Journal of
Mathematical
Inequalities
Volume 20, Number 1 (2026), 17–28
doi:10.7153/jmi-2026-20-02

A LOWER BOUND FOR THE BETA FUNCTION

TIEHONG ZHAO AND MIAOKUN WANG*

(*Communicated by M. Krnić*)

Abstract. We establish a novel lower bound for Euler's beta function $B(x,y)$, asserting that the inequality

$$B(x,y) > \frac{x+y}{xy}\left(1-\frac{2xy}{x+y+1}\right)$$

holds for all $(x,y) \in (0,1] \times (0,1]$. This bound refines the lower bound derived by P. Ivády [12, Theorem, (3.2)] in the case where $0 < x+y < 1$.

1. Introduction

The classical gamma function and its logarithmic derivative (referred to as the psi function) are defined respectively as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad \text{and} \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad x > 0.$$

The derivatives of ψ, i.e., $\psi', \psi'', \cdots$, are termed polygamma functions. The beta function, a well-known Euler's integral of the first kind, is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

for $x,y > 0$. It bears a close relationship with the gamma function through the elegant identity

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The utility of the beta function is often overshadowed by that of the gamma function, partly perhaps because it can be evaluated in terms of the gamma function. In fact, it has various applications not only in the theory of special functions, but it also plays a role in other fields, for instance, statistics, mathematical physics, and graph theory; see [1, 10, 18].

In recent years, numerous papers on notable inequalities related to the gamma and polygamma functions have been published (see [5, 7, 9, 15, 16, 17, 19, 26] and the

---

Mathematics subject classification (2020): 26D07, 33B15.
Keywords and phrases: Beta function, gamma function, psi function, inequalities.

* Corresponding author.

references therein); only a few inequalities concerning the beta function can be found in the literature [3, 4, 6, 8, 11].

In 2000, Dragomir et al. [8] proved that the inequality

$$0 \leqslant \frac{1}{xy} - B(x,y) \leqslant \frac{1}{4}$$

holds for all $x,y \geqslant 1$. Subsequently, Alzer [3] refined the upper bound $1/4$ to the sharpest possible constant $\max\limits_{x\geqslant 1}\{\Delta(x)\} = 0.08731\cdots$, where $\Delta(x) = 1/x^2 - \Gamma^2(x)/\Gamma(2x)$ for $x \geqslant 1$.

For all $x,y \in (0,1]$, sharp rational bounds for the difference $1/(xy) - B(x,y)$ were established in [4], given by

$$\frac{1}{xy}\left[1 - \alpha\frac{(1-x)(1-y)}{(1+x)(1+y)}\right] \leqslant B(x,y) \leqslant \frac{1}{xy}\left[1 - \beta\frac{(1-x)(1-y)}{(1+x)(1+y)}\right], \tag{1.1}$$

where $\alpha = \frac{2}{3}\pi^2 - 4 = 2.57973\cdots$ and $\beta = 1$ are the sharpest possible constants.

In 2012, Ivády [12] strengthened the right-hand side of the inequality from [4] (referred to herein as (1.1)). Although the original proof contained errors (later corrected in [13]), Ivády presented a double inequality for $B(x,y)$: for all $x,y \in (0,1]$,

$$\frac{x+y-xy}{xy} \leqslant B(x,y) \leqslant \frac{x+y}{xy(1+xy)} \tag{1.2}$$

holds, with equality if and only if $x = y = 1$. The right-hand side of this inequality (referred to herein as (1.2)) improves the right-hand side of (1.1), while the left-hand sides of (1.2) and (1.1) are not comparable for all $x,y \in (0,1]$. However, for all $x,y \in (0,1]$ with $x+y \leqslant 1$, the left-hand side of (1.2) is also better than that of (1.1). Indeed, due to $\alpha > 5/2$, for $x+y \leqslant 1$,

$$\begin{aligned}
&x+y-xy-\left[1-\alpha\frac{(1-x)(1-y)}{(1+x)(1+y)}\right] > x+y-xy-\left[1-\frac{5}{2}\frac{(1-x)(1-y)}{(1+x)(1+y)}\right]\\
&= \frac{(1-x)(1-y)(3-2x-2y-2xy)}{2(1+x)(1+y)} \geqslant \frac{(1-x)(1-y)\left[3-2x-2y-(x+y)^2\right]}{2(1+x)(1+y)}\\
&= \frac{(1-x)(1-y)(1-x-y)(3+x+y)}{2(1+x)(1+y)} \geqslant 0.
\end{aligned}$$

In 2016, Ivády [14] further improved the lower bound in (1.2), showing that

$$B(x,y) \geqslant \frac{1}{xy}\frac{x+y-xy/2}{1+xy/2} \tag{1.3}$$

for all $x,y \in (0,1]$.

Against this backdrop, the primary objective of this paper is to present a new lower bound for $B(x,y)$ applicable to all $x,y \in (0,1]$. This result is formally stated in the following theorem.

THEOREM 1.1. *For all $x,y \in (0,1]$, the following inequality holds:*

$$B(x,y) > \frac{x+y}{xy}\left(1-\frac{2xy}{x+y+1}\right).$$

First, we make a key observation: for $x+y \geqslant 1$ and $xy > 0$, the inequality

$$1-\frac{xy}{x+y} \geqslant 1-\frac{2xy}{x+y+1}$$

holds. Combining this with inequality (1.2), we derive

$$B(x,y) \geqslant \frac{x+y}{xy}\left(1-\frac{xy}{x+y}\right) \geqslant \frac{x+y}{xy}\left(1-\frac{2xy}{x+y+1}\right), \tag{1.4}$$

where the equalities cannot hold simultaneously. This allows us to restrict the proof of Theorem 1.1 to the case where $0 < x+y < 1$.

REMARK 1.2. Assuming Theorem 1.1 holds, for all $x,y \in (0,1]$ with $x+y \leqslant 1$, we have

$$B(x,y) > \frac{x+y}{xy}\left(1-\frac{2xy}{x+y+1}\right) \geqslant \frac{x+y}{xy}\left(1-\frac{xy}{x+y}\right) = \frac{x+y-xy}{xy}.$$

This directly implies that our lower bound is sharper than the one in (1.2) for this domain. Furthermore, for all $x,y \in (0,1]$ with $x+y \leqslant 1/2$, the inequality

$$\begin{aligned} B(x,y) &> \frac{x+y}{xy}\left(1-\frac{2xy}{x+y+1}\right) \\ &= \frac{1}{xy}\frac{x+y-xy/2}{1+xy/2} + \frac{1-2(x+y)+(x+y)(x+y-2xy)}{(1+x+y)(2+xy)} \\ &\geqslant \frac{1}{xy}\frac{x+y-xy/2}{1+xy/2} \end{aligned}$$

holds. This demonstrates that the lower bound for $B(x,y)$ in Theorem 1.1 is also sharper than the one given in (1.3) when $x,y \in (0,1]$ and $x+y \leqslant 1/2$.

2. Lemmas

To prove the main theorem, we first establish the following four lemmas. The first lemma provides a lower bound for the difference between two psi functions, which is due to H. Alzer [2, Theorem 7].

LEMMA 2.1. *Let $n \geqslant 0$ be an integer and $s \in (0,1)$. For all $x > 0$, the following inequality holds:*

$$\psi(x+1)-\psi(x+s) > (1-s)\left[\frac{1}{x+s+n} + \sum_{i=0}^{n-1}\frac{1}{(x+i+1)(x+i+s)}\right].$$

The second lemma is critical for proving the main theorem, as it establishes rational bounds for the derivatives of psi function. In [20], Yang introduced the function

$$\mathscr{L}(x,a)=\frac{1}{90a^2+2}\log\left(x^2+x+\frac{3a+1}{3}\right)+\frac{45a^2}{90a^2+2}\log\left(x^2+x+\frac{15a-1}{45a}\right),$$

which is used to approximate the psi function. For $a=2/5$ or $4/5$, the partial derivatives of $\mathscr{L}(x,a)$ are given by

$$\begin{aligned}
\mathscr{L}_x(x,2/5)&=\frac{3(1+2x)(61+90x+90x^2)}{2(11+15x+15x^2)(5+18x+18x^2)},\\
\mathscr{L}_x(x,4/5)&=\frac{3(1+2x)(199+180x+180x^2)}{2(17+15x+15x^2)(11+36x+36x^2)},\\
\mathscr{L}_{xx}(x,2/5)&=-\frac{3\begin{pmatrix}4993+36546x+110526x^2+196560x^3\\+219780x^4+145800x^5+48600x^6\end{pmatrix}}{2(11+15x+15x^2)^2(5+18x+18x^2)^2},\\
\mathscr{L}_{xx}(x,4/5)&=-\frac{3\begin{pmatrix}46537+322206x+784446x^2+1118880x^3\\+1045440x^4+583200x^5+194400x^6\end{pmatrix}}{2(17+15x+15x^2)^2(11+36x+36x^2)^2}.
\end{aligned}$$

Notably, [20, Lemma 1] proves that the mappings $a\mapsto\mathscr{L}_x(x,a)$ and $a\mapsto\mathscr{L}_{xx}(x,a)$ are decreasing and increasing on $(1/15,\infty)$, respectively. Combining this result with [28, Corollary 3.3], we obtain the following lemma:

LEMMA 2.2. *Let* $a_1=\frac{40+3\sqrt{205}}{105}=0.79003\cdots$, $a_2=\frac{45-4\pi^2+3\sqrt{4\pi^4-80\pi^2+405}}{30(\pi^2-9)}=0.47053\cdots$, *and let* $a_3=0.43218\cdots$ *be the unique solution of the equation* $\mathscr{L}_{xx}(0,a)=\psi''(1)$. *Then the inequalities*

$$\mathscr{L}_x(x,4/5)<\mathscr{L}_x(x,a_1)<\psi'(x+1)<\mathscr{L}_x(x,a_2)<\mathscr{L}_x(x,2/5), \tag{2.5}$$
$$\mathscr{L}_{xx}(x,2/5)<\mathscr{L}_{xx}(x,a_3)<\psi''(x+1)<\mathscr{L}_{xx}(x,a_1)<\mathscr{L}_{xx}(x,4/5) \tag{2.6}$$

hold for all $x>0$ *with the sharpest possible constants* a_1,a_2 *and* a_3.

LEMMA 2.3. *Let* $0<x<(\sqrt{3}+1)/2$, *and define the function*

$$f(x)=\log\left[\frac{\Gamma(x+1)^2}{\Gamma(2x+1)}\right]-\log\left(1-\frac{2x^2}{1+2x}\right).$$

Then $f(x)>0$ *for all* $x\in(0,(\sqrt{3}+1)/2)$.

Proof. We begin by computing the first derivative of $f(x)$. Differentiation gives

$$f'(x)=2\left[\psi(x+1)-\psi(2x+1)+\frac{2x(1+x)}{(1+2x)(1+2x-2x^2)}\right]\triangleq 2\hat{f}(x), \tag{2.7}$$

where $\hat{f}(x)$ denotes the expression in the brackets for brevity.

Next, we calculate the first derivative of $\hat{f}(x)$:

$$\hat{f}'(x) = \psi'(x+1) - 2\psi'(2x+1) + \frac{2(1+2x+2x^2+8x^3+4x^4)}{(1+2x)^2(1+2x-2x^2)^2}.$$

By virtue of the inequality in (2.5) (i.e., $\psi'(x+1) > \mathscr{L}_x(x,4/5)$ and $\psi'(2x+1) < \mathscr{L}_x(2x,2/5)$), substituting these bounds into $\hat{f}'(x)$ yields

$$\begin{aligned}\hat{f}'(x) &> \mathscr{L}_x(x,4/5) - 2\mathscr{L}_x(2x,2/5) + \frac{2(1+2x+2x^2+8x^3+4x^4)}{(1+2x)^2(1+2x-2x^2)^2} \\ &= \frac{\begin{pmatrix} 5533+37994x+92054x^2+935456x^3+11448028x^4+67479864x^5 \\ +232646232x^6+513934848x^7+747879552x^8+717361920x^9 \\ +442143360x^{10}+158630400x^{11}+18662400x^{12} \end{pmatrix}}{2\begin{bmatrix} (1+2x)^2(1+2x-2x^2)^2(17+15x+15x^2) \\ \times(11+36x+36x^2)(11+30x+60x^2)(5+36x+72x^2) \end{bmatrix}} > 0.\end{aligned}$$

This strict positivity implies that $\hat{f}(x)$ is strictly increasing on $(0,\infty)$. Additionally, we note that $\hat{f}(0) = f(0) = 0$ (a result implicitly referenced in (2.7)). For any $0 < x < (\sqrt{3}+1)/2$, since $\hat{f}(x)$ is strictly increasing, we have $\hat{f}(x) > \hat{f}(0) = 0$.

Recall that $f'(x) = 2\hat{f}(x)$, so $f'(x) > 0$ for all $0 < x < (\sqrt{3}+1)/2$. This means that $f(x)$ is strictly increasing on $\big(0,(\sqrt{3}+1)/2\big)$. Consequently, for all $0 < x < (\sqrt{3}+1)/2$, we obtain $f(x) > f(0) = 0$, which completes the proof. □

The next lemma provides a straightforward criterion for determining the sign of a specific class of polynomials, referred to as Positive-Negative type (abbreviated as PN type) polynomials (see [23, 25, 27]).The counterpart of a PN type polynomial is termed a Negative-Positive type polynomial. Additional details regarding these polynomials and their series extensions are available in [21, 22, 24].

LEMMA 2.4. ([25, Lemma 4]) *Let $P_n(x)$ be a Positive-Negative (PN) type polynomial of degree n, defined as*

$$P_n(x) = \sum_{k=0}^{m} a_k x^k - \sum_{k=m+1}^{n} a_k x^k,$$

where $a_k \geqslant 0$ for all $k \geqslant 0$, and there exist at least two integers $0 \leqslant k_1 \leqslant m$ and $m+1 \leqslant k_2 \leqslant n$ such that $a_{k_1} \neq 0$ and $a_{k_2} \neq 0$. Then there exists a unique $x_0 \in (0,\infty)$ such that $P_n(x_0) = 0$, with $P_n(x) > 0$ for all $x \in (0,x_0)$ and $P_n(x) < 0$ for all $x \in (x_0,\infty)$.

REMARK 2.5. For a PN type polynomial $P_n(x)$, the following property holds trivially: if $P_n(x_1) > 0$ for some $x_1 > 0$, then $P_n(x) > 0$ for all $x \in (0,x_1)$; if $P_n(x_2) < 0$ for some $x_2 > 0$, then $P_n(x) < 0$ for all $x \in (x_2,\infty)$. Correspondingly, for a NP type polynomial $P_n(x)$, the signs of $P_n(x)$ on the intervals $(0,x_1)$ and (x_2,∞) are reversed.

Using Lemma 2.4, we present the following examples of PN and NP type polynomials, which are essential for proving the main theorem.

EXAMPLE 2.1. Consider the following PN type polynomials:

$$p_0(x) = 1802 + 51043x - 183763x^2 - 789936x^3 - 613605x^4,$$

$$p_1(x) = \left[\begin{array}{l} 158486648 + 11733123789x + 31469130662x^2 + 40356637167x^3 \\ +19492784598x^4 - 25810079475x^5 - 68587238430x^6 \\ -84246205050x^7 - 76169872800x^8 - 54370440000x^9 \\ -28066500000x^{10} - 9112500000x^{11} \end{array}\right],$$

$$p_2(x) = \left[\begin{array}{l} 9221 + 85665x + 368643x^2 + 918717x^3 + 1422360x^4 \\ +1354842x^5 + 652860x^6 - 76140x^7 - 291600x^8 - 145800x^9 \end{array}\right],$$

$$p_3(x) = \left[\begin{array}{l} 3699375712 + 26523455964x + 60608029362x^2 + 68646948321x^3 \\ +32033337786x^4 - 25862152266x^5 - 50136084000x^6 \\ -35657677440x^7 - 11393978400x^8 \end{array}\right],$$

$$p_4(x) = 37 + 90x - 93x^2 - 636x^3 - 810x^4 - 540x^5.$$

Calculating the values of these polynomials at specific points gives: $p_0(3/20) = 75107551/32000$, $p_1(1/5) = 64124455182553/15625$, $p_2(1) = 4298768$, $p_3(1) = 68461255039$ and $p_4(9/25) = 21101408/1953125$. By Lemma 2.4, we conclude:

$$p_0(x) > 0 \quad \text{for} \quad x \in [0, 3/20], \tag{2.8}$$

$$p_1(x) > 0 \quad \text{for} \quad x \in [0, 1/5], \tag{2.9}$$

$$p_2(x) > 0 \quad \text{for} \quad x \in [0, 1], \tag{2.10}$$

$$p_3(x) > 0 \quad \text{for} \quad x \in [0, 1], \tag{2.11}$$

$$p_4(x) > 0 \quad \text{for} \quad x \in [0, 9/25]. \tag{2.12}$$

EXAMPLE 2.2. We use the following NP type polynomials in subsequent proofs:

$$q_0(x) = -11 - 5x + 11x^2 + 5x^3, \qquad q_1(x) = -5 + 129x + 131x^2 + 33x^3,$$

$$q_2(x) = -65 + 254x + 242x^2 + 49x^3, \quad q_3(x) = -84 + 222x + 157x^2 + 19x^3,$$

$$q_4(x) = -45 + 101x + 43x^2 + 2x^3, \qquad q_5(x) = -11 + 23x + 4x^2.$$

Evaluating these polynomials at $x = 1/2$ yields: $q_0(1/2) = -81/8$, $q_1(1/2) = 771/8$, $q_2(1/2) = 1029/8$, $q_3(1/2) = 549/8$, $q_4(1/2) = 33/2$ and $q_5(1/2) = 3/2$. By Lemma 2.4, we deduce two key results:

1. $q_0(x) < 0$ for all $x \in (0, 1/2)$;
2. For each $j \in \{1,2,3,4,5\}$, $q_j(x)$ has a unique root in the interval $(0, 1/2)$.

Let x_j denote the positive root of $q_j(x)$ for $j \in \{1,2,3,4,5\}$. Simple numerical computation gives the ordering: $x_1 = 0.03733\cdots < x_2 = 0.2114\cdots < x_3 = 0.3085\cdots < x_4 = 0.3822\cdots < x_5 = 0.4439\cdots$. By Remark 2.5, this ordering implies that if $q_j(x) < 0$ for some $x \in (0, 1/2]$, then $q_{j+1}(x) < 0$ for the same x (for all $1 \leqslant j \leqslant 4$).

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 for the case where $0 < x + y \leqslant 1$. Without loss of generality, we assume $x \leqslant y$ (this symmetry assumption simplifies the analysis without compromising generality).

Proof of Theorem 1.1. Let $0 < x \leqslant y \leqslant 1 - x$ (which implies $0 < x \leqslant 1/2$), and define the auxiliary function

$$\begin{aligned} F(x,y) &= \log\left[\frac{xyB(x,y)}{x+y}\right] - \log\left(1 - \frac{2xy}{x+y+1}\right) \\ &= \log\left[\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)}\right] - \log\left(1 - \frac{2xy}{x+y+1}\right). \end{aligned} \tag{3.13}$$

First, we compute the partial derivatives of $F(x,y)$ with respect to x and y. For the partial derivative with respect to x:

$$\frac{\partial F}{\partial x} = \psi(x+1) - \psi(x+y+1) + \frac{2y(1+y)}{(1+x+y)(1+x+y-2xy)}. \tag{3.14}$$

For the partial derivative with respect to y (by symmetry with respect to x and y):

$$\begin{aligned} \frac{\partial F}{\partial y} &= \psi(y+1) - \psi(x+y+1) + \frac{2x(1+x)}{(1+x+y)(1+x+y-2xy)} \\ &= \psi(y+1) - \psi(y+x) - \frac{1}{x+y} + \frac{2x(1+x)}{(1+x+y)(1+x+y-2xy)}. \end{aligned} \tag{3.15}$$

We divide into two cases to complete the proof.

Case 1: $1/5 \leqslant x \leqslant 1/2$ and $x \leqslant y \leqslant 1 - x$.
Substitute $n = 3$ into Lemma 2.1. Combining this with equation (3.15), we obtain

$$\begin{aligned} \frac{\partial F}{\partial y} > (1-x)&\left[\frac{1}{x+y+3} + \sum_{i=0}^{2} \frac{1}{(y+i+1)(y+i+x)}\right] \\ &- \frac{1}{x+y} + \frac{2x(1+x)}{(1+x+y)(1+x+y-2xy)} \\ = &\frac{xQ(x,y)}{(1+x+y-2xy)\prod_{j=1}^{3}\left[(y+j)(x+y+j)\right]}, \end{aligned}$$

where $Q(x,y) = -q_0(x) + \sum_{k=1}^{5} q_k(x)y^k - (1-2x)y^6$, and $q_k(x)$ (for $0 \leqslant k \leqslant 5$) are the NP type polynomials defined in Example 2.2. From Example 2.2, $Q(x,y)$ is a PN type polynomial in y for all $x \in (0, 1/2]$. This, together with Lemma 2.4 (the sign criterion for PN type polynomials) and the fact that

$$Q(x, 1-x) = \frac{4}{625}(1-x)\begin{bmatrix} 252 + (5x-1)(7137 + 5300x^2) \\ +(1-x)(5x-1)(24365 + 375x^2) \end{bmatrix} > 0$$

leads to the conclusion that $Q(x,y) > 0$ for $1/5 \leqslant x \leqslant 1/2$ and $x \leqslant y \leqslant 1-x$. Thus, $\frac{\partial F}{\partial y} > 0$ for all x,y in this case, $F(x,y)$ is strictly increasing in y. Therefore, $F(x,y) \geqslant F(x,x)$ for $x \leqslant y \leqslant 1-x$. By Lemma 2.3, $F(x,x) = f(x) > 0$, so we conclude $F(x,y) > 0$ for $1/5 \leqslant x \leqslant 1/2$ and $x \leqslant y \leqslant 1-x$.

Case 2: $0 < x < 1/5$ and $x \leqslant y \leqslant 1-x$.
Let D denote the trapezoidal domain $\{(x,y)|\, x<y<1-x,\ 0<x<1/5\}$, with ∂D representing its boundary. Our goal here is to show that $F(x,y)$ has no extreme values (i.e. no critical points) in the interior of D.

To do this, define the auxiliary function $G(x,y)$ as the difference of the partial derivatives of $F(x,y)$:

$$G(x,y) = \frac{\partial F(x,y)}{\partial x} - \frac{\partial F(x,y)}{\partial y} = \psi(x+1) - \psi(y+1) - \frac{2(x-y)}{1+x+y-2xy}. \tag{3.16}$$

We continue to divide into three subcases.

Subcase A: $y \geqslant x + 9/25$.
First, compute the partial derivatives of $G(x,y)$:

$$\begin{aligned} \frac{\partial G}{\partial x} &= \psi'(x+1) - \frac{2(1+2y-2y^2)}{(1+x+y-2xy)^2}, \\ \frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x} &= \frac{12(y-x)}{(1+x+y-2xy)^3} > 0. \end{aligned} \tag{3.17}$$

The mixed partial derivative $\frac{\partial^2 G}{\partial x \partial y} > 0$ implies that $\frac{\partial G}{\partial x}$ is strictly increasing in y. Thus, for $y \geqslant x + 9/25$, we have

$$\frac{\partial G(x,y)}{\partial x} > \frac{\partial G(x, x+9/25)}{\partial x} = \psi'(x+1) - \frac{913+350x-1250x^2}{2(17+16x-25x^2)^2} \triangleq g(x). \tag{3.18}$$

In what follows, we show that $g(x) > 0$ for all $x \in (0, 1/5)$:

- For $x \in (0, 1/10]$: By (2.5) (which gives $\psi'(x+1) > \mathscr{L}_x(x, 4/5)$) and (2.8) (which confirms $p_0(x) > 0$), substituting these bounds into $g(x)$ yields

$$\begin{aligned} g(x) &> \mathscr{L}_x(x,4/5) - \frac{913+350x-1250x^2}{2(17+16x-25x^2)^2} \\ &= \frac{p_0(x) + 307230x^5 + 823500x^6 + 675000x^7}{2(17+15x+15x^2)(17+16x-25x^2)^2(11+36x+36x^2)} > 0. \end{aligned}$$

- For $x \in (1/10, 1/5)$: By (2.6) (which gives $\psi''(x+1) < \mathscr{L}_{xx}(x, 4/5)$) and (2.9) (which confirms $p_1(x) > 0$), we analyze the derivative of $g(x)$:

$$\begin{aligned} g'(x) &= \psi''(x+1) + \frac{11633 - 21600x - 13125x^2 + 31250x^3}{(17+16x-25x^2)^3} \\ &< \mathscr{L}_{xx}(x, 4/5) + \frac{11633 - 21600x - 13125x^2 + 31250x^3}{(17+16x-25x^2)^3} \\ &= -\frac{127679911(10x-1) + xp_1(x)}{2(17+15x+15x^2)^2(17+16x-25x^2)^3(11+36x+36x^2)^2} < 0. \end{aligned}$$

This implies that $g(x)$ is strictly decreasing on $(1/10, 1/5)$, so $g(x) > g(1/5) = 0.001914\cdots > 0$.

Combining these results, $\frac{\partial G}{\partial x} > 0$ for all $x \in (0, 1/5)$ by (3.18). Next, consider the second derivative of $G(0,y)$ with respect to y: by (2.6), (2.10) and (3.16), for $y \in (0,1)$,

$$\begin{aligned} \frac{d^2}{dy^2} G(0,y) &= -\frac{4}{(1+y)^3} - \psi''(1+y) < -\frac{4}{(1+y)^3} - \mathscr{L}_{yy}(y, 2/5) \\ &= -\frac{p_2(y)}{2(1+y)^3(11+15y+15y^2)^2(5+18y+18y^2)^2} < 0. \end{aligned}$$

This shows that the function $y \mapsto G(0,y)$ is strictly concave on $(0,1)$, so $G(0,y) \geqslant \min\{G(0,0), G(0,1)\} = 0$. Since $\frac{\partial G}{\partial x} > 0$, we have $G(x,y) > G(0,y) \geqslant 0$ for $0 < x < 1/5$ and $x + 9/25 \leqslant y < 1$.

Subcase B: $9/25 < y < x + 9/25$.

First, correct the domain constraint implied by the subcase: since $9/25 < y < x + 9/25$, rearranging gives $y - 9/25 < x < 1/5$ (consistent with the overall Case 2 domain $0 < x < 1/5$).

From equation (3.17) (not explicitly shown here but referenced in the original proof), the partial derivative $\frac{\partial G}{\partial y}$ is strictly increasing in x for $0 < x < y$. By the bound for $\psi'(y+1)$ from Lemma 2.2 (i.e., $\psi'(y+1) < \mathscr{L}_y(y, 2/5)$ from (2.5)), substituting this into $\frac{\partial G}{\partial y}$ yields:

$$\begin{aligned} \frac{\partial G(x,y)}{\partial y} &> \frac{\partial G(y - 9/25, y)}{\partial y} = \frac{13 + 2150y - 1250y^2}{2(8+34y-25y^2)^2} - \psi'(y+1) \\ &> \frac{13 + 2150y - 1250y^2}{2(8+34y-25y^2)^2} - \mathscr{L}_y(y, 2/5) \\ &= \frac{5275352 + (25y-9)\begin{bmatrix} 4404553 + 18643550y + 55576875y^2 \\ +88996875y^3 + 9375000y^4 \\ +843750y^4(1-y)(57+50y) \end{bmatrix}}{6250(11+15y+15y^2)(5+18y+18y^2)(8+34y-25y^2)^2} > 0. \end{aligned}$$

This confirms $\frac{\partial G}{\partial y} > 0$, meaning $G(x,y)$ is strictly increasing in y for fixed x. Next, analyze the concavity of $G(x, 9/25)$ (the value of $G(x,y)$ at $y = 9/25$) with respect to x. By (2.6) (which gives $\psi''(x+1) < \mathscr{L}_{xx}(x,4/5)$) and (2.11) (which confirms $p_3(x) > 0$), the second derivative of $G(x,9/25)$ with respect to x satisfies:

$$\begin{aligned}\frac{d^2}{dx^2}G(x,9/25) &= \psi''(x+1) + \frac{25564}{(34+7x)^3} < \mathscr{L}_{xx}(x,4/5) + \frac{25564}{(34+7x)^3}\\ &= -\frac{p_3(x)+200037600x^9}{2(34+7x)^3(17+15x+15x^2)^2(11+36x+36x^2)^2} < 0\end{aligned}$$

for all $x \in (0,1/5)$. This strict negativity implies $G(x,9/25)$ is strictly concave on $(0,1/5)$. For strictly concave functions, the minimum on an interval is attained at one of the endpoints. Thus:

$$\begin{aligned}G(x,y) > G(x,9/25) &> \min\left\{G(0,9/25), G(1/5,9/25)\right\}\\ &= \min\{0.0554\cdots, 0.04015\cdots\} > 0.\end{aligned}$$

Subcase C: $x < y \leqslant 9/25$.

Recall from Subcase B that the mapping $x \mapsto \frac{\partial G(x,y)}{\partial y}$ is strictly increasing for $0 < x < y$. For fixed y, this monotonicity implies $\frac{\partial G(x,y)}{\partial y} > \frac{\partial G(0,y)}{\partial y}$ (since $x > 0$). By (2.5) (the bound $\psi'(y+1) < \mathscr{L}_y(y,2/5)$ from Lemma 2.2), substitute this into $\frac{\partial G(0,y)}{\partial y}$:

$$\begin{aligned}\frac{\partial G(x,y)}{\partial y} > \frac{\partial G(0,y)}{\partial y} &= \frac{2}{(y+1)^2} - \psi'(y+1) > \frac{2}{(y+1)^2} - \mathscr{L}_y(y,2/5)\\ &= \frac{p_4(y)}{2(1+y)^2(11+15y+15y^2)(5+18y+18y^2)} > 0.\end{aligned}$$

The final inequality holds because $p_4(y) > 0$ for all $y \in (0,9/25]$ – a result confirmed by (2.12) in Example 2.1. Since $\frac{\partial G(x,y)}{\partial y} > 0$, $G(x,y)$ is strictly increasing in y for fixed x. Given the subcase domain $x < y \leqslant 9/25$, the minimum value of $G(x,y)$ in y is attained at $y = x$. By equation (3.16) (the definition of $G(x,y)$): $G(x,x) = \frac{\partial F(x,x)}{\partial x} - \frac{\partial F(x,x)}{\partial y} = 0$ (by symmetry of $F(x,y)$ in x and y when $x = y$). Thus, $G(x,y) > G(x,x) = 0$ for all $0 < x < y \leqslant 9/25$.

In conclusion, $G(x,y) > 0$ for all $(x,y) \in D$. This implies $F(x,y)$ has no extreme values in D: if there existed a critical point $(x_0,y_0) \in D$, we would have $\frac{\partial F(x_0,y_0)}{\partial x} = \frac{\partial F(x_0,y_0)}{\partial y} = 0$, which would force $G(x_0,y_0) = 0$ – a contradiction to $G(x,y) > 0$ in D.

Next, consider the boundary ∂D of D. The function $F(x,y)$ has the following properties on ∂D:

(i) When $x + y = 1$: $F(x,y) > 0$ by equations (1.4) and (3.13).

(ii) When $x = 0$: From (3.13), it is straightforward to verify $F(0,y) = 0$.

(iii) When $x = y$: By Lemma 2.3, $F(x,x) = f(x) > 0$ for all $x \in (0,1)$.

(iv) When $x = 1/5$: Case 1 already established $F(1/5,y) > 0$ for $1/5 \leqslant y \leqslant 4/5$.

Since $F(x,y)$ is continuous on the closure $\bar{D}$ of D, it attains its minimum value on $\bar{D}$. As $F(x,y)$ has no critical points in D, its minimum must lie on ∂D. Thus,

$$F(x,y) \geqslant \min_{(x,y)\in\partial D} \{F(x,1-x), F(0,y), F(x,x), F(1/5,y)\} = 0,$$

with equality if and only if $x = 0$ (which is not in the domain $(0,1]\times(0,1]$ of Theorem 1.1).

This completes the proof of Theorem 1.1. □

Acknowledgements. This work was supported by Huzhou Natural Science Foundation of China under Grant No. 2025YZ37.

REFERENCES

[1] P. ÅHAG, R. CZYŻ, *An inequality for the beta function with application to pluripotential theory*, J. Ineuqual. Appl. **2009**, Article number: 901397, (2009).

[2] H. ALZER, *On some inequalities for the gamma and psi functions*, Math. Comp. **66**(217), 373–389, (1997).

[3] H. ALZER, *Sharp inequalities for the beta function*, Indagationes Math. **12**, 15–21, (2001).

[4] H. ALZER, *Some beta function inequalities*, Proc. of the Royal Soc. of Edinburgh **133A**, 731–745, (2003).

[5] O. ATALE, *Newman's inequalities and log-convexity properties of gamma function*, Bull. Int. Math. Virtual Inst. **12** (3), 469–476, (2022).

[6] P. CERONE, *Special functions: approximations and bounds*, Appl. Anal. Discrete Math. **1** (1), 72–91, (2007).

[7] C.-P. CHEN, C. MORTICI, *Asymptotic expansions and inequalities relating to the gamma function*, Appl. Anal. Discrete Math. **16** (2), 379–399, (2022).

[8] S. S. DRAGOMIR, R. P. AGARWAL, N. S. BARNETT, *Inequalities for beta and gamma functions via some classical and new integral inequalities*, J. Inequal. Appl. **5**, 103–165 (2000).

[9] S. DAS, A. SWAMINATHAN, *A harmonic mean inequality for the polygamma function*, Math. Inequal. Appl. **23** (1), 71–76, (2020).

[10] J. DWYER, *Circuits in complete graphs and their relations to rapidly-increasing special functions*, `www.algana.co.uk/publications/Circuits.pdf`.

[11] S. G. FROM, S. RATNASINGAM, *Some new inequalities for the beta function and certain ratios of beta functions*, Results Appl. Math. **15**, Paper No. 100302, 17 pp, (2022).

[12] P. IVÁDY, *On a beta function inequality*, J. Math. Inequal. **6** (3), 333–341, (2012).

[13] P. IVÁDY, *Corrigendum to: "On a beta function inequality"*, J. Math. Inequal. **10** (2), 597–601, (2016).

[14] P. IVÁDY, *An improvement of a beta function inequality*, J. Math. Sci.: Adv. Appl. **40**, 35–53, (2016).

[15] L.-C. LIANG, L.-F. ZHENG, A. WAN, *A class of completely monotonic functions involving the polygamma functions*, J. Inequal. Appl. Paper No. 12, 16 pp, (2022).

[16] R. PÉREZ-MARCO, *On the definition of the Euler gamma function*, Enseign. Math. **68** (1–2), 135–160, (2022).

[17] F. QI, *Lower bound of sectional curvature of Fisher-Rao manifold of beta distributions and complete monotonicity of functions involving polygamma functions*, Results Math. **76** (4), Paper No. 217, 16 pp, (2021).

[18] B. R. RAO, *Gautschi-type inequalities for incomplete moments of a multivariate distribution with applications*, Metron. **39**, 43–51, (1981).

[19] L. YIN, *Monotonic properties for ratio of the generalized (p,k)-polygamma functions*, J. Math. Inequal. **16** (3), 915–921, (2022).

[20] Z.-H. YANG, *The monotonicity and convexity of a function involving digamma one and their applications*, (2014), arXiv:1408.2245 [math.CA].

[21] Z.-H. YANG, *Sharp approximations for the complete elliptic integrals of the second kind by one-parameter means*, J. Math. Anal. Appl. **467** (1), 446–461, (2018).

[22] Z.-H. YANG, J.-F. TIAN, *Monotonicity rules for the ratio of two Laplace transforms with applications*, J. Math. Anal. Appl. **470** (2), 821–845, (2019).

[23] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU, W. ZHANG, *On rational bounds for the gamma function*, J. Inequal. Appl. **2017**, 17 pp, Paper No. 210, (2017).

[24] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU, W. ZHANG, *On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind*, J. Math. Anal. Appl. **462** (2), 1714–1726, (2018).

[25] Z.-H. YANG, J.-F. TIAN, *Monotonicity and sharp inequalities related to gamma function*, J. Math. Inequal. **12** (1), 1–22, (2018).

[26] Z.-H. YANG, J.-F. TIAN, *Complete monotonicity of the remainder of the asymptotic series for the ratio of two gamma functions*, J. Math. Anal. Appl. **517** (2), Paper No. 126649, 15 pp, (2023).

[27] Z.-H. YANG, J.-F. TIAN, M.-K. WANG, *A positive answer to Bhatia-Li conjecture on the monotonicity for a new mean in its parameter*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **114** (3), Paper No. 126, (2020).

[28] T.-H. ZHAO, Z.-H. YANG, Y.-M. CHU, *Monotonicity properties of a function involving the psi function with applications*, J. Inequal. Appl. **2015**, Article No. 193 (2015).

(Received May 5, 2023)

Tiehong Zhao
School of Mathematics
Hangzhou Normal University
Hangzhou 311121, China
e-mail: `tiehong.zhao@hznu.edu.cn`

Miaokun Wang
Department of Mathematics
Huzhou University
Huzhou 313000, China
e-mail: `wangmiaokun@zjhu.edu.cn`