|\frac{\partial(\mathbf{x})}{\partial(\theta,\psi)}\right|\right)^{\frac{1}{q}} d\psi d\theta \\
&= R^2\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}}\int_0^{2\omega_p} C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta = 4\omega_p R^2,
\end{aligned}
$$

as required. □

LEMMA 2. *Let* $p=\frac{q}{q-1}$, $q>1$. *The volume of the* p*-Sphere is* $vol_p(A,R):=\frac{4}{3}\omega_p R^3$, $p>1$.

Proof. Consider the transformation $\mathbf{h}:[0,R]\times\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]\times[0,2\omega_p]\to B_p(A,R)$ $\mathbf{h}=(h_1,h_2,h_3)$, given by

$$
\mathbf{h}:\begin{cases}
h_1(r,\theta,\psi)=a_1+rC_{\frac{p-1}{p}}(\theta)\,C_{\frac{p-1}{p}}(\psi)\\
h_2(r,\theta,\psi)=a_2+rC_{\frac{p-1}{p}}(\theta)\,S_{\frac{p-1}{p}}(\psi)\\
h_3(r,\theta,\psi)=a_3+rS_{\frac{p-1}{p}}(\theta)
\end{cases}
$$

where $p>1$, $r\geqslant 0$, $-\frac{\omega_p}{2}\leqslant\theta\leqslant\frac{\omega_p}{2}$ and $0\leqslant\psi\leqslant 2\omega_p$. Moreover,

$$
\begin{aligned}
&\det\left(\frac{\partial(h_1,h_2,h_3)}{\partial(r,\theta,\psi)}\right)\\
&=\begin{vmatrix}
C_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi) & C_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi) & S_{\frac{p-1}{p}}(\theta)\\
-rS^{p-1}_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi) & -rS^{p-1}_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi) & rC^{p-1}_{\frac{p-1}{p}}(\theta)\\
-rC_{\frac{p-1}{p}}(\theta)S^{p-1}_{\frac{p-1}{p}}(\psi) & rC_{\frac{p-1}{p}}(\theta)C^{p-1}_{\frac{p-1}{p}}(\psi) & 0
\end{vmatrix}\\
&=C_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi)\begin{vmatrix}
-rS^{p-1}_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi) & rC^{p-1}_{\frac{p-1}{p}}(\theta)\\
rC_{\frac{p-1}{p}}(\theta)C^{p-1}_{\frac{p-1}{p}}(\psi) & 0
\end{vmatrix}\\
&\quad -C_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi)\begin{vmatrix}
-rS^{p-1}_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi) & rC^{p-1}_{\frac{p-1}{p}}(\theta)\\
-rC_{\frac{p-1}{p}}(\theta)S^{p-1}_{\frac{p-1}{p}}(\psi) & 0
\end{vmatrix}\\
&\quad +S_{\frac{p-1}{p}}(\theta)\begin{vmatrix}
-rS^{p-1}_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi) & -rS^{p-1}_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi)\\
-rC_{\frac{p-1}{p}}(\theta)S^{p-1}_{\frac{p-1}{p}}(\psi) & rC_{\frac{p-1}{p}}(\theta)C^{p-1}_{\frac{p-1}{p}}(\psi)
\end{vmatrix}\\
&=-r^2C^p_{\frac{p-1}{p}}(\theta)C^p_{\frac{p-1}{p}}(\psi)C_{\frac{p-1}{p}}(\theta)-r^2C^p_{\frac{p-1}{p}}(\theta)S^p_{\frac{p-1}{p}}(\psi)C_{\frac{p-1}{p}}(\theta)\\
&\quad -r^2S^p_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\theta)C^p_{\frac{p-1}{p}}(\psi)-r^2S^p_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\theta)S^p_{\frac{p-1}{p}}(\psi)\\
&=-r^2C_{\frac{p-1}{p}}(\theta).
\end{aligned}\tag{3}
$$

Thus, changing variables yields that

$$
\begin{aligned}
\mathrm{vol}_p(B_p)=\iiint\limits_{B_p(A,R)}dx_1dx_2dx_3&=\int_0^R\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}}\int_0^{2\omega_p}\left|\det\left(\frac{\partial(h_1,h_2,h_3)}{\partial(r,\theta,\psi)}\right)\right|d\psi d\theta dr\\
&=\int_0^R\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}}\int_0^{2\omega_p}r^2C_{\frac{p-1}{p}}(\theta)\,d\psi d\theta dr=\frac{4}{3}\omega_pR^3,
\end{aligned}
$$

as required. □

The subsequent inequality, of Hadamard type, is valid.

THEOREM 2. *Let $p=\frac{q}{q-1}$, $q>1$. If the mapping $f:B_p(A,R)\to\mathbb{R}$ is convex on the p-ball $B_p(A,R)$ $(p>1)$, then the inequality*

$$
f(A)\leqslant\frac{1}{vol_p(B)}\iiint\limits_{B_p(A,R)}f(x_1,x_2,x_3)\,dx_1dx_2dx_3\leqslant\frac{1}{\mathfrak{S}_p(A,R)}\iint\limits_{S_p(A,R)}f(x_1,x_2,x_3)\,dS_p\tag{4}
$$

holds, where

$$S_p(A,R) := \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1-a_1|^p + |x_2-a_2|^p + |x_3-a_3|^p = R^p, p>1\right\}$$

is the p-sphere centered at the point A with radius R, and

$$vol_p(B) = \frac{4}{3}\omega_p R^3, \qquad \mathfrak{S}_p(A,R) := 4\omega_p R^2, \qquad p \gtreqqless 1.$$

The above inequalities are sharp.

Proof. Define the mapping of the Euclidean plane $\mathbb{R}^3$ onto itself, $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$ characterized by: $\mathbf{g} = (g_1,g_2,g_3)$, $g_j(\mathbf{x}) = -x_j + 2a_j$, $j = 1,2,3$. Therefore, the image of the p-ball $B_p(A,R)$ is onto itself.

It is easy to observe that

$$\frac{\partial(g_1,g_2,g_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1.$$

Changing the variables, we have

$$\begin{aligned}\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 &= \iiint\limits_{B_p(A,R)} f(g_1(\mathbf{x}),g_2(\mathbf{x}),g_3(\mathbf{x}))\left|\frac{\partial(g_1,g_2,g_3)}{\partial(x_1,x_2,x_3)}\right| dx_1dx_2dx_3 \\ &= \iiint\limits_{B_p(A,R)} f(-x_1+2a_1,-x_2+2a_2,-x_3+2a_3)\,dx_1dx_2dx_3.\end{aligned}$$

The convexity of f on $B_p(A,R)$ ensures that

$$\frac{1}{2}\left[f(x_1,x_2,x_3) + f(-x_1+2a_1,-x_2+2a_2,-x_3+2a_3)\right] \geqslant f(a_1,a_2,a_3),$$

which gives, by integration on the p-ball $B_p(A,R)$, that

$$\begin{aligned}&\frac{1}{2}\left[\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 + \iiint\limits_{B_p(A,R)} f(-x_1+2a_1,-x_2+2a_2,-x_3+2a_3)\,dx_1dx_2dx_3\right] \\ &\geqslant f(a_1,a_2,a_3)\iiint\limits_{B_p(A,R)} dx_1dx_2dx_3 \\ &= \mathrm{vol}_p(B(A,R)) \cdot f(a_1,a_2,a_3)\end{aligned}$$

since

$$\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 = \iiint\limits_{B_p(A,R)} f(-x_1+2a_1,-x_2+2a_2)\,dx_1dx_2dx_3.$$

Thus, the first inequality in (4) follows.

Now, consider the transformation $\mathbf{h} : [0,R] \times \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right] \times [0, 2\omega_p] \to B_p(A,R)$ $\mathbf{h} = (h_1, h_2, h_3)$, given by

$$\mathbf{h} : \begin{cases} h_1(r,\theta,\psi) = a_1 + rC_{\frac{p-1}{p}}(\theta)\, C_{\frac{p-1}{p}}(\psi) \\ h_2(r,\theta,\psi) = a_2 + rC_{\frac{p-1}{p}}(\theta)\, S_{\frac{p-1}{p}}(\psi) \\ h_3(r,\theta,\psi) = a_3 + rS_{\frac{p-1}{p}}(\theta) \end{cases}$$

where $r \geqslant 0$ and $0 \leqslant \theta \leqslant 2\omega_p$. Changing variables yields that (see (3)):

$$\begin{aligned} &\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\, dx_1 dx_2 dx_3 \\ &= \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(h_1,h_2,h_3) \left| \det\left(\frac{\partial(h_1,h_2,h_3)}{\partial(r,\theta,\psi)} \right) \right| d\psi\, d\theta\, dr \\ &= \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(h_1(r,\theta,\psi), h_2(r,\theta,\psi), h_3(r,\theta,\psi))\, r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi\, d\theta\, dr. \end{aligned}$$

Now, since f is convex we have

$$\begin{aligned} &f(h_1(r,\theta,\psi), h_2(r,\theta,\psi), h_3(r,\theta,\psi)) \\ &= f\left(a_1 + rC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + rC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + rS_{\frac{p-1}{p}}(\theta)\right) \\ &= f\left(\frac{r}{R}\left(a_1 + rC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + rC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + rS_{\frac{p-1}{p}}(\theta)\right)\right. \\ &\qquad\qquad \left. + \left(1 - \frac{r}{R}\right)(a_1,a_2,a_3)\right) \\ &\leqslant \frac{r}{R} f\left(a_1 + RC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + RC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + RS_{\frac{p-1}{p}}(\theta)\right) \\ &\qquad + \left(1 - \frac{r}{R}\right) f(a_1,a_2,a_3). \end{aligned}$$

Consequently, we have

$$\begin{aligned} &f\left(a_1 + rC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + rC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + rS_{\frac{p-1}{p}}(\theta)\right) r^2 C_{\frac{p-1}{p}}(\theta) \\ &\leqslant \frac{r}{R} f\left(a_1 + RC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + RC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + RS_{\frac{p-1}{p}}(\theta)\right) r^2 C_{\frac{p-1}{p}}(\theta) \\ &\qquad + \left(1 - \frac{r}{R}\right) f(a_1,a_2,a_3)\, r^2 C_{\frac{p-1}{p}}(\theta). \end{aligned}$$

Integrating the above inequality on $[0,R]\times\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]\times[0,2\omega_p]$, we get

$$\begin{aligned}
&\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \\
&\leqslant \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(h_1(R,\theta,\psi),h_2(R,\theta,\psi),h_3(R,\theta,\psi))\,r^2 C_{\frac{p-1}{p}}(\theta)\,d\psi d\theta dr \\
&\qquad + \int_0^R \int_{-\frac{\omega_p}{2}}^{-\frac{\omega_p}{2}} \int_0^{2\omega_p} \left(1-\frac{r}{R}\right) f(a_1,a_2,a_3)\,r^2 C_{\frac{p-1}{p}}(\theta) d\psi d\theta dr \\
&= \frac{R^3}{4}\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(h_1(R,\theta,\psi),h_2(R,\theta,\psi),h_3(R,\theta,\psi))\,C_{\frac{p-1}{p}}(\theta)\,d\psi d\theta \qquad (5)\\
&\qquad + \frac{\omega_p R^3}{3} f(a_1,a_2,a_3).
\end{aligned}$$

For sake of simplicity, let us denote

$$\mathbf{P} := \frac{R^3}{4}\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(h_1(R,\theta,\psi),h_2(R,\theta,\psi),h_3(R,\theta,\psi))\,C_{\frac{p-1}{p}}(\theta)\,d\psi d\theta. \qquad (6)$$

Hence, to determine the p-sphere volume described above we compute the fundamental vector product and then integrate its q-length over the region T. For this purpose, consider the parameterized p-surface $\lambda:\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]\times[0,2\omega_p]\to\mathbb{R}^2$ defined by

$$S_p(A,R):\begin{cases} x_1(\theta,\psi)=a_1+RC_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi) \\ x_2(\theta,\psi)=a_2+RC_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi) \\ x_3(\theta,\psi)=a_3+RS_{\frac{p-1}{p}}(\theta) \end{cases};$$

where $R>0$, $\theta\in\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]$, and $\psi\in[0,2\omega_p]$.

Therefore, (2) implies that

$$\begin{aligned}
\mathbf{M} &:= \iint\limits_{S_p(A,R)} f(x_1,x_2,x_3)\,dS_p \qquad (7)\\
&= \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(\theta,\psi),x_2(\theta,\psi),x_3(\theta,\psi))\left(\left|\frac{\partial(\mathbf{x})}{\partial(\theta,\psi)}\right|\right)^{\frac{1}{q}} d\psi d\theta \\
&= R^2\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(R,\theta,\psi),x_2(R,\theta,\psi),x_3(R,\theta,\psi))\,C_{\frac{p-1}{p}}(\theta)\,d\psi d\theta,
\end{aligned}$$

thus (6) and this implies that

$$\mathbf{P}=\frac{R}{4}\cdot\mathbf{M}.$$

So that, the inequality (5) implies that

$$\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \leqslant \frac{\omega_p R^3}{3} f(a_1,a_2,a_3) + \frac{R}{4}\cdot \mathbf{M}$$

$$= \frac{\omega_p R^3}{3} f(a_1,a_2,a_3) + \frac{R}{4} \iint\limits_{S_p(A,R)} f(x_1,x_2,x_3)\,dS_p.$$

Dividing the previous inequality by $\mathrm{vol}_p(B) = \frac{4}{3}R^3\omega_p$, we get

$$\frac{1}{\mathrm{vol}_p(B)} \iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \leqslant \frac{1}{4} f(a_1,a_2,a_3) + \frac{3}{4}\cdot \frac{1}{\mathfrak{S}_p(A,R)} \iint\limits_{S_p(A,R)} f(x_1,x_2,x_3)\,dS_p, \quad (8)$$

but the left-hand side inequality tells us

$$f(a_1,a_2,a_3) \leqslant \frac{1}{\mathrm{vol}_p(B)} \iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3$$

therefore, (8) implies that

$$\frac{1}{\mathrm{vol}_p(B)} \iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \leqslant \frac{1}{\mathfrak{S}_p(A,R)} \iint\limits_{S_p(A,R)} f(x_1,x_2,x_3)\,dS_p$$

which is the right-hand inequality in (4), and this finishes the proof completely. □

THEOREM 3. *Let $p = \frac{q}{q-1}$, $q > 1$, $A = (a_1,a_2,a_3) \in \mathbb{R}^3$ be any point, and $f : B_p(A,R) \to \mathbb{R}$ be a function defined in the open p-ball $B_p(A,R)$ satisfying the Hölder condition*

$$|f(a_1,a_2,a_3) - f(y_1,y_2,y_3)| \leqslant L_1\cdot|a_1-y_1|^{\beta_1} + L_2\cdot|a_2-y_2|^{\beta_2} + L_3\cdot|a_3-y_3|^{\beta_3}, \quad (9)$$

for all $(y_1,y_2,y_3) \in B_p(A,R)\setminus\{A\}$, and for some constants $L_1,L_2,L_3 > 0$ and $\beta_1,\beta_2.\beta_3 \in (-1,\infty]$. Then, we have

$$\left| f(A) - \frac{1}{vol_p(B)} \iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \right| \leqslant \frac{6}{\omega_p}\cdot\left[L_1\cdot\frac{R^{\beta_1}}{\beta_1+3}\cdot\frac{\Gamma\left(\frac{\beta_1+1}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\beta_1+3}{p}\right)} + L_2\cdot\frac{R^{\beta_2}}{\beta_2+3}\cdot\frac{\Gamma\left(\frac{\beta_2+1}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\beta_2+3}{p}\right)} + L_3\cdot\frac{\omega_p\cdot R^{\beta_2}}{2(\beta_3+1)(\beta_3+3)} \right]. \quad (10)$$

The constant $\frac{6}{\omega_p}$ *is the best possible for all* $\beta_1,\beta_2,\beta_3>0$.

Proof. The triangle integral inequality and (9) implies that

$$\left|f(A)-\frac{1}{\mathrm{vol}_p(B)}\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3\right|$$
$$\leqslant \frac{1}{\mathrm{vol}_p(B)}\iiint\limits_{B_p(A,R)} |f(a_1,a_2,a_3)-f(x_1,x_2,x_3)|\,dx_1dx_2dx_3$$
$$\leqslant \frac{1}{\mathrm{vol}_p(B)}\Bigg[L_1\cdot\iiint\limits_{B_p(A,R)}|x_1-a_1|^{\beta_1}\,dx_1dx_2dx_3+L_2\cdot\iiint\limits_{B_p(A,R)}|x_2-a_2|^{\beta_2}\,dx_1dx_2dx_3 \qquad (11)$$
$$+L_3\cdot\iiint\limits_{B_p(A,R)}|x_3-a_3|^{\beta_3}\,dx_1dx_2dx_3\Bigg].$$

Making of use the change of variables

$$\begin{cases} x_1(r,\theta,\psi)=a_1+rC_{\frac{p-1}{p}}(\theta)\,C_{\frac{p-1}{p}}(\psi)\\ x_2(r,\theta,\psi)=a_2+rC_{\frac{p-1}{p}}(\theta)\,S_{\frac{p-1}{p}}(\psi)\\ x_3(r,\theta,\psi)=a_3+rS_{\frac{p-1}{p}}(\theta)\end{cases}$$

where $r>0$, $\theta\in\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]$, and $\psi\in[0,2\omega_p]$.

Thus, from (3) we have

$$\left|\det\left(\frac{\partial(h_1,h_2,h_3)}{\partial(r,\theta,\psi)}\right)\right|=r^2C_{\frac{p-1}{p}}(\theta).$$

So that, we have

$$\iiint\limits_{B_p(A,R)}|x_1-a_1|^{\beta_1}\,dx_1dx_2dx_3$$
$$=\int_0^R\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}}\int_0^{2\omega_p} r^{\beta_1}\left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_1}\left|C_{\frac{p-1}{p}}(\psi)\right|^{\beta_1}r^2C_{\frac{p-1}{p}}(\theta)\,d\psi d\theta dr$$
$$=\int_0^R r^{\beta_1+2}dr\cdot\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}}\left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_1}C_{\frac{p-1}{p}}(\theta)\,d\theta\cdot\int_0^{2\omega_p}\left|C_{\frac{p-1}{p}}(\psi)\right|^{\beta_1}d\psi$$
$$=\frac{R^{\beta_1+3}}{\beta_1+3}\cdot 2\int_0^{\frac{\omega_p}{2}}\left(C_{\frac{p-1}{p}}(\theta)\right)^{\beta_1+1}d\theta\cdot 4\int_0^{\frac{\omega_p}{2}}\left(C_{\frac{p-1}{p}}(\psi)\right)^{\beta_1}d\psi$$
$$=8\cdot\frac{R^{\beta_1+3}}{\beta_1+3}\cdot\frac{\Gamma\left(\frac{\beta_1+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_1+3}{p}\right)}\cdot\frac{\Gamma\left(\frac{\beta_1+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_1+2}{p}\right)},$$

$$\begin{aligned}
&\iiint\limits_{B_p(A,R)} |x_2-a_2|^{\beta_2}\, dx_1dx_2dx_3 \\
&= \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} r^{\beta_2} \left| C_{\frac{p-1}{p}}(\theta)\right|^{\beta_2} \left| S_{\frac{p-1}{p}}(\psi)\right|^{\beta_2} r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta dr \\
&= \int_0^R r^{\beta_2+2} dr \cdot \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| C_{\frac{p-1}{p}}(\theta)\right|^{\beta_2} C_{\frac{p-1}{p}}(\theta)\, d\theta \cdot \int_0^{2\omega_p} \left| S_{\frac{p-1}{p}}(\psi)\right|^{\beta_2} d\psi \\
&= \frac{R^{\beta_2+3}}{\beta_2+3} \cdot 2\int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\theta)\right)^{\beta_2+1} d\theta \cdot 4\int_0^{\frac{\omega_p}{2}} \left(S_{\frac{p-1}{p}}(\psi)\right)^{\beta_2} d\psi \\
&= 8 \cdot \frac{R^{\beta_2+3}}{\beta_2+3} \cdot \frac{\Gamma\left(\frac{\beta_2+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_2+3}{p}\right)} \cdot \frac{\Gamma\left(\frac{\beta_2+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_2+2}{p}\right)},
\end{aligned}$$

and

$$\begin{aligned}
\iiint\limits_{B_p(A,R)} |x_3-a_3|^{\beta_3}\, dx_1dx_2dx_3 &= \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} r^{\beta_2} \left| S_{\frac{p-1}{p}}(\theta)\right|^{\beta_3} r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta dr \\
&= 4\omega_p \cdot \frac{R^{\beta_2+3}}{(\beta_3+1)(\beta_3+3)}.
\end{aligned}$$

Substituting in (11) we get the required inequality in (10). The sharpness of the constant $\frac{6}{\omega_p}$ follows by considering the function $f : B(S,R) \to \mathbb{R}$ defined as

$$f(x_1,x_2,x_3) = L_1 \cdot |x_1-a_1|^{\beta_1} + L_2 \cdot |x_2-a_2|^{\beta_2} + L_3 \cdot |x_3-a_3|^{\beta_3},$$

for all $\beta_1, \beta_2, \beta_3 > 0$. □

COROLLARY 1. *Let $A = (a_1,a_2,a_3) \in \mathbb{R}^3$ be any point and $f : B(A,R) \to \mathbb{R}$ be a function defined in the open ball $B(A,R)$ satisfying the Hölder condition*

$$|f(a_1,a_2,a_3) - f(y_1,y_2,y_3)| \leqslant L_1 \cdot |a_1-y_1|^{\beta_1} + L_2 \cdot |a_2-y_2|^{\beta_2} + L_3 \cdot |a_1-y_1|^{\beta_3}, \tag{12}$$

for all $(y_1,y_2,y_3) \in B(A,R) \setminus \{A\}$, and for some constants $L_1,L_2,L_3 > 0$ and $\beta_1,\beta_2.\beta_3 \in (-1,\infty]$. Then, we have

$$\begin{aligned}
&\left| f(A) - \frac{1}{vol(B)} \iiint\limits_{B(A,R)} f(x_1,x_2,x_3)\, dx_1dx_2dx_3 \right| \\
&\qquad \leqslant \frac{6}{\pi} \cdot \left[L_1 \cdot \frac{R^{\beta_1}}{\beta_1+3} \cdot \frac{\pi\Gamma\left(\frac{\beta_1+1}{2}\right)}{4\Gamma\left(\frac{\beta_1+3}{2}\right)} + L_2 \cdot \frac{R^{\beta_2}}{\beta_2+3} \cdot \frac{\pi\Gamma\left(\frac{\beta_2+1}{2}\right)}{4\Gamma\left(\frac{\beta_2+3}{2}\right)} \right. \\
&\qquad\qquad \left. + L_3 \cdot \frac{\pi \cdot R^{\beta_2}}{2(\beta_3+1)(\beta_3+3)} \right].
\end{aligned} \tag{13}$$

Proof. Setting $p=2$ in (10). $\square$

COROLLARY 2. *Let* $p=\frac{q}{q-1}$, $q>1$, $A=(a_1,a_2,a_3)\in\mathbb{R}^3$ *be any point, and* $f: B(A,R)\to\mathbb{R}$ *be a function defined in the open ball* $B(A,R)$ *satisfying the Lipschitz condition*

$$|f(a_1,a_2,a_3)-f(y_1,y_2,y_3)| \leqslant L_1\cdot|a_1-y_1|+L_2\cdot|a_2-y_2|+L_3\cdot|a_1-y_1|, \qquad (14)$$

for all $(y_1,y_2,y_3)\in B(A,R)\backslash\{A\}$, *and for some constants* $L_1,L_2,L_3>0$. *Then, we have*

$$\left|f(A)-\frac{1}{vol(B)}\iiint\limits_{B(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3\right| \leqslant \frac{3R}{8}\cdot(L_1+L_2+L_3). \qquad (15)$$

Proof. Setting $\beta_1=\beta_2=\beta_3=1$ in (13). $\square$

THEOREM 4. *Let* $p=\frac{q}{q-1}$, $q>1$, $A=(a_1,a_2,a_3)\in\mathbb{R}^3$ *be any point, and* $f: B_p(A,R)\to\mathbb{R}$ *be a function defined in the open* p*-ball* $B_p(A,R)$ *satisfying the Hölder condition*

$$\begin{aligned}|f(a_1,a_2,a_3)&-f(y_1,y_2,y_3)|\\ &\leqslant L_1\cdot|a_1-y_1|^{\beta_1}|a_2-y_2|^{\beta_2}+L_2\cdot|a_2-y_2|^{\beta_3}|a_3-y_3|^{\beta_4}\\ &\qquad +L_3\cdot|a_1-y_1|^{\beta_5}|a_3-y_3|^{\beta_6}, \qquad (16)\end{aligned}$$

for all $(y_1,y_2,y_3)\in B_p(A,R)\backslash\{A\}$, *and for some constants* $L_1,L_2,L_3>0$ *and* $\overline{\beta_1,\beta_6}\in(-1,\infty]$. *Then, we have*

$$\begin{aligned}&\left|f(A)-\frac{1}{vol_p(B)}\iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3\right|\\ &\leqslant \frac{6\Gamma^2\left(\frac{1}{p}\right)}{p^2\omega_p}\cdot\left[L_1\cdot\frac{R^{\beta_1+\beta_2}}{\beta_1+\beta_2+3}\cdot\frac{\Gamma\left(\frac{\beta_1+1}{p}\right)\Gamma\left(\frac{\beta_2+1}{p}\right)}{\Gamma\left(\frac{\beta_1+\beta_2+3}{p}\right)}\right.\\ &\quad +L_2\cdot\frac{R^{\beta_3+\beta_4}}{\beta_3+\beta_4+3}\cdot\frac{\Gamma\left(\frac{\beta_3+\beta_4+2}{p}\right)}{\Gamma\left(\frac{\beta_3+\beta_4+3}{p}\right)}\cdot\frac{\Gamma\left(\frac{\beta_3+1}{p}\right)}{\Gamma\left(\frac{\beta_3+2}{p}\right)}\\ &\quad \left.+L_3\cdot\frac{R^{\beta_5+\beta_6}}{\beta_5+\beta_6+3}\cdot\frac{\Gamma\left(\frac{\beta_5+\beta_6+2}{p}\right)}{\Gamma\left(\frac{\beta_5+\beta_6+3}{p}\right)}\cdot\frac{\Gamma\left(\frac{\beta_5+1}{p}\right)}{\Gamma\left(\frac{\beta_5+2}{p}\right)}\right]. \qquad (17)\end{aligned}$$

The constant $\frac{6\Gamma^2\left(\frac{1}{p}\right)}{p^2\omega_p}$ *is sharp for all* $\overline{\beta_1,\beta_6}>0$.

Proof. On utilizing the triangle integral inequality and (16), we get

$$\begin{aligned}
&\left| f(A) - \frac{1}{\mathrm{vol}_p(B)} \iiint_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \right| \\
&\leqslant \frac{1}{\mathrm{vol}_p(B)} \iiint_{B_p(A,R)} |f(a_1,a_2,a_3) - f(x_1,x_2,x_3)|\,dx_1dx_2dx_3 \\
&\leqslant \frac{1}{\mathrm{vol}_p(B)} \left[L_1 \cdot \iiint_{B_p(A,R)} |x_1-a_1|^{\beta_1} |x_2-a_2|^{\beta_2}\,dx_1dx_2dx_3 \right. \\
&\qquad + L_2 \cdot \iiint_{B_p(A,R)} |x_2-a_2|^{\beta_3} |x_3-a_3|^{\beta_4}\,dx_1dx_2dx_3 \\
&\qquad \left. + L_3 \cdot \iiint_{B_p(A,R)} |x_1-a_1|^{\beta_5} |x_3-a_3|^{\beta_6}\,dx_1dx_2dx_3 \right].
\end{aligned} \tag{18}$$

Making of use the change of variables

$$\begin{cases}
x_1(r,\theta,\psi) = a_1 + rC_{\frac{p-1}{p}}(\theta)\,C_{\frac{p-1}{p}}(\psi) \\
x_2(r,\theta,\psi) = a_2 + rC_{\frac{p-1}{p}}(\theta)\,S_{\frac{p-1}{p}}(\psi) \\
x_3(r,\theta,\psi) = a_3 + rS_{\frac{p-1}{p}}(\theta)
\end{cases}$$

where $r > 0$, $\theta \in \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$, and $\psi \in [0, 2\omega_p]$.

Thus, from (3) we have

$$\left| \det \left(\frac{\partial(h_1,h_2,h_3)}{\partial(r,\theta,\psi)} \right) \right| = r^2 C_{\frac{p-1}{p}}(\theta).$$

Therefore, we have

$$\begin{aligned}
&\iiint_{B_p(A,R)} |x_1-a_1|^{\beta_1} |x_2-a_2|^{\beta_2}\,dx_1dx_2dx_3 \\
&= \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} r^{\beta_1+\beta_2} \left| C_{\frac{p-1}{p}}(\theta) \right|^{\beta_1} \left| C_{\frac{p-1}{p}}(\psi) \right|^{\beta_1} \\
&\quad \times \left| C_{\frac{p-1}{p}}(\theta) \right|^{\beta_2} \left| S_{\frac{p-1}{p}}(\psi) \right|^{\beta_2} r^2 C_{\frac{p-1}{p}}(\theta)\,d\psi d\theta dr \\
&= \int_0^R r^{\beta_1+\beta_2+2} dr \cdot \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| C_{\frac{p-1}{p}}(\theta) \right|^{\beta_1+\beta_2} C_{\frac{p-1}{p}}(\theta)\,d\theta \\
&\quad \times \int_0^{2\omega_p} \left| C_{\frac{p-1}{p}}(\psi) \right|^{\beta_1} \left| S_{\frac{p-1}{p}}(\psi) \right|^{\beta_2} d\psi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{R^{\beta_1+\beta_2+3}}{\beta_1+\beta_2+3} \cdot 2\int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\theta)\right)^{\beta_1+\beta_2+1} d\theta \\
&\quad \times 4\int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\psi)\right)^{\beta_1} \left(S_{\frac{p-1}{p}}(\psi)\right)^{\beta_2} d\psi \\
&= 8 \cdot \frac{R^{\beta_1+\beta_2+3}}{\beta_1+\beta_2+3} \cdot \frac{\Gamma\left(\frac{\beta_1+\beta_2+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_1+\beta_2+3}{p}\right)} \cdot \frac{\Gamma\left(\frac{\beta_1+1}{p}\right)\Gamma\left(\frac{\beta_2+1}{p}\right)}{p\Gamma\left(\frac{\beta_1+\beta_2+2}{p}\right)} \\
&= 8 \cdot \frac{R^{\beta_1+\beta_2+3}}{\beta_1+\beta_2+3} \cdot \frac{\Gamma\left(\frac{\beta_1+1}{p}\right)\Gamma\left(\frac{\beta_2+1}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\beta_1+\beta_2+3}{p}\right)},
\end{aligned}
$$

$$
\begin{aligned}
&\iiint\limits_{B_p(A,R)} |x_2-a_2|^{\beta_3} |x_3-a_3|^{\beta_4} dx_1dx_2dx_3 \\
&= \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} r^{\beta_3+\beta_4} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_3} \left|S_{\frac{p-1}{p}}(\psi)\right|^{\beta_3} \left|S_{\frac{p-1}{p}}(\theta)\right|^{\beta_4} r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta dr \\
&= \int_0^R r^{\beta_3+\beta_4+2} dr \cdot \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_3} \left|S_{\frac{p-1}{p}}(\theta)\right|^{\beta_4} C_{\frac{p-1}{p}}(\theta)\, d\theta \cdot \int_0^{2\omega_p} \left|S_{\frac{p-1}{p}}(\psi)\right|^{\beta_3} d\psi \\
&= \frac{R^{\beta_3+\beta_4+3}}{\beta_3+\beta_4+3} \cdot 2\int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\theta)\right)^{\beta_3+1} \left(S_{\frac{p-1}{p}}(\theta)\right)^{\beta_4} d\theta \cdot 4\int_0^{\frac{\omega_p}{2}} \left(S_{\frac{p-1}{p}}(\psi)\right)^{\beta_3} d\psi \\
&= 8 \cdot \frac{R^{\beta_3+\beta_4+3}}{\beta_3+\beta_4+3} \cdot \frac{\Gamma\left(\frac{\beta_3+\beta_4+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_3+\beta_4+3}{p}\right)} \cdot \frac{\Gamma\left(\frac{\beta_3+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_3+2}{p}\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\iiint\limits_{B_p(A,R)} |x_1-a_1|^{\beta_3} |x_3-a_3|^{\beta_4} dx_1dx_2dx_3 \\
&= \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} r^{\beta_5+\beta_6} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_5} \left|C_{\frac{p-1}{p}}(\psi)\right|^{\beta_5} \left|S_{\frac{p-1}{p}}(\theta)\right|^{\beta_6} r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta dr \\
&= \int_0^R r^{\beta_5+\beta_6+2} dr \cdot \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_5} \left|S_{\frac{p-1}{p}}(\theta)\right|^{\beta_6} C_{\frac{p-1}{p}}(\theta)\, d\theta \cdot \int_0^{2\omega_p} \left|C_{\frac{p-1}{p}}(\psi)\right|^{\beta_5} d\psi \\
&= \frac{R^{\beta_4+\beta_6+3}}{\beta_5+\beta_6+3} \cdot 2\int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\theta)\right)^{\beta_5+1} \left(S_{\frac{p-1}{p}}(\theta)\right)^{\beta_6} d\theta \cdot 4\int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\psi)\right)^{\beta_5} d\psi \\
&= 8 \cdot \frac{R^{\beta_5+\beta_6+3}}{\beta_5+\beta_6+3} \cdot \frac{\Gamma\left(\frac{\beta_5+\beta_6+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_5+\beta_6+3}{p}\right)} \cdot \frac{\Gamma\left(\frac{\beta_5+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_5+2}{p}\right)}.
\end{aligned}
$$

Substituting in (18) and dividing on $\mathrm{vol}_p(B) = \frac{4}{3}\omega_p R^3$ we get the required inequality in (17). The sharpness of the constant $\frac{6\Gamma^2\left(\frac{1}{p}\right)}{p^2\omega_p}$ follows by considering the function $f : B(S,R) \to \mathbb{R}$ defined as

$$f(x_1,x_2,x_3) = L_1 \cdot |x_1 - a_1|^{\beta_1} |x_2 - a_2|^{\beta_2} + L_2 \cdot |x_2 - a_2|^{\beta_3} |x_3 - a_3|^{\beta_4} + L_3 \cdot |x_1 - a_1|^{\beta_5} |x_3 - a_3|^{\beta_6},$$

for all $\overline{\beta_1,\beta_6} > 0$. □

COROLLARY 3. *Let $p = \frac{q}{q-1}$, $q > 1$, $A = (a_1,a_2,a_3) \in \mathbb{R}^3$ be any point, and $f : B_p(A,R) \to \mathbb{R}$ be a function defined in the open p-ball $B_p(A,R)$ satisfying the Hölder condition*

$$|f(a_1,a_2,a_3) - f(y_1,y_2,y_3)| \leqslant L_1 \cdot |a_1 - y_1|^{\alpha} |a_2 - y_2|^{\alpha} + L_2 \cdot |a_2 - y_2|^{\beta} |a_3 - y_3|^{\beta} + L_3 \cdot |a_1 - y_1|^{\gamma} |a_3 - y_3|^{\gamma}, \quad (19)$$

for all $(y_1,y_2,y_3) \in B_p(A,R) \setminus \{A\}$, and for some constants $L_1,L_2,L_3 > 0$ and $\alpha,\beta,\gamma \in (-1,\infty]$. Then, we have

$$\left| f(A) - \frac{1}{vol_p(B)} \iiint_{B_p(A,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \right| \leqslant \frac{6\Gamma^2\left(\frac{1}{p}\right)}{p^2\omega_p} \cdot \left[L_1 \cdot \frac{R^{2\alpha}}{2\alpha+3} \cdot \frac{\Gamma^2\left(\frac{\alpha+1}{p}\right)}{\Gamma\left(\frac{2\alpha+3}{p}\right)} + L_2 \cdot \frac{R^{2\beta}}{2\beta+3} \cdot \frac{\Gamma\left(\frac{2\beta+2}{p}\right)}{\Gamma\left(\frac{2\beta+3}{p}\right)} \cdot \frac{\Gamma\left(\frac{\beta+1}{p}\right)}{\Gamma\left(\frac{\beta+2}{p}\right)} + L_3 \cdot \frac{R^{2\gamma}}{2\gamma+3} \cdot \frac{\Gamma\left(\frac{2\gamma+2}{p}\right)}{\Gamma\left(\frac{2\gamma+3}{p}\right)} \cdot \frac{\Gamma\left(\frac{\gamma+1}{p}\right)}{\Gamma\left(\frac{\gamma+2}{p}\right)} \right]. \quad (20)$$

The constant $\frac{6\Gamma^2\left(\frac{1}{p}\right)}{p^2\omega_p}$ is sharp for all $\overline{\beta_1,\beta_6} > 0$.

Proof. Setting $\beta_1 = \beta_2 = \beta_3 = \alpha$, $\beta_3 = \beta_4 = \beta_6 = \beta$, and $\beta_7 = \beta_8 = \gamma$ in (17). □

COROLLARY 4. *Let $p = \frac{q}{q-1}$, $q > 1$, $A = (a_1,a_2,a_3) \in \mathbb{R}^3$ be any point, and $f : B_p(A,R) \to \mathbb{R}$ be a function defined in the open p-ball $B_p(A,R)$ satisfying the Hölder condition*

$$|f(a_1,a_2,a_3) - f(y_1,y_2,y_3)| \leqslant L_1 \cdot |a_1 - y_1|\,|a_2 - y_2| + L_2 \cdot |a_2 - y_2|\,|a_3 - y_3| + L_3 \cdot |a_1 - y_1|\,|a_3 - y_3|, \quad (21)$$

for all $(y_1,y_2,y_3) \in B_p(A,R) \setminus \{A\}$, *and for some constants* $L_1,L_2,L_3 > 0$. *Then, we have*

$$\left| f(A) - \frac{1}{vol_p(B)} \iiint_{B_p(A,R)} f(x_1,x_2,x_3)\, dx_1 dx_2 dx_3 \right| \leqslant \frac{6R^2\Gamma^2\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)}{5p^2\omega_p\Gamma\left(\frac{5}{p}\right)} \cdot \left[L_1 \cdot \Gamma\left(\frac{2}{p}\right) + (L_2+L_3) \cdot \frac{\Gamma\left(\frac{4}{p}\right)}{\Gamma\left(\frac{3}{p}\right)} \right]. \quad (22)$$

The constant $\frac{6\Gamma^2\left(\frac{1}{p}\right)}{p^2\omega_p}$ *is sharp for all* $\overline{\beta_1,\beta_6} > 0$.

Proof. Setting $\alpha = \beta = \gamma = 1$ in (20). □

COROLLARY 5. *Let* $A = (a_1,a_2,a_3) \in \mathbb{R}^3$ *be any point, and* $f : B(A,R) \to \mathbb{R}$ *be a function defined in the open ball* $B(A,R)$ *satisfying the Hölder condition*

$$|f(a_1,a_2,a_3) - f(y_1,y_2,y_3)| \leqslant L_1 \cdot |a_1-y_1||a_2-y_2| + L_2 \cdot |a_2-y_2||a_3-y_3| + L_3 \cdot |a_1-y_1||a_3-y_3|, \quad (23)$$

for all $(y_1,y_2,y_3) \in B(A,R) \setminus \{A\}$, *and for some constants* $L_1,L_2,L_3 > 0$. *Then, we have*

$$\left| f(A) - \frac{1}{vol(B)} \iiint_{B(A,R)} f(x_1,x_2,x_3)\, dx_1 dx_2 dx_3 \right| \leqslant \frac{2R^2}{5\sqrt{\pi}} \cdot \left[L_1 + (L_2+L_3) \cdot \frac{2}{\sqrt{\pi}} \right]. \quad (24)$$

The constant $\frac{6\Gamma^2\left(\frac{1}{p}\right)}{p^2\omega_p}$ *is sharp for all* $\overline{\beta_1,\beta_6} > 0$.

Proof. Setting $p = 2$ in (22). □

THEOREM 5. *Let* $p = \frac{q}{q-1}$ $(q > 1)$ *and* $f : B_p(A,R) \to \mathbb{R}$ *be a function defined in the open* p*-ball* $B_p(A,R)$ *satisfying the condition*

$$\begin{aligned} &\Big| f\left(a_1 + rC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + rC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + rS_{\frac{p-1}{p}}(\theta)\right) \\ &\quad - f\left(a_1 + RC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + RC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + RS_{\frac{p-1}{p}}(\theta)\right)\Big| \\ &\leqslant L_1 (R-r)^{\beta_1} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_2} \left|C_{\frac{p-1}{p}}(\psi)\right|^{\beta_3} + L_2 (R-r)^{\beta_4} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_5} \left|S_{\frac{p-1}{p}}(\psi)\right|^{\beta_6} \\ &\qquad + L_3 (R-r)^{\beta_7} \left|S_{\frac{p-1}{p}}(\theta)\right|^{\beta_8} \end{aligned} \quad (25)$$

for all $r \in [0,R]$, $\theta \in \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$, $\psi \in [0, 2\omega_p]$, *and some constants* $L_1, L_2, L_3 > 0$ *and* $\beta_i \in (-1, \infty]$ $\left(i = \overline{1,8}\right)$. *Then, we have*

$$\left| \frac{1}{vol_p(B)} \iiint\limits_{B_p(A,R)} f(x_1,x_2,x_3)\, dx_1 dx_2 dx_3 - \frac{1}{\mathfrak{S}_p(A,R)} \iint\limits_{S_p(A,R)} f(x_1,x_2,x_3)\, dS_p \right| \tag{26}$$
$$\leqslant \frac{6}{\omega_p} \left[L_1 \frac{2\beta_1 R^{\beta_1}}{(\beta_1+1)(\beta_1+2)(\beta_1+3)} \times \frac{\Gamma\left(\frac{\beta_2+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_2+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_3+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_3+2}{p}\right)} \right.$$
$$+ L_2 \frac{2\beta_4 R^{\beta_4}}{(\beta_4+1)(\beta_4+2)(\beta_4+3)} \times \frac{\Gamma\left(\frac{\beta_5+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_5+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_6+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_6+2}{p}\right)}$$
$$\left. + L_3 \frac{\beta_7 R^{\beta_7}}{(\beta_7+1)(\beta_7+2)(\beta_7+3)} \times \frac{1}{\beta_8+1} \right].$$

Proof. Using the change of variables

$$\begin{cases} x_1(r,\theta,\psi) = a_1 + rC_{\frac{p-1}{p}}(\theta)\, C_{\frac{p-1}{p}}(\psi) \\ x_2(r,\theta,\psi) = a_2 + rC_{\frac{p-1}{p}}(\theta)\, S_{\frac{p-1}{p}}(\psi) \\ x_3(r,\theta,\psi) = a_3 + rS_{\frac{p-1}{p}}(\theta) \end{cases}$$

where $r > 0, \theta \in \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$, and $\psi \in [0, 2\omega_p]$. For which we have

$$\left| \det\left(\frac{\partial(x_1,x_2,x_3)}{\partial(r,\theta,\psi)} \right) \right| = r^2 C_{\frac{p-1}{p}}(\theta).$$

The triangle integral inequality and (25) implies that

$$\left| \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(r,\theta,\psi), x_2(r,\theta,\psi), x_3(r,\theta,\psi))\, r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta dr \right. \tag{27}$$
$$\left. - \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi))\, r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta dr \right|$$
$$= \left| \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} [f(x_1(r,\theta,\psi), x_2(r,\theta,\psi), x_3(r,\theta,\psi)) \right.$$
$$\left. - \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi))]\, r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta dr \right|$$
$$\leqslant \int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} |f(x_1(r,\theta,\psi), x_2(r,\theta,\psi), x_3(r,\theta,\psi))$$
$$- f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi))|\, r^2 \left| C_{\frac{p-1}{p}}(\theta) \right| d\psi d\theta dr$$

$$\begin{aligned}
&\leqslant L_1 \int_0^R r^2 (R-r)^{\beta_1} dr \times \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| C_{\frac{p-1}{p}}(\theta)\right|^{\beta_2} \left| C_{\frac{p-1}{p}}(\theta)\right| d\theta \times \int_0^{2\omega_p} \left| C_{\frac{p-1}{p}}(\psi)\right|^{\beta_3} d\psi \\
&\quad + L_2 \int_0^R r^2 (R-r)^{\beta_4} dr \times \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| C_{\frac{p-1}{p}}(\theta)\right|^{\beta_5} \left| C_{\frac{p-1}{p}}(\theta)\right| d\theta \times \int_0^{2\omega_p} \left| S_{\frac{p-1}{p}}(\psi)\right|^{\beta_6} d\psi \\
&\quad + L_3 \int_0^R r^2 (R-r)^{\beta_7} dr \times \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| S_{\frac{p-1}{p}}(\theta)\right|^{\beta_8} \left| C_{\frac{p-1}{p}}(\theta)\right| d\theta \\
&= L_1 \frac{2\beta_1 R^{\beta_1+3}}{(\beta_1+1)(\beta_1+2)(\beta_1+3)} \\
&\qquad \times 2\cdot \int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\theta)\right)^{\beta_2} C_{\frac{p-1}{p}}(\theta)\, d\theta \times 4\cdot \int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\psi)\right)^{\beta_3} d\psi \\
&\quad + L_2 \frac{2\beta_4 R^{\beta_4+3}}{(\beta_4+1)(\beta_4+2)(\beta_4+3)} \\
&\qquad \times 2\cdot \int_0^{\frac{\omega_p}{2}} \left(C_{\frac{p-1}{p}}(\theta)\right)^{\beta_5} C_{\frac{p-1}{p}}(\theta)\, d\theta \times 4\cdot \int_0^{\frac{\omega_p}{2}} \left(S_{\frac{p-1}{p}}(\psi)\right)^{\beta_6} d\psi \\
&\quad + 2\omega_p \cdot L_3 \frac{2\beta_7 R^{\beta_7+3}}{(\beta_7+1)(\beta_7+2)(\beta_7+3)} \times 2\cdot \int_0^{\frac{\omega_p}{2}} \left(S_{\frac{p-1}{p}}(\theta)\right)^{\beta_8} C_{\frac{p-1}{p}}(\theta)\, d\theta \\
&= L_1 \frac{16\beta_1 R^{\beta_1+3}}{(\beta_1+1)(\beta_1+2)(\beta_1+3)} \times \frac{\Gamma\left(\frac{\beta_2+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_2+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_3+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_3+2}{p}\right)} \\
&\quad + L_2 \frac{16\beta_4 R^{\beta_4+3}}{(\beta_4+1)(\beta_4+2)(\beta_4+3)} \times \frac{\Gamma\left(\frac{\beta_5+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_5+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_6+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_6+2}{p}\right)} \\
&\quad + L_3 \frac{8\beta_7 R^{\beta_7+3}}{(\beta_7+1)(\beta_7+2)(\beta_7+3)} \times \frac{1}{\beta_8+1}.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
&\int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi))\, r^2 C_{\frac{p-1}{p}}(\theta)\, d\psi\, d\theta\, dr \\
&= \frac{R^3}{3} \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi))\, C_{\frac{p-1}{p}}(\theta)\, d\psi\, d\theta \\
&= \frac{R}{3}\cdot \mathbf{M} \qquad \text{(by (7))} \\
&= \frac{R}{3} \iint\limits_{S_p(A,R)} f(x_1,x_2,x_3)\, dS_p
\end{aligned}$$

then the most left side of previous inequality could be written as (see (6)):

$$\left|\int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f\left(x_1\left(r,\theta,\psi\right),x_2\left(r,\theta,\psi\right),x_3\left(r,\theta,\psi\right)\right) r^2 C_{\frac{p-1}{p}}\left(\theta\right) d\psi d\theta dr \right.$$
$$\left. -\int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f\left(x_1\left(R,\theta,\psi\right),x_2\left(R,\theta,\psi\right),x_3\left(R,\theta,\psi\right)\right) r^2 C_{\frac{p-1}{p}}\left(\theta\right) d\psi d\theta dr\right|$$
$$=\left|\int_0^R \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f\left(x_1\left(r,\theta,\psi\right),x_2\left(r,\theta,\psi\right),x_3\left(r,\theta,\psi\right)\right) r^2 C_{\frac{p-1}{p}}\left(\theta\right) d\psi d\theta dr \right.$$
$$\left. -\frac{R^3}{3}\int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f\left(x_1\left(R,\theta,\psi\right),x_2\left(R,\theta,\psi\right),x_3\left(R,\theta,\psi\right)\right) C_{\frac{p-1}{p}}\left(\theta\right) d\psi d\theta\right|$$
$$=\left|\iiint\limits_{B_p(A,R)} f\left(x_1,x_2,x_3\right) dx_1 dx_2 dx_3 - \frac{R}{3}\frac{1}{\mathrm{vol}\left(B_p\left(A,R\right)\right)}\iint\limits_{S_p(A,R)} f\left(x_1,x_2,x_3\right) dS_p\right|. \tag{28}$$

Dividing (28) by $\mathrm{vol}_p\left(B\right)=\frac{4}{3}\omega_p R^3$, and then combining (27) and (28) we get the required results in (26). □

COROLLARY 6. *Let $p=\frac{q}{q-1}$ $(q>1)$ and $f:B_p\left(A,R\right)\to\mathbb{R}$ be a function defined in the open p-ball $B_p\left(A,R\right)$ satisfying the condition*

$$\left|f\left(a_1+rC_{\frac{p-1}{p}}\left(\theta\right)C_{\frac{p-1}{p}}\left(\psi\right),a_2+rC_{\frac{p-1}{p}}\left(\theta\right)S_{\frac{p-1}{p}}\left(\psi\right),a_3+rS_{\frac{p-1}{p}}\left(\theta\right)\right)\right.$$
$$\left.-f\left(a_1+RC_{\frac{p-1}{p}}\left(\theta\right)C_{\frac{p-1}{p}}\left(\psi\right),a_2+RC_{\frac{p-1}{p}}\left(\theta\right)S_{\frac{p-1}{p}}\left(\psi\right),a_3+RS_{\frac{p-1}{p}}\left(\theta\right)\right)\right|$$
$$\leqslant L_1\left(R-r\right)^{\alpha}\left|C_{\frac{p-1}{p}}\left(\theta\right)\right|^{\alpha}\left|C_{\frac{p-1}{p}}\left(\psi\right)\right|^{\alpha}+L_2\left(R-r\right)^{\beta}\left|C_{\frac{p-1}{p}}\left(\theta\right)\right|^{\beta}\left|S_{\frac{p-1}{p}}\left(\psi\right)\right|^{\beta}$$
$$+L_3\left(R-r\right)^{\gamma}\left|S_{\frac{p-1}{p}}\left(\theta\right)\right|^{\gamma} \tag{29}$$

for all $r\in[0,R]$, $\theta\in\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]$, $\psi\in[0,2\omega_p]$, and some constants $L_1,L_2,L_3>0$ and $\alpha,\beta,\gamma\in(-1,\infty]$. Then, we have

$$\left|\frac{1}{vol_p\left(B\right)}\iiint\limits_{B_p(A,R)} f\left(x_1,x_2,x_3\right) dx_1 dx_2 dx_3 - \frac{1}{\mathfrak{S}_p\left(A,R\right)}\iint\limits_{S_p(A,R)} f\left(x_1,x_2,x_3\right) dS_p\right| \tag{30}$$
$$\leqslant\frac{6}{\omega_p}\left[L_1\frac{2\alpha R^{\alpha}}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)}\times\frac{\Gamma\left(\frac{\alpha+1}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\alpha+3}{p}\right)}\right.$$
$$\left.+L_2\frac{2\beta R^{\beta}}{\left(\beta+1\right)\left(\beta+2\right)\left(\beta+3\right)}\times\frac{\Gamma\left(\frac{\beta+1}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\beta+3}{p}\right)}+L_3\frac{\gamma R^{\gamma}}{\left(\gamma+1\right)\left(\gamma+2\right)\left(\gamma+3\right)}\times\frac{1}{\gamma+1}\right].$$

Proof. Setting $\alpha=\beta_1=\beta_2=\beta_3$, $\beta=\beta_3=\beta_4=\beta_6$, and $\gamma=\beta_7=\beta_8$ in (26). □

COROLLARY 7. *Let $p=\frac{q}{q-1}$ $(q>1)$ and $f:B_p(A,R)\to\mathbb{R}$ be a function defined in the open p-ball $B_p(A,R)$ satisfying the condition*

$$\begin{aligned}&\left|f\left(a_1+rC_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi),a_2+rC_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi),a_3+rS_{\frac{p-1}{p}}(\theta)\right)\right.\\&\quad\left.-f\left(a_1+RC_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi),a_2+RC_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi),a_3+RS_{\frac{p-1}{p}}(\theta)\right)\right|\\&\qquad\leqslant L_1(R-r)\left|C_{\frac{p-1}{p}}(\theta)\right|\left|C_{\frac{p-1}{p}}(\psi)\right|+L_2(R-r)\left|C_{\frac{p-1}{p}}(\theta)\right|\left|S_{\frac{p-1}{p}}(\psi)\right|\\&\qquad\qquad+L_3(R-r)\left|S_{\frac{p-1}{p}}(\theta)\right| \qquad (31)\end{aligned}$$

for all $r\in[0,R]$, $\theta\in\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]$, $\psi\in[0,2\omega_p]$, and some constants $L_1,L_2,L_3>0$. Then, we have

$$\left|\frac{1}{vol_p(B)}\iiint\limits_{B_p(A,R)}f(x_1,x_2,x_3)\,dx_1dx_2dx_3-\frac{1}{\mathfrak{S}_p(A,R)}\iint\limits_{S_p(A,R)}f(x_1,x_2,x_3)\,dS_p\right| \qquad (32)$$
$$\leqslant\frac{R}{2\omega_p}\left[L_1\cdot\frac{\Gamma\left(\frac{2}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{4}{p}\right)}+L_2\cdot\frac{\Gamma\left(\frac{2}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{4}{p}\right)}+\frac{1}{4}L_3\right].$$

Proof. Setting $\alpha=\beta=\gamma=1$ in (30). □

COROLLARY 8. *Let $f:B(A,R)\to\mathbb{R}$ be a function defined in the open ball $B(A,R)$ satisfying the condition*

$$\begin{aligned}&\left|f\left(a_1+rC_{\frac12}(\theta)C_{\frac12}(\psi),a_2+rC_{\frac12}(\theta)S_{\frac12}(\psi),a_3+rS_{\frac12}(\theta)\right)\right.\\&\quad\left.-f\left(a_1+RC_{\frac12}(\theta)C_{\frac12}(\psi),a_2+RC_{\frac12}(\theta)S_{\frac12}(\psi),a_3+RS_{\frac12}(\theta)\right)\right|\\&\qquad\leqslant L_1(R-r)\left|C_{\frac12}(\theta)\right|\left|C_{\frac12}(\psi)\right|+L_2(R-r)\left|C_{\frac12}(\theta)\right|\left|S_{\frac12}(\psi)\right|\\&\qquad\qquad+L_3(R-r)\left|S_{\frac12}(\theta)\right| \qquad (33)\end{aligned}$$

for all $r\in[0,R]$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\psi\in[0,2\pi]$, and some constants $L_1,L_2,L_3>0$. Then, we have

$$\left|\frac{1}{vol(B)}\iiint\limits_{B(A,R)}f(x_1,x_2,x_3)\,dx_1dx_2dx_3-\frac{1}{\mathfrak{S}(A,R)}\iint\limits_{S(A,R)}f(x_1,x_2,x_3)\,dS\right| \qquad (34)$$
$$\leqslant\frac{R}{8\pi}\left((L_1+L_2)\cdot\pi+L_3\right).$$

Proof. Setting $p = 2$ in (32). $\square$

THEOREM 6. *Let* $p = \frac{q}{q-1}$ $(q > 1)$ *and* $f : B_p(A,R) \to \mathbb{R}$ *be a function defined in the open* p*-ball* $B_p(A,R)$ *satisfying the condition*

$$\begin{aligned}
&\left| f(a_1,a_2,a_3) - f\left(a_1 + RC_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi), a_2 + RC_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi), a_3 + RS_{\frac{p-1}{p}}(\theta)\right)\right| \\
&\quad \leqslant L_1 R^{\beta_1} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_2} \left|C_{\frac{p-1}{p}}(\psi)\right|^{\beta_3} + L_2 R^{\beta_4} \left|C_{\frac{p-1}{p}}(\theta)\right|^{\beta_5} \left|S_{\frac{p-1}{p}}(\psi)\right|^{\beta_6} \\
&\qquad + L_3 R^{\beta_7} \left|S_{\frac{p-1}{p}}(\theta)\right|^{\beta_8} \qquad (35)
\end{aligned}$$

for all $r \in [0,R]$, $\theta \in \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$, $\psi \in [0, 2\omega_p]$, *and some constants* $L_1, L_2, L_3 > 0$ *and* $\beta_i \in (-1,\infty]$ $\left(i = \overline{1,8}\right)$. *Then, we have*

$$\begin{aligned}
&\left| f(A) - \frac{1}{\mathfrak{S}_p(A,R)} \iint\limits_{S_p(A,R)} f(x_1,x_2,x_3)\, dS_p \right| \qquad (36) \\
&\leqslant \frac{1}{4\omega_p}\left[L_1 R^{\beta_1} \frac{\Gamma\left(\frac{\beta_2+2}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\beta_2+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_3+1}{p}\right)}{\Gamma\left(\frac{\beta_3+2}{p}\right)} + L_2 R^{\beta_4} \frac{\Gamma\left(\frac{\beta_5+2}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\beta_5+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_6+1}{p}\right)}{\Gamma\left(\frac{\beta_6+2}{p}\right)} \right. \\
&\qquad \left. + L_3 R^{\beta_7} \frac{1}{\beta_8+1} \right].
\end{aligned}$$

Proof. The triangle integral inequality and (35) implies that

$$\begin{aligned}
&\Big| \mathfrak{S}_p(A,R) f(a_1,a_2,a_3) \\
&\quad \left. - \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi)) R^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta \right| \qquad (37) \\
&= \left| \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(a_1,a_2,a_3) R^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta \right. \\
&\quad \left. - \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi)) R^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta \right| \\
&= \left| \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} \left[f(a_1,a_2,a_3) - f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi)) \right] \right. \\
&\quad \left. \times R^2 C_{\frac{p-1}{p}}(\theta)\, d\psi d\theta \right|
\end{aligned}$$

$$\leqslant \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \int_0^{2\omega_p} \left| f(a_1,a_2,a_3) - f(x_1(R,\theta,\psi), x_2(R,\theta,\psi), x_3(R,\theta,\psi)) \right| R^2 \left| C_{\frac{p-1}{p}}(\theta) \right| d\psi d\theta$$

$$\leqslant L_1 R^{\beta_1+2} \times \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| C_{\frac{p-1}{p}}(\theta) \right|^{\beta_2} \left| C_{\frac{p-1}{p}}(\theta) \right| d\theta \times \int_0^{2\omega_p} \left| C_{\frac{p-1}{p}}(\psi) \right|^{\beta_3} d\psi$$
$$+ L_2 R^{\beta_4+2} \times \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| C_{\frac{p-1}{p}}(\theta) \right|^{\beta_5} \left| C_{\frac{p-1}{p}}(\theta) \right| d\theta \times \int_0^{2\omega_p} \left| S_{\frac{p-1}{p}}(\psi) \right|^{\beta_6} d\psi$$
$$+ L_3 R^{\beta_7+2} \times \int_{-\frac{\omega_p}{2}}^{\frac{\omega_p}{2}} \left| S_{\frac{p-1}{p}}(\theta) \right|^{\beta_8} \left| C_{\frac{p-1}{p}}(\theta) \right| d\theta$$

$$= L_1 R^{\beta_1+2} \frac{\Gamma\left(\frac{\beta_2+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_2+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_3+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_3+2}{p}\right)}$$
$$+ L_2 R^{\beta_4+2} \frac{\Gamma\left(\frac{\beta_5+2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_5+3}{p}\right)} \times \frac{\Gamma\left(\frac{\beta_6+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{\beta_6+2}{p}\right)} + L_3 R^{\beta_7+2} \frac{1}{\beta_8+1}.$$

Dividing (37) by $\mathfrak{S}_p(A,R) = 4\omega_p R^2$, and we get the required results in (36). □

COROLLARY 9. *Let $p = \frac{q}{q-1}$ $(q>1)$ and $f : B_p(A,R) \to \mathbb{R}$ be a function defined in the open p-ball $B_p(A,R)$ satisfying the condition*

$$\left| f(a_1,a_2,a_3) - f\left(a_1 + RC_{\frac{p-1}{p}}(\theta) C_{\frac{p-1}{p}}(\psi), a_2 + RC_{\frac{p-1}{p}}(\theta) S_{\frac{p-1}{p}}(\psi), a_3 + RS_{\frac{p-1}{p}}(\theta)\right) \right|$$
$$\leqslant L_1 R^{\alpha} \left| C_{\frac{p-1}{p}}(\theta) \right|^{\alpha} \left| C_{\frac{p-1}{p}}(\psi) \right|^{\alpha} + L_2 R^{\beta} \left| C_{\frac{p-1}{p}}(\theta) \right|^{\beta} \left| S_{\frac{p-1}{p}}(\psi) \right|^{\beta}$$
$$+ L_3 R^{\gamma} \left| S_{\frac{p-1}{p}}(\theta) \right|^{\gamma} \tag{38}$$

for all $r \in [0,R]$, $\theta \in \left[-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$, $\psi \in [0, 2\omega_p]$, and some constants $L_1, L_2, L_3 > 0$ and $\alpha, \beta, \gamma \in (-1, \infty]$ Then, we have

$$\left| f(A) - \frac{1}{\mathfrak{S}_p(A,R)} \iint_{S_p(A,R)} f(x_1,x_2,x_3)\, dS_p \right| \tag{39}$$
$$\leqslant \frac{1}{4\omega_p} \left[L_1 R^{\alpha} \frac{\Gamma\left(\frac{\alpha+1}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\alpha+3}{p}\right)} + L_2 R^{\beta} \frac{\Gamma\left(\frac{\beta+1}{p}\right)\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{\beta+3}{p}\right)} + L_3 R^{\gamma} \frac{1}{\gamma+1} \right].$$

Proof. Setting $\beta_1 = \beta_2 = \beta_3 = \alpha$, $\beta_3 = \beta_4 = \beta_6 = \beta$, and $\beta_7 = \beta_8 = \gamma$ in (36). □

COROLLARY 10. *Let $p=\frac{q}{q-1}$ $(q>1)$ and $f:B_p(A,R)\to\mathbb{R}$ be a function defined in the open p-ball $B_p(A,R)$ satisfying the condition*

$$\left|f(a_1,a_2,a_3)-f\left(a_1+RC_{\frac{p-1}{p}}(\theta)C_{\frac{p-1}{p}}(\psi),a_2+RC_{\frac{p-1}{p}}(\theta)S_{\frac{p-1}{p}}(\psi),a_3+RS_{\frac{p-1}{p}}(\theta)\right)\right| \leqslant L_1R\left|C_{\frac{p-1}{p}}(\theta)\right|\left|C_{\frac{p-1}{p}}(\psi)\right|+L_2R\left|C_{\frac{p-1}{p}}(\theta)\right|\left|S_{\frac{p-1}{p}}(\psi)\right|+L_3R\left|S_{\frac{p-1}{p}}(\theta)\right| \tag{40}$$

for all $r\in[0,R]$, $\theta\in\left[-\frac{\omega_p}{2},\frac{\omega_p}{2}\right]$, $\psi\in[0,2\omega_p]$, and some constants $L_1,L_2,L_3>0$. Then, we have

$$\left|f(A)-\frac{1}{\mathfrak{S}_p(A,R)}\iint\limits_{S_p(A,R)}f(x_1,x_2,x_3)\,dS_p\right| \leqslant \frac{R}{4\omega_p}\left[L_1\frac{\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{4}{p}\right)}+L_2\frac{\Gamma^2\left(\frac{1}{p}\right)}{p^2\Gamma\left(\frac{4}{p}\right)}+\frac{1}{2}L_3\right]. \tag{41}$$

Proof. Setting $\alpha=\beta=\gamma=1$ in (39). □

COROLLARY 11. *Let $f:B(A,R)\to\mathbb{R}$ be a function defined in the open ball $B(A,R)$ satisfying the condition*

$$\left|f(a_1,a_2,a_3)-f\left(a_1+RC_{\frac{1}{2}}(\theta)C_{\frac{1}{2}}(\psi),a_2+RC_{\frac{1}{2}}(\theta)S_{\frac{1}{2}}(\psi),a_3+RS_{\frac{1}{2}}(\theta)\right)\right| \leqslant L_1R\left|C_{\frac{1}{2}}(\theta)\right|\left|C_{\frac{1}{2}}(\psi)\right|+L_2R\left|C_{\frac{1}{2}}(\theta)\right|\left|S_{\frac{1}{2}}(\psi)\right|+L_3R\left|S_{\frac{1}{2}}(\theta)\right| \tag{42}$$

for all $r\in[0,R]$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\psi\in[0,2\pi]$, and some constants $L_1,L_2,L_3>0$. Then, we have

$$\left|f(A)-\frac{1}{\mathfrak{S}(A,R)}\iint\limits_{S(A,R)}f(x_1,x_2,x_3)\,dS\right|\leqslant\frac{R}{8\pi}\left(L_1\frac{\pi}{2}+L_2\frac{\pi}{2}+L_3\right). \tag{43}$$

Proof. Setting $p=2$ in (41). □

REFERENCES

[1] A. BARANI, *Hermite-Hadamard and Ostrowski Type Inequalities on Hemispheres*, Mediterr. J. Math., **13**, (2016), 4253–4263.
[2] N. S. BARNETT, F.-C. CIRSTEA, S. S. DRAGOMIR, *Inequalities for the integral mean of Hölder continuous functions defined on disks in a plane*, RGMIA research report collection, **5**, 1 (2001), Article No. 7.
[3] M. BESSENYEI, *The Hermite-Hadamard inequality on simplices*, Amer. Math. Mon., **115**, 4 (2008), 339–345.
[4] M. BESSENYEI, *The Hermite–Hadamard inequality in Beckenbach's setting*, J. Math. Anal. Appl., **364**, (2) (2010), 366–383.
[5] J. DE LA CAL, J. CÁRCAMO, *Multidimensional Hermite-Hadamard inequalities and the convex order*, J. Math. Anal. Appl., **324**, (2006), 248–261.
[6] J. DE LA CAL, J. CÁRCAMO, L. ESCAURIAZA, *A general multidimensional Hermite-Hadamard type inequality*, J. Math. Anal. Appl., **356**(2) (2009), 659–663.
[7] Y. CHEN, *Hadamard's inequality on a triangle and on a polygon*, Tamkang J. Math., **35**, (3), (2004), 247–254.
[8] S. S. DRAGOMIR, *On Hadamard's inequality for the convex mappings defined on a ball in the space and applications*, Math. Inequal. Appl., **3**, 2 (2000), 177–187.
[9] M. R. DELAVAR, *Sharp trapezoid and mid-point type inequalities on closed balls in* $\mathbb{R}^3$, J. Inequal. Appl., **2020**, 114 (2020).
[10] R. EULER, J. SADEK, *The* π*s go full circle*, Math. Mag, **72**, (1999), 59–63.
[11] A. GUESSAB, B. SEMISALOV, *Optimal general Hermite-Hadamard-type inequalities in a ball and their applications in multidimensional numerical integration*, Filomati, **170**, (2021), 83–108.
[12] M. JLELI, B. SAMET, *On Hermite-Hadamard-type inequalities for subharmonic functions Over circular ring domains*, Numeri. Func. Anal. Optim., **44**, (13) (2023), 1395–1408.
[13] P. LINDQVIST, J. PEETRE, p*-Arclength of the* q*-circle*, Math. Student, **72**, 1–4 (2003), 139–145.
[14] P. LINDQVIST, J. PEETRE, *Comments on Erik Lundberg's 1879 thesis, especially on the work of Göran Dillner and his influence on Lundberg*, Mem. dell'Istituto Lombardo, Accad. Sci. e Lett., Classe Sci. Mat. Nat., **XXXI**, Fasc. 1, Milano, 2004.
[15] P. LINDQVIST, J. PEETRE, *Two Remarkable Identities, Called Twos, for Inverses to Some Abelian Integrals*, AMM, **105**, 5 (2001), 403–410.
[16] E. LUNDBERG, *Om hypergoniometriska funktioner af komplexa variabla*, Stockholm 1879, English translation: On hypergoniometric functions of complex variables, http://www.maths.th.se/matematiklu/personal/jaak/engJP.html.
[17] M. MATŁOKA, *On Hadamard's inequality for* h*-convex function on a disk*, Appl. Math. Comp., **235**, (2014), 118–123.
[18] A. MERCER, *Hadamard's inequality for a triangle, a regular polygon and a circle*, Math. Inequal. Appl., **5** (5), (2002), 219–223.
[19] F.-C. MITROI, E. SYMEONIDIS, *The converse of the Hermite-Hadamard inequality on simplices*, Exposit. Math., **30** (4), (2012), 389–396.
[20] C. P. NICULESCU, *The Hermite-Hadamard inequality for convex functions of a vector variable*, Math. Inequal. Appl., **5** (4), (2002), 619–623.
[21] S. STEINERBERGER, *The Hermite–Hadamard inequality in higher dimensions*, J. Geom. Anal., **30**, (2020), 466–483.
[22] X. WANG, J. RUANA, X. MA, *On the Hermite–Hadamard inequalities for* h*-Convex functions on balls and ellipsoids*, Filomat, **33**, (18) (2019), 5871–5886.

(Received September 14, 2024)

Mohammad W. Alomari
Department of Mathematics
Faculty of Science and Information Technology
Jadara University
P.O. Box 733, Irbid, P.C. 21110, Jordan
e-mail: malomari@jadara.edu.jo
mwomath@gmail.com