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# ON AN OPEN PROBLEM CONCERNING THE RECIPROCAL SUM RELATED TO THE RIEMANN ZETA–FUNCTION

SHUN-WEI XU, BO ZHANG AND CHAO-PING CHEN*

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*Abstract.* In 2016, Lin studied the computational problem of the reciprocal sum related to the Riemann zeta function. More precisely, the author proved that, for any positive integer $n$,

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} \right\rfloor = n-1 \quad \text{and} \quad \left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} \right\rfloor = 2n(n-1),$$

where $\lfloor x \rfloor$ is the floor function, that is, it denotes the greatest integer less than or equal to $x$. At the same time, Lin also proposed the following open problem: Whether there exists an explicit computational formula for $\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^s} \right)^{-1} \right\rfloor$, where $s$ is an integer with $s \geqslant 4$. In this paper, we present the asymptotic expansion of $\left( \sum_{k=n}^{\infty} 1/k^{j+1} \right)^{-1}$ in terms of $1/n$. Based on this expansion, we answer the open problem of Lin for $s=4$ and $s=5$. Using our method and Maple software one can study the open problem of Lin for the cases $s \geqslant 6$.

## 1. Introduction

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1. \tag{1.1}$$

This function plays a central role in the applications of complex analysis to number theory. The number-theoretic properties of $\zeta(s)$ are exhibited by the following result known as Euler's formula, which gives a relationship between the set of primes and the set of positive integers:

$$\zeta(s) = \prod_{p} \left(1 - p^{-s}\right)^{-1}, \qquad \Re(s) > 1, \tag{1.2}$$

where the product is taken over all primes. It is readily seen that $\zeta(s) \neq 0$ $(\Re(s) = \sigma \geqslant 1)$, and the Riemann's functional equation for $\zeta(s)$:

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s) \sin\left(\frac{1}{2}\pi s\right) \zeta(1-s) \tag{1.3}$$

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* Corresponding author.

shows that $\zeta(s) \neq 0$ $(\sigma \leqslant 0)$ except for the trivial zeros in

$$\zeta(-2n) = 0, \qquad n \in \mathbb{N} := \{1, 2, \ldots\}. \tag{1.4}$$

Furthermore, in view of the following known relation:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \qquad \Re(s) > 0 \quad \text{and} \quad s \neq 1, \tag{1.5}$$

we find that $\zeta(s) < 0$ $(s \in \mathbb{R};\ 0 < s < 1)$. The assertion that all the non-trivial zeros of $\zeta(s)$ have real part $\frac{1}{2}$ is popularly known as the Riemann hypothesis which was conjectured (but not proven) in the memoir of Riemann [21]. This hypothesis is still one of the most challenging mathematical problems today (see Edwards [8]), which was unanimously chosen to be one of the seven greatest unsolved mathematical puzzles of our time, so-called the millennium problems (see Devlin [6]).

Let $\{F_n\}_{n\geqslant 0}$ and $\{L_n\}_{n\geqslant 0}$ be the sequence of Fibonacci numbers and the sequence of Lucas numbers defined, respectively, by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geqslant 0,$$

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geqslant 0.$$

It is well-known that $F_n$ and $L_n$ can be expressed explicitly as

$$F_n = \frac{1}{\sqrt{5}} \left\{ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right\} \quad \text{and} \quad L_n = \left( \frac{1+\sqrt{5}}{2} \right)^n + \left( \frac{1-\sqrt{5}}{2} \right)^n.$$

These two sequences have many important positions, please refer to [12, 22, 28, 29] and the references therein.

The Fibonacci and Lucas zeta functions $\zeta_F(s)$ and $\zeta_L(s)$ are defined, respectively, by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad \text{and} \quad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s}. \tag{1.6}$$

These series are absolutely convergent for $\Re(s) > 0$ and can be considered as an analogue of the Riemann zeta function $\zeta(s)$ in (1.1). When $s = 1$, the number

$$\sum_{n=1}^{\infty} F_n^{-1} = 3.35988566624317755317201130291892717968890513373 2\ldots$$

is called the reciprocal Fibonacci constant and André-Jeannin [3] proved that this constant is an irrational number. Navas [19] discussed the analytic continuation of these series. Duverney et al. [7] proved that $\zeta_F(2m)$ (for $m = 1, 2, 3, \ldots$) are all transcendental numbers. Elsner et al. [9] proved the algebraic independence of the numbers in the collections $\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ and $\zeta_L(2)$, $\zeta_L(4)$, $\zeta_L(6)$ as well as express algebraically even 'zeta values' $\zeta_F(2s)$ (and $\zeta_L(2s)$) for $s \geqslant 4$ in terms of the three algebraically independent numbers in the corresponding collection. Similar algebraic independence results are shown for the alternating versions of (1.6). Elsner et al. [10] proved the algebraic independence of the reciprocal sums of odd terms in Fibonacci numbers $\sum_{n=1}^{\infty} F_{2n-1}^{-1}$, $\sum_{n=1}^{\infty} F_{2n-1}^{-2}$, $\sum_{n=1}^{\infty} F_{2n-1}^{-3}$ and write each $\sum_{n=1}^{\infty} F_{2n-1}^{-s}$ $(s \geqslant 4)$ as an explicit rational function of these three numbers over $\mathbb{Q}$. Similar results are obtained for various series including the reciprocal sums of odd terms in Lucas numbers.

Ohtsuka and Nakamura [20] studied the partial infinite sums of the reciprocal Fibonacci numbers, and the partial infinite sums of the reciprocals of the square of the Fibonacci numbers. More precisely, these authors proved the following two identities:

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geqslant 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd and } n \geqslant 1, \end{cases} \tag{1.7}$$

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ is even and } n \geqslant 2; \\ F_{n-1}F_n, & \text{if } n \text{ is odd and } n \geqslant 1, \end{cases} \tag{1.8}$$

where $\lfloor x \rfloor$ is the floor function, that is, it denotes the greatest integer less than or equal to $x$. Wang [23] gave the similar results for partial infinite sums including reciprocal cubical-Fibonacci numbers.

The sequence $P_n$ of Pell numbers is defined by the recurrence relation:

$$P_{n+1} = 2P_n + P_{n-1},$$

with seed values $P_0 = 0$ and $P_1 = 1$. From the characteristic equations $x^2 - x - 1 = 0$ we also have the computational formula

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

Holliday and Komatsu [11], Zhang and Wang [34,35] independently gave analogues of the results (1.9) and (1.10) for the Pell numbers:

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2}, & \text{if } n \text{ is even and } n \geqslant 2; \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \text{ is odd and } n \geqslant 1, \end{cases} \tag{1.9}$$

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2P_{n-1}P_n - 1, & \text{if } n \text{ is even and } n \geqslant 2; \\ 2P_{n-1}P_n, & \text{if } n \text{ is odd and } n \geqslant 1. \end{cases} \tag{1.10}$$

Xu and Wang [24] gave the similar results for partial infinite sums including reciprocal cubical-Pell numbers.

For some recent developments of (1.7)–(1.10), see the works by (for example) Liu and Zhao [18], Wu and Zhang [30,31], Zhang and Wu [36], Kuhapatanakul [13,14], Yuan et al. [33], Wang and Wen [25], Wang and Zhang [26,27], Liu and Wang [17], and Choo [5].

Inspired by (1.7)–(1.10), in 2016 Lin [15] studied the computational problem of the reciprocal sum related to the Riemann zeta function. More precisely, the author proved that, for any positive integer $n$,

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} \right\rfloor = n-1 \quad \text{and} \quad \left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} \right\rfloor = 2n(n-1).$$

Also in [15], Lin proposed the following open problem.

OPEN PROBLEM 1.1. Whether there exists an explicit computational formula for

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^s} \right)^{-1} \right\rfloor, \tag{1.11}$$

where $s$ is an integer with $s \geqslant 4$.

This problem is interesting, because it is actually an effective approximation for the partial sum of the Riemann zeta function $\zeta(s)$.

Lin and Li [16] gave an interesting computational formula for (1.11) with $s = 4$. More precisely, these authors proved that, for any positive integer $n$,

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^4} \right)^{-1} \right\rfloor = \begin{cases} 24m^3 - 18m^2 + \left\lfloor \frac{3(5m-1)}{2} \right\rfloor, & \text{if} \quad n = 2m; \\ 24m^3 - 54m^2 + \left\lfloor \frac{3(58m-17)}{4} \right\rfloor, & \text{if} \quad n = 2m-1. \end{cases} \tag{1.12}$$

Xu [32] considered the open problem of Lin for $s = 4$ and $s = 5$ and obtained the following results:

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^4} \right)^{-1} \right\rfloor = -1 + 4n - 5n^2 + 3n^3 + \left\lfloor \frac{(2n+1)(n-1)}{4} \right\rfloor \tag{1.13}$$

and

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^5} \right)^{-1} \right\rfloor = -5n + 9n^2 - 8n^3 + 4n^4 + \left\lfloor \frac{(n+1)(n-2)}{3} \right\rfloor. \tag{1.14}$$

As far as we know, the above open problem has not yet been solved for $s \geqslant 6$. In this paper, we present the asymptotic expansion of $\left(\sum_{k=n}^{\infty} 1/k^{j+1}\right)^{-1}$ in terms of $1/n$. Based on this expansion, we answer the open problem of Lin for $s = 4$ and $s = 5$ (Theorems 3.1 and 4.1). Using our method and Maple software one can study the open problem 1.1 for the cases $s \geqslant 6$. For example, we present (without proof) an explicit computational formula for $\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^6}\right)^{-1} \right\rfloor$ (Eq. (4.19)).

The numerical values given have been calculated using the computer program MAPLE 13.

## 2. Lemmas

The following lemmas will be used in our present investigation.

LEMMA 2.1. (see [4, Lemma 5]) *Let $g$ be a function with asymptotical expansion $(c_0 \neq 0)$:*

$$g(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad x \to \infty.$$

*Then for all real $r$ it holds*

$$\big(g(x)\big)^r \sim \sum_{n=0}^{\infty} P_n(r) x^{-n}, \tag{2.1}$$

*where*

$$P_0(r) = c_0^r \quad \text{and} \quad P_n(r) = \frac{1}{nc_0} \sum_{k=1}^{n} [k(1+r)-n] c_k P_{n-k}(r), \qquad n \geqslant 1. \tag{2.2}$$

In particular, the choice $r=-1$ in (2.1) yields

$$\big(g(x)\big)^{-1} \sim \sum_{n=0}^{\infty} p_n x^{-n}, \tag{2.3}$$

where

$$p_0 = c_0^{-1} \quad \text{and} \quad p_n = -\frac{1}{c_0} \sum_{k=1}^{n} c_k p_{n-k}, \qquad n \geqslant 1. \tag{2.4}$$

Euler's gamma function:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi (or digamma) function. The derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\{\psi(x)\}, \qquad n \in \mathbb{N}$$

are called the polygamma functions.

LEMMA 2.2. (see [2, Theorem 9]) *Let $k \geqslant 1$ and $n \geqslant 0$ be integers. Then for all real numbers $x > 0$:*

$$S_k(2n;x) < (-1)^{k+1} \psi^{(k)}(x) < S_k(2n+1;x), \tag{2.5}$$

*where*

$$S_k(p;x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}},$$

*and* $B_n$ $(n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\})$ *are the Bernoulli numbers defined by the following generating function:*

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi.$$

The inequality (2.5) can be wtitten as

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{j=1}^{2n} \frac{B_{2j}}{(2j)!}\frac{\Gamma(k+2j)}{x^{k+2j}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{j=1}^{2n+1} \frac{B_{2j}}{(2j)!}\frac{\Gamma(k+2j)}{x^{k+2j}}. \tag{2.6}$$

From the series expansion and asymptotic formula for the polygamma functions $\psi^{(n)}(x)$ (see [1, p. 260]):

$$\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty}(x+k)^{-n-1}$$

and

$$\psi^{(n)}(x) \sim (-1)^{n-1}\left\{\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{k=1}^{\infty} B_{2k}\frac{(2k+n-1)!}{(2k)!x^{2k+n}}\right\}, \qquad x \to \infty,$$

we find, as $n \to \infty$,

$$\sum_{k=n}^{\infty}\frac{1}{k^{j+1}} = \frac{(-1)^{j+1}}{j!}\psi^{(j)}(n) \sim \frac{1}{n^j}\left\{\frac{1}{j} + \frac{1}{2n} + \sum_{\ell=2}^{\infty}\frac{(\ell+j-1)!B_\ell}{j!\ell!n^\ell}\right\}, \tag{2.7}$$

or alternatively

$$\sum_{k=n}^{\infty}\frac{1}{k^{j+1}} \sim \frac{1}{n^j}\sum_{\ell=0}^{\infty}\frac{a_\ell}{n^\ell}, \tag{2.8}$$

with the coefficients $a_\ell \equiv a_\ell(j)$ given by

$$a_0 = \frac{1}{j}, \quad a_1 = \frac{1}{2}, \quad a_\ell = \frac{(\ell+j-1)!B_\ell}{j!\ell!}, \qquad \ell \geqslant 2. \tag{2.9}$$

By (2.3), from (2.8) we obtain the asymptotic expansion of $\left(\sum_{k=n}^{\infty} 1/k^{j+1}\right)^{-1}$ given by Lemma 2.3.

LEMMA 2.3. *Let $j \geqslant 1$ be a given integer. Then the following asymptotic expansion holds:*

$$\left(\sum_{k=n}^{\infty}\frac{1}{k^{j+1}}\right)^{-1} \sim n^j \sum_{\ell=0}^{\infty} b_\ell n^{-\ell}, \qquad n\to\infty, \tag{2.10}$$

*with the coefficients $b_\ell \equiv b_\ell(j)$ given by*

$$b_0 = j \quad \text{and} \quad b_\ell = -j\sum_{k=1}^{\ell} a_k b_{\ell-k}, \qquad \ell \geqslant 1, \tag{2.11}$$

*where $a_\ell \equiv a_\ell(j)$ are given in (2.9). Namely,*

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{k^{j+1}}\right)^{-1} \sim n^j\Bigg\{ & j-\frac{j^2}{2n}+\frac{j^2(2j-1)}{12n^2}-\frac{j^3(j-2)}{24n^3}+\frac{j^2(6j^3-29j^2+16j+6)}{720n^4}\\ &-\frac{j^3(j-4)(2j^2-10j-3)}{1444n^5}+\frac{j^2(2j-3)(6j^4-76j^3+291j^2+264j+80)}{60480n^6}\\ &-\frac{j^3(j-6)(3j^4-42j^3+248j^2+238j+80)}{120960n^7}+\dots\Bigg\}, \qquad n\to\infty.\end{aligned} \tag{2.12}$$

## 3. A solution of the open problem 1.1 for $s = 4$

Xu [32], Lin and Li [16] have given a computational formula for (1.11) with $s=4$. Here we present a solution of the open problem 1.1 for $s=4$. Our formula is different from formulas in [16, 32].

The choice $j=3$ in (2.10) yields

$$\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1} = 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}-\frac{9}{16n}-\frac{9}{32n^2}+\frac{55}{64n^3}+O\left(\frac{1}{n^4}\right), \qquad n\to\infty. \tag{3.1}$$

Based on (3.1), we find a computational formula for (1.11) with $s=4$ given by Theorem 3.1.

THEOREM 3.1. *For $n=1$, we have*

$$\left\lfloor\left(\sum_{k=1}^{\infty}\frac{1}{k^4}\right)^{-1}\right\rfloor = \left\lfloor\frac{90}{\pi^4}\right\rfloor = 0, \tag{3.2}$$

*and for $n \geqslant 2$ we have*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}\right\rfloor = \left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right\rfloor. \tag{3.3}$$

*Proof.* Direct computations yield

$$\left(\sum_{k=2}^{\infty}\frac{1}{k^4}\right)^{-1}=\frac{90}{-90+\pi^4}=12.147\ldots,\quad \left(3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right)_{n=2}=12.375,$$

$$\left(\sum_{k=3}^{\infty}\frac{1}{k^4}\right)^{-1}=\frac{720}{-765+8\pi^4)}=50.445\ldots,\quad \left(3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right)_{n=3}=50.625,$$

$$\left(\sum_{k=4}^{\infty}\frac{1}{k^4}\right)^{-1}=\frac{6480}{-6965+72\pi^4}=133.733\ldots,\left(3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right)_{n=4}=133.875,$$

Hence, (3.3) is valid for $n=2,3$, and $4$.

We now show that (3.3) is aslo valid for all $n\geqslant 5$. It suffices to show that, for $n\geqslant 5$,

$$\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right\rfloor<\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}<\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right\rfloor+1. \tag{3.4}$$

Direct computations yield (where $m\in\mathbb{N}$)

$$3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}=\begin{cases}192m^3+72m^2+15m+1+\frac{1}{8}, & \text{if } n=4m+1;\\ 192m^3+216m^2+87m+12+\frac{3}{8}, & \text{if } n=4m+2;\\ 192m^3+360m^2+231m+50+\frac{5}{8}, & \text{if } n=4m+3;\\ 192m^3+504m^2+447m+133+\frac{7}{8}, & \text{if } n=4m+4\end{cases} \tag{3.5}$$

and

$$3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}=\begin{cases}192m^3+72m^2+15m+1, & \text{if } n=4m+1;\\ 192m^3+216m^2+87m+12+\frac{1}{4}, & \text{if } n=4m+2;\\ 192m^3+360m^2+231m+50+\frac{1}{2}, & \text{if } n=4m+3;\\ 192m^3+504m^2+447m+133+\frac{3}{4}, & \text{if } n=4m+4.\end{cases} \tag{3.6}$$

We have, by (3.5) and (3.6),

$$\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right\rfloor=\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}\right\rfloor$$
$$=\begin{cases}192m^3+72m^2+15m+1, & \text{if } n=4m+1;\\ 192m^3+216m^2+87m+12, & \text{if } n=4m+2;\\ 192m^3+360m^2+231m+50, & \text{if } n=4m+3;\\ 192m^3+504m^2+447m+133, & \text{if } n=4m+4.\end{cases} \tag{3.7}$$

We see from (3.7) that, for $n\geqslant 5$,

$$\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right\rfloor=\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}\right\rfloor. \tag{3.8}$$

Thus, (3.4) can be written for $n\geqslant 5$ as

$$\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}\right\rfloor<\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}<\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}\right\rfloor+1. \tag{3.9}$$

By (3.6), we have, for $n\geqslant 5$,

$$\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}\right\rfloor\leqslant 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}. \tag{3.10}$$

By (3.5) and (3.8), we have, for $n\geqslant 5$,

$$3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}<\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}\right\rfloor+1=\left\lfloor 3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}\right\rfloor+1. \tag{3.11}$$

In order to prove (3.9), it suffices to show by (3.10) and (3.11) that, for $n\geqslant 5$,

$$3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}<\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}<3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}. \tag{3.12}$$

Noting that

$$\sum_{k=n}^{\infty}\frac{1}{k^{j+1}}=\frac{(-1)^{j+1}}{j!}\psi^{(j)}(n) \tag{3.13}$$

holds, (3.12) can be written for $n\geqslant 4$ as

$$\frac{1}{3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}}<\frac{1}{3!}\psi^{(3)}(n)<\frac{1}{3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}}. \tag{3.14}$$

The choice $(k,n)=(3,1)$ in (2.6) yields, for $x>0$,

$$\frac{1}{3n^3}+\frac{1}{2n^4}+\frac{1}{3n^5}-\frac{1}{6n^7}<\frac{1}{3!}\psi^{(3)}(x)<\frac{1}{3n^3}+\frac{1}{2n^4}+\frac{1}{3n^5}-\frac{1}{6n^7}+\frac{2}{9n^9}. \tag{3.15}$$

Using (3.15), we find, for $n\geqslant 5$,

$$\begin{aligned}\frac{1}{3!}\psi^{(3)}(n)&-\frac{1}{3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}}\\&>\frac{1}{3n^3}+\frac{1}{2n^4}+\frac{1}{3n^5}-\frac{1}{6n^7}-\frac{1}{3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{9}{8}}\\&=\frac{478+275(n-5)+51(n-5)^2+3(n-5)^3}{6n^7(2n-1)(4n^2-4n+3)}>0\end{aligned}$$

and

$$\begin{aligned}\frac{1}{3!}\psi^{(3)}(n)&-\frac{1}{3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}}\\&<\frac{1}{3n^3}+\frac{1}{2n^4}+\frac{1}{3n^5}-\frac{1}{6n^7}+\frac{2}{9n^9}-\frac{1}{3n^3-\frac{9}{2}n^2+\frac{15}{4}n-\frac{5}{4}}\\&=-\frac{1}{18n^9(12n^3-18n^2+15n-5)}\Big\{4520+16410(n-5)+13287(n-5)^2\\&\qquad+4767(n-5)^3+876(n-5)^4+81(n-5)^5+3(n-5)^6\Big\}<0.\end{aligned}$$

Hence, (3.14) holds for $n\geqslant 5$. The proof is complete. □

REMARK 3.1. The formula (3.3) can be written for $n\geqslant 2$ as

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}\right\rfloor=\begin{cases}192m^3+72m^2+15m+1, & \text{if } n=4m+1\ (m\in\mathbb{N});\\ 192m^3+216m^2+87m+12, & \text{if } n=4m+2\ (m\in\mathbb{N}_0);\\ 192m^3+360m^2+231m+50, & \text{if } n=4m+3\ (m\in\mathbb{N}_0);\\ 192m^3+504m^2+447m+133, & \text{if } n=4m+4\ (m\in\mathbb{N}_0).\end{cases} \tag{3.16}$$

## 4. A solution of the open problem 1.1 for $s=5$

The choice $j=4$ in (2.10) yields

$$\left(\sum_{k=n}^{\infty}\frac{1}{k^5}\right)^{-1}=4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}+\frac{88}{27n^2}+\frac{88}{27n^3}+O\left(\frac{1}{n^4}\right) \tag{4.1}$$

as $n\to\infty$. Based on (4.1), we find a computational formula for (1.11) with $s=5$ given by Theorem 4.1. Our formula is different from that in [32].

THEOREM 4.1. *For $n = 1, 2,$ and $3$, we have*

$$\left\lfloor \left( \sum_{k=1}^{\infty} \frac{1}{k^5} \right)^{-1} \right\rfloor = \left\lfloor \frac{1}{\zeta(5)} \right\rfloor = 0, \tag{4.2}$$

$$\left\lfloor \left( \sum_{k=2}^{\infty} \frac{1}{k^5} \right)^{-1} \right\rfloor = \left\lfloor \frac{1}{\zeta(5)-1} \right\rfloor = 27, \tag{4.3}$$

$$\left\lfloor \left( \sum_{k=3}^{\infty} \frac{1}{k^5} \right)^{-1} \right\rfloor = \left\lfloor \frac{32}{32\zeta(5)-33} \right\rfloor = 176, \tag{4.4}$$

*and for $n \geqslant 4$ we have*

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^5} \right)^{-1} \right\rfloor = \left\lfloor 4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{9} \right\rfloor. \tag{4.5}$$

*Proof.* It suffices to show that, for $n \geqslant 4$,

$$\left\lfloor 4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{9} \right\rfloor < \left( \sum_{k=n}^{\infty} \frac{1}{k^5} \right)^{-1} < \left\lfloor 4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{9} \right\rfloor + 1. \tag{4.6}$$

Direct computations yield

$$4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{9} = \begin{cases} 324m^4 + 432m^3 + 228m^2 + 40m - 1 + \frac{7}{9}, & \text{if } n = 3m+1; \\ 324m^4 + 864m^3 + 876m^2 + 384m + 58 + \frac{4}{9}, & \text{if } n = 3m+2; \\ 324m^4 + 1296m^3 + 1956m^2 + 1304m + 319 + \frac{7}{9}, & \text{if } n = 3m+3 \end{cases} \tag{4.7}$$

and

$$4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n = \begin{cases} 324m^4 + 432m^3 + 228m^2 + 40m, & \text{if } n = 3m+1; \\ 324m^4 + 864m^3 + 876m^2 + 384m + 58 + \frac{2}{3}, & \text{if } n = 3m+2; \\ 324m^4 + 1296m^3 + 1956m^2 + 1304m + 320, & \text{if } n = 3m+3. \end{cases} \tag{4.8}$$

For $n=3m+1$ $(m\in\mathbb{N})$, we have, by (4.7) and (4.8),

$$\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}\right\rfloor = 324m^4+432m^3+228m^2+40m-1$$
$$=\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n\right\rfloor -1 = 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-1.$$

We then obtain that, for $n=3m+1$ $(m\in\mathbb{N})$,

$$4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n = \left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}\right\rfloor +1. \tag{4.9}$$

For $n=3m+2$ $(m\in\mathbb{N})$, we have, by (4.7) and (4.8),

$$\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}\right\rfloor = 324m^4+864m^3+876m^2+384m+58$$
$$=\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n\right\rfloor.$$

We then obtain that, for $n=3m+2$ $(m\in\mathbb{N})$,

$$4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n < \left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n\right\rfloor +1$$
$$=\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}\right\rfloor +1. \tag{4.10}$$

For $n=3m+3$ $(m\in\mathbb{N})$, we have, by (4.7) and (4.8),

$$\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}\right\rfloor +1 = 324m^4+1296m^3+1956m^2+1304m+320$$
$$=\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n\right\rfloor = 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n. \tag{4.11}$$

Hence, for all $n\geqslant 4$ $(m\in\mathbb{N})$, we have, by (4.9), (4.10) and (4.11),

$$4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n \leqslant \left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}\right\rfloor +1. \tag{4.12}$$

By (4.7), we have, for $n\geqslant 4$,

$$\left\lfloor 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}\right\rfloor < 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9}. \tag{4.13}$$

In order to prove (4.6), it suffices to show by (4.12) and (4.13) that, for $n\geqslant 4$,

$$4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n-\frac{2}{9} < \left(\sum_{k=n}^{\infty}\frac{1}{k^5}\right)^{-1} < 4n^4-8n^3+\frac{28}{3}n^2-\frac{16}{3}n. \tag{4.14}$$

Noting that (3.13) holds, (4.14) can be written for $n \geqslant 4$ as

$$\frac{1}{4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n} < -\frac{1}{4!}\psi^{(4)}(n) < \frac{1}{4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{9}}. \tag{4.15}$$

The choice $(k,n) = (4,2)$ on the left-hand side of (2.6) and $(k,n) = (4,1)$ on the right-hand side of (2.6) yield, for $x > 0$,

$$\begin{aligned}\frac{1}{4x^4} + \frac{1}{2x^5} + \frac{5}{12x^6} - \frac{7}{24x^8} + \frac{1}{2n^{10}} - \frac{11}{8n^{12}} &< -\frac{1}{4!}\psi^{(4)}(x)\\ &< \frac{1}{4x^4} + \frac{1}{2x^5} + \frac{5}{12x^6} - \frac{7}{24x^8} + \frac{1}{2n^{10}}.\end{aligned} \tag{4.16}$$

Using (4.16), we find, for $n \geqslant 4$,

$$\begin{aligned}&-\frac{1}{4!}\psi^{(4)}(n) - \frac{1}{4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n}\\ &\quad > \frac{1}{4x^4} + \frac{1}{2x^5} + \frac{5}{12x^6} - \frac{7}{24x^8} + \frac{1}{2n^{10}} - \frac{11}{8n^{12}} - \frac{1}{4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n}\\ &\quad = \frac{1}{24n^{12}(n-1)(3n^2-3n+4)}\Big\{648 + 13305(n-4) + 16610(n-4)^2\\ &\quad\quad + 8721(n-4)^3 + 2416(n-4)^4 + 371(n-4)^5 + 30(n-4)^6 + (n-4)^7\Big\} > 0\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{4!}\psi^{(4)}(n) - \frac{1}{4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{9}}\\ &\quad < \frac{1}{4x^4} + \frac{1}{2x^5} + \frac{5}{12x^6} - \frac{7}{24x^8} + \frac{1}{2n^{10}} - \frac{1}{4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{9}}\\ &\quad = -\frac{88n^4 + 264n^3 - 511n^2 + 288n + 12}{24n^{10}(18n^4 - 36n^3 + 42n^2 - 24n - 1)} < 0.\end{aligned}$$

Hence, (4.15) holds for $n \geqslant 4$. The proof is complete. □

REMARK 4.1. Let $m \in \mathbb{N}$. The formula (4.5) can be written for $n \geqslant 4$ as

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^5}\right)^{-1} \right\rfloor = \begin{cases} 324m^4 + 432m^3 + 228m^2 + 40m - 1, & \text{if } n = 3m+1;\\ 324m^4 + 864m^3 + 876m^2 + 384m + 58, & \text{if } n = 3m+2;\\ 324m^4 + 1296m^3 + 1956m^2 + 1304m + 319, & \text{if } n = 3m+3. \end{cases} \tag{4.17}$$

REMARK 4.2. Using our method and Maple software one can study the open problem 1.1 for the cases $s \geqslant 6$. For example, the choice $j = 5$ in (2.10) yields

$$\left(\sum_{k=n}^{\infty} \frac{1}{k^6}\right)^{-1} = 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n + \frac{25}{96} + \frac{1625}{192n} + O\left(\frac{1}{n^2}\right). \tag{4.18}$$

Based on (4.18), we here present (without proof) the following computational formula for (1.11) with $s = 6$:

For $n = 1$ and $n = 2$, we have

$$\left\lfloor \left( \sum_{k=1}^{\infty} \frac{1}{k^6} \right)^{-1} \right\rfloor = \left\lfloor \frac{945}{\pi^6} \right\rfloor = 0, \quad \left\lfloor \left( \sum_{k=2}^{\infty} \frac{1}{k^6} \right)^{-1} \right\rfloor = \left\lfloor \frac{945}{\pi^6 - 945} \right\rfloor = 57,$$

and for $n \geqslant 3$, we find

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{k^6} \right)^{-1} \right\rfloor = \left\lfloor 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n + \frac{25}{96} + \frac{1625}{192n} \right\rfloor. \tag{4.19}$$

Some computer experiments indicate that formula (4.19) is true.

## REFERENCES


[1] M. ABRAMOWITZ AND I.A. STEGUN (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Dover, New York, 1965.

[2] H. ALZER, *On some inequalities for the gamma and psi functions*, Math. Comput. **66** (1997), 373–389.

[3] R. ANDRÉ-JEANNIN, *Irrationalité de la somme des inverses de certaines suites récurrentes*, C. R. Acad. Sci. Paris I **308** (1989), 539–541.

[4] C.-P. CHEN, N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic formulae associated with the Wallis power function and digamma function*, J. Classical Anal. **2** (2013), 151–166.

[5] Y. CHOO, *On the reciprocal sums of generalized Fibonacci numbers*, Int. J. Math. Anal. **10** (28) (2016), 1365–1373.

[6] K. DEVLIN, *The Millennium Problems*, Basic Books, New York, 2002.

[7] D. DUVERNEY, K. NISHIOKA, K. NISHIOKA AND I. SHIOKAWA, *Transcendence of Rogers-Ramanujan continued fraction and reciprocal sums of Fibonacci numbers*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 140–142.

[8] H. M. EDWARDS, *Riemann's Zeta Function*, Academic Press, New York, 1974; Dover Publications, Mineola, New York, 2001.

[9] C. ELSNER, S. SHIMOMURA AND I. SHIOKAWA, *Algebraic relations for reciprocal sums of Fibonacci numbers*, Acta Arith. **130** (2007), 37–60.

[10] C. ELSNER, S. SHIMOMURA AND I. SHIOKAWA, *Algebraic relations for reciprocal sums of odd terms in Fibonacci numbers*, Ramanujan J. **17** (2008), 429–446.

[11] S. H. HOLLIDAY AND T. KOMATSU, *On the sum of reciprocal generalized Fibonacci numbers*, Integers **11** (2011), 441–455.

[12] T. KOSHY, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[13] K. KUHAPATANAKUL, *On the sums of reciprocal generalized Fibonacci numbers*, J. Integer Seq. **16** (7) (2013), Article 13.7.1.

[14] K. KUHAPATANAKUL, *Alternating sums of reciprocal generalized Fibonacci numbers*, SpringerPlus **3** (1) (2014), 485, http://www.springerplus.com/content/3/1/485.

[15] X. LIN, *Some identities related to the Riemann zeta-function*, J. Inequal. Appl. **2016** (2016), 32.

[16] X. LIN AND X. LI, *A reciprocal sum related to the Riemann $\zeta$-function*, J. Math. Inequal. **11** (2017), 209–215.

[17] R. LIU AND ANDREW YZ. WANG, *Sums of products of two reciprocal Fibonacci numbers*, Adv. Difference Equ. **2016** (2016), 136.

[18] R. LIU AND F. Z. ZHAO, *On the sums of reciprocal hyperfibonacci numbers and hyperlucas numbers*, J. Integer Seq. **15** (4) (2012), Article 12.4.5.

[19] L. NAVAS, *Analytic continuation of the Fibonacci Dirichlet series*, Fibonacci Q. **39** (2001), 409–418.

[20] H. OHTSUKA AND S. NAKAMURA, *On the sum of reciprocal Fibonacci numbers*, Fibonacci Q. **46/47** (2008/2009), 153–159.

[21] B. RIEMANN, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Akad. Berlin **1859** (1859), 671–680.
[22] N. N. VOROBIEV, *Fibonacci Numbers*, Springer, Basel, 2002.
[23] T. WANG, *On the infinite sum of reciprocal Fibonacci numbers*, Acta. Math. Sin. Chin. Ser. **55** (3) (2012), 517–524 (in Chinese).
[24] Z. XU AND T. WANG, *The infinite sum of the cubes of reciprocal Pell numbers*, Adv. Differ. Equ. **2013** (2013), 184.
[25] ANDREW YZ. WANG AND P. WEN, *On the partial finite sums of the reciprocals of the Fibonacci numbers*, J. Inequal. Appl. **2015** (2015), 73.
[26] ANDREW YZ. WANG AND F. ZHANG, *The reciprocal sums of even and odd terms in the Fibonacci sequence*, J. Inequal. Appl. **2015** (2015), 376.
[27] ANDREW YZ. WANG AND F. ZHANG, *The reciprocal sums of the Fibonacci* $3$*-subsequences*, Adv. Difference Equ. **2016** (2016), 27.
[28] Wikipedia contributors, *Fibonacci number*, Wikipedia, The Free Encyclopedia, https://en.wikipedia.org/wiki/Fibonacci_number.
[29] Wikipedia contributors, *Lucas numbers*, Wikipedia, The Free Encyclopedia, https://en.wikipedia.org/wiki/Lucas_number.
[30] Z. WU AND W. ZHANG, *The sums of the reciprocals of Fibonacci polynomials and Lucas polynomials*, J. Inequal. Appl. **2012**, (2012), 134.
[31] Z. WU AND W. ZHANG, *Several identities involving the Fibonacci polynomials and Lucas polynomials*, J. Inequal. Appl. **2013** (2013), 205.
[32] H. XU, *Some computational formulas related the Riemann zeta-function tails*, J. Inequal. Appl. **2016** (2016), 132.
[33] P. YUAN, Z. HE, AND J. ZHOU, *On the sum of reciprocal generalized Fibonacci numbers*, Abstr. Appl. Anal. **2014** (2014), Art. ID 402540.
[34] W. ZHANG AND T. WANG, *The infinite sum of reciprocal Pell numbers*, Appl. Math. Comput. **218** (2012), 6164–6167.
[35] W. ZHANG AND T. WANG, *The infinite sum of reciprocal of the square of the Pell numbers*, J. Weinan Teach. Univ. **26** (2011), 39–42.
[36] H. ZHANG AND Z. WU, *On the reciprocal sums of the generalized Fibonacci sequences*, Adv. Difference Equ. **2013** (2013), 377.

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*Shun-Wei Xu*
*School of Mathematics and Information Science*
*Henan Polytechnic University*
*Jiaozuo City 454003, Henan Province, China*
*e-mail:* xswmath@163.com

*Bo Zhang*
*School of Mathematics and Information Science*
*Henan Polytechnic University*
*Jiaozuo City 454003, Henan Province, China*
*e-mail:* zhangbohpu@sohu.com

*Chao-Ping Chen*
*School of Mathematics and Information Science*
*Henan Polytechnic University*
*Jiaozuo City 454003, Henan Province, China*
*e-mail:* chenchaoping@sohu.com