Journal of
Mathematical
Inequalities
Volume 20, Number 1 (2026), 29–50
doi:10.7153/jmi-2026-20-03

HERMITE–HADAMARD AND GRADIENT INEQUALITIES FOR CONVEX FUNCTIONS

SHIGERU FURUICHI, NICUŞOR MINCULETE, HAMID REZA MORADI
AND MOHAMMAD SABABHEH

(*Communicated by M. Krnić*)

Abstract. The Hermite-Hadamard inequality is one of the most exciting and applicable inequalities that govern convex functions. In this paper, we investigate convex and differentiable convex functions and present several new orderings among such functions. The results include refinements of the well-known gradient inequality for convex functions, a delicate refinement of the celebrated Hermite-Hadamard inequality, and many other interesting consequences.

As applications of the obtained results, we present some inequalities for scalar means, operator means, relative and Shannon entropies for scalars and operators.

1. Motivation and background

Let $f : I \to \mathbb{R}$ be a given function on an interval I. It is customary in the literature to compare between the quantities $f((1-\nu)a+\nu b)$ and $(1-\nu)f(a)+\nu f(b)$ for $a,b \in I$ and $0 \leqslant \nu \leqslant 1$.

In [24], it is shown that if $f : [a,b] \to \mathbb{R}$ is a twice differentiable function such that there exist real constants m and M so that $m \leqslant f'' \leqslant M$, then

$$m\frac{\nu(1-\nu)}{2}(a-b)^2 \leqslant (1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b) \leqslant M\frac{\nu(1-\nu)}{2}(a-b)^2 \tag{1.1}$$

for all $\nu \in [0,1]$.

Notice that if $m \geqslant 0$ in the above inequality, we reach the weaker inequality

$$f((1-\nu)a+tb) \leqslant (1-\nu)f(a)+\nu f(b). \tag{1.2}$$

A function $f : I \to \mathbb{R}$ that satisfies (1.2) for all $a,b \in I$ and $0 \leqslant \nu \leqslant 1$ is called a convex function.

Noting that if $a < x < b$, one has the simple identities

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \quad \text{and} \quad a+b-x = \frac{x-a}{b-a}a + \frac{b-x}{b-a}b$$

Mathematics subject classification (2020): Primary 26B25, 26D15; Secondary 26D10.
Keywords and phrases: Hermite-Hadamard inequality, Jensen inequality, convex function.

and using (1.2), one can easily show that a convex function $f:[a,b]\to\mathbb{R}$ satisfies

$$f(x)+f(a+b-x)\leqslant f(a)+f(b),\ a\leqslant x\leqslant b. \tag{1.3}$$

Extending this latter inequality, the Jensen-Mercer inequality states that [21]

$$f\left(a+b-\sum_{i=1}^{n}w_i x_i\right)\leqslant f(a)+f(b)-\sum_{i=1}^{n}w_i f(x_i) \tag{1.4}$$

for all $x_1,\ldots,x_n\in[a,b]$, where f is a convex function on $[a,b]$ and $w_i\geqslant 0$ with $\sum_{i=1}^{n}w_i=1$. In fact, this is equivalent to the celebrated Jensen inequality, which extends (1.2) to the form

$$f\left(\sum_{i=1}^{n}w_i x_i\right)\leqslant\sum_{i=1}^{n}w_i f(x_i),$$

for the same parameters.

We notice that the definition of convexity given in (1.2) and the consequent inequalities of Mercer and Jensen do not assume differentiability of the function. Adding the differentiability assumption allows for obtaining other interesting relations. For example, if $f:I\to\mathbb{R}$ is a differentiable convex function, then we have the well-known gradient inequality

$$f(a)+(b-a)f'(a)\leqslant f(b), \tag{1.5}$$

where $a,b\in I$. We should remark here differentiability assumption on convex functions is not a harsh assumption. This is due to the fact that differentiable convex functions are dense in the class of convex functions, as one can check in [1].

Convex functions have been crucial in many fields, such as convex optimization, mathematical inequalities, operator theory, functional analysis, mathematical physics, and applied mathematics. We refer the reader to [14, 26, 33, 34, 35] as a sample of the research that employed convexity in such fields.

One of the most impressive inequalities for convex functions is the Hermite-Hadamard inequality (H-H inequality). It provides a necessary and sufficient condition for a function to be convex. This superior result of Hermite and Hadamard reads as follows:

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leqslant\frac{f(a)+f(b)}{2}$$

or

$$f\left(\frac{a+b}{2}\right)\leqslant\int_{0}^{1}f((1-t)a+tb)\,dt\leqslant\frac{f(a)+f(b)}{2},$$

where $f:[a,b]\to\mathbb{R}$ is convex.

In 1906, L. Fejér proved the following integral inequalities which are known in the literature as Fejér inequality:

$$f\left(\frac{a+b}{2}\right)\int_{0}^{1}p(u(t))\,dt\leqslant\int_{0}^{1}p(u(t))f(u(t))\,dt\leqslant\frac{f(a)+f(b)}{2}\int_{0}^{1}p(u(t))\,dt,$$

where $f:[a,b]\to\mathbb{R}$ is convex, $u(t)=(1-t)a+tb$ and $p:[a,b]\to(0,\infty)$ is Riemann integrable function and symmetric about $\frac{a+b}{2}$ (i.e. $p(a+b-x)=p(x)$), (see e.g. [24]). It is clear that this inequality is a generalization of the Hermite-Hadamard inequality.

In recent years, a considerable amount of research has been devoted to studying integral inequalities, such as the Hermite-Hadamard inequality, for convex functions. We refer the reader to [2, 5, 9, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 29, 30, 32, 36, 38, 39, 40] for such references.

In this paper, we present several new inequalities for convex functions. This includes gradient inequalities and Hermite-Hadamard-type inequalities. Refinements of the existing results will be presented, and some applications will be introduced.

Among many other results, we prove

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) \\
& \leqslant \int_0^1 f\left(\frac{1}{2}\left((1-\nu)a+\nu b+\frac{a+b}{2}\right)\right) d\nu \\
& \leqslant \int_0^1 \left(\int_0^1 f\left((1-t)\,\frac{a+b}{2}+t\left((1-\nu)\,a+\nu b\right)\right) dt\right) d\nu \\
& \leqslant \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\int_0^1 f\left((1-\nu)\,a+\nu b\right) d\nu\right) \\
& \leqslant \int_0^1 f\left((1-\nu)\,a+\nu b\right) d\nu \\
& \leqslant \int_0^1 f\left((1-\nu)\,f\left(\frac{(2-\nu)a+\nu b}{2}\right)+\nu f\left(\frac{(1-\nu)\,a+(1+\nu)\,b}{2}\right)\right) d\nu \\
& \leqslant \int_0^1 \left((1-\nu)\int_0^1 f\left((1-t)\left((1-\nu)\,a+\nu b\right)+ta\right) dt+\nu\int_0^1 f\left((1-t)\left((1-\nu)\,a+\nu b\right)+tb\right) dt\right) d\nu \\
& \leqslant \frac{1}{2}\left(\int_0^1 f\left((1-\nu)\,a+\nu b\right) d\nu+\frac{f(a)+f(b)}{2}\right) \\
& \leqslant \frac{f(a)+f(b)}{2},
\end{aligned}
$$

as a delicate refinement of the H-H inequality. Furthermore, we prove that

$$
f'(a)\,(b-a)+f(a)\leqslant 2f\left(\frac{a+b}{2}\right)-f(a)\leqslant f(b)
$$

as a refinement of (1.5). Another version that involves bounds of the second derivative, if exists, will be shown by the form

$$\frac{m}{2}(a-b)^2 \leqslant f(b)-f(a)-(b-a)f'(a) \leqslant \frac{M}{2}(a-b)^2,$$

where $m \leqslant f'' \leqslant M$. After that, operator applications that involve the operator relative entropy, the arithmetic and geometric means will be proved by the form

$$A\nabla B - \frac{1}{4}(A\sharp B)A^{-1}S(A|B)A^{-1}(A\sharp B) + \frac{1}{4}S(A|B) \leqslant \int_0^1 A\sharp_p B dp.$$

Many other applications will be presented as well.

2. Auxiliary results for convex functions

In (1.3), we have an upper bound of the quantity $f(x)+f(a+b-x)$. In the following lemma, we employ (1.5) to present lower and upper bounds of $f((1-\nu)a+\nu b)$ for differentiable convex functions in a way that allows obtaining a lower bound of $f(x)+f(a+b-x)$.

LEMMA 2.1. *Let $f: J \to \mathbb{R}$ be a differentiable convex function and let $a,b \in J$. Then*

$$f(a)+\nu(b-a)f'(a) \leqslant f((1-\nu)a+\nu b) \leqslant f(a)+\nu(b-a)f'((1-\nu)a+\nu b) \quad (2.1)$$

for all $\nu \in [0,1]$.

Proof. Using (1.5), we deduce the double inequality

$$f(a)+(b-a)f'(a) \leqslant f(b) \leqslant f(a)+(b-a)f'(b). \quad (2.2)$$

If, in relation (2.2), we replace b by $(1-\nu)a+\nu b$, we obtain the desired result. □

REMARK 2.1. If we take $\nu = \frac{1}{2}$ in relation (2.1), then we prove the following inequality:

$$f(a)+\left(\frac{b-a}{2}\right)f'(a) \leqslant f\left(\frac{a+b}{2}\right) \leqslant f(a)+\left(\frac{b-a}{2}\right)f'\left(\frac{a+b}{2}\right). \quad (2.3)$$

The first inequality of (2.3), together with (1.3), implies

$$2f(a)+(b-a)f'(a) \leqslant 2f\left(\frac{a+b}{2}\right) \leqslant f(x)+f(a+b-x) \leqslant f(a)+f(b),$$

where $a \leqslant x \leqslant b$.

Now we have the following reverse of the Jensen-Mercer inequality (1.4).

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function. Then*

$$f(a)+\left(b-\sum_{i=1}^{n}w_ix_i\right)f'(a)\leqslant f\left(a+b-\sum_{i=1}^{n}w_ix_i\right)\leqslant f(a)+f(b)-\sum_{i=1}^{n}w_if(x_i)$$

for all $x_i\in[a,b]$ and $w_i\geqslant 0$, with $\sum_{i=1}^{n}w_i=1$.

Proof. Using inequality (1.5) when b is replaced by $a+b-\sum_{i=1}^{n}w_ix_i$ and from relation (1.4), we obtain the inequality from the statement. □

The following result has been demonstrated in [27, Remark 2.3], but we present its proof for the reader's convenience. This result provides a considerable refinement of (1.5).

LEMMA 2.2. *Let $f:J\to\mathbb{R}$ be a differentiable convex function and let $a,b\in J$. Then*

$$f'(a)(b-a)+f(a)\leqslant 2f\left(\frac{a+b}{2}\right)-f(a)\leqslant f(b).$$

Proof. We will present two proofs.

(I) It is well known that that if $f:J\to\mathbb{R}$ is a convex function, then for any $a,b\in J$, one has [7]

$$\begin{aligned}f((1-v)a+vb)&\leqslant(1-v)f(a)+vf(b)-2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\\&\leqslant(1-v)f(a)+vf(b)\end{aligned}\tag{2.4}$$

where $r=\min\{v,1-v\}$ and $0<v<1$. From (2.4), we have

$$\begin{aligned}\frac{f(a+v(b-a))-f(a)}{v}&\leqslant f(b)-f(a)-\frac{2r}{v}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\\&\leqslant f(b)-f(a).\end{aligned}\tag{2.5}$$

Now, assume that f is differentiable. If $v\to 0^+$ in (2.5), we obtain

$$f'(a)(b-a)\leqslant -2f(a)+2f\left(\frac{a+b}{2}\right)\leqslant f(b)-f(a),$$

which proves the desired result.

(II) In (1.5), replace b by $\frac{a+b}{2}$ to get

$$f(a)+\left(\frac{b-a}{2}\right)f'(a)\leqslant f\left(\frac{a+b}{2}\right),$$

which means that

$$f'(a)(b-a)+f(a)\leqslant 2f\left(\frac{a+b}{2}\right)-f(a).$$

Since we have $f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}$, we deduce $2f\left(\frac{a+b}{2}\right) - f(a) \leqslant f(b)$. This completes the proof. □

PROPOSITION 2.1. *Let $f : J \to \mathbb{R}$ be a differentiable convex function and let $a,b \in J$. Then, for $a \leqslant x \leqslant b$,*

$$\begin{aligned} f'(a)\left(\frac{b-a}{2}\right) + f(a) &\leqslant f\left(\frac{a+x}{2}\right) + f\left(a+\frac{b-x}{2}\right) - f(a) \\ &\leqslant \frac{1}{2}\left(f(x) + f(a+b-x)\right) \leqslant \frac{1}{2}\left(f(a)+f(b)\right). \end{aligned}$$

Proof. From Lemma 2.2, replacing b with x and $a+b-x$, we obtain

$$f'(a)(x-a) + f(a) \leqslant 2f\left(\frac{a+x}{2}\right) - f(a) \leqslant f(x)$$

and

$$f'(a)(b-x) + f(a) \leqslant 2f\left(a+\frac{b-x}{2}\right) - f(a) \leqslant f(a+b-x).$$

By adding the two relations above, we get

$$\begin{aligned} f'(a)\left(\frac{b-a}{2}\right) + f(a) &\leqslant f\left(\frac{a+x}{2}\right) + f\left(a+\frac{b-x}{2}\right) - f(a) \\ &\leqslant \frac{1}{2}\left(f(x) + f(a+b-x)\right). \end{aligned}$$

Now from (1.3), we deduce the result. □

REMARK 2.2. If we take $x = \frac{a+b}{2}$ in Proposition 2.1, we get

$$f'(a)\left(\frac{b-a}{2}\right) + f(a) \leqslant 2f\left(\frac{3a+b}{2}\right) - f(a) \leqslant f\left(\frac{a+b}{2}\right).$$

We have the following ordering for convex functions using Lemma 2.2. This provides a refinement of (1.2).

PROPOSITION 2.2. *Let $f : J \to \mathbb{R}$ be a convex function and let $a,b \in J$. Then the following inequality holds:*

$$\begin{aligned} & f((1-v)a+vb) \\ & \leqslant (1-v)f\left(\frac{(2-v)a+vb}{2}\right) + vf\left(\frac{(1-v)a+(1+v)b}{2}\right) \\ & \leqslant \frac{1}{2}\left(f((1-v)a+vb) + (1-v)f(a) + vf(b)\right), \end{aligned}$$

for every $0 \leqslant v \leqslant 1$.

Proof. From the convexity of f, we obtain the first inequality in the following:

$$\begin{aligned} f((1-\nu)a+\nu b) &= f\left((1-\nu)\left(\frac{(1-\nu)a+\nu b}{2}\right)+\nu\left(\frac{(1-\nu)a+(1+\nu)b}{2}\right)\right) \\ &\leqslant (1-\nu)f\left(\frac{(2-\nu)a+\nu b}{2}\right)+\nu f\left(\frac{(1-\nu)a+(1+\nu)b}{2}\right). \end{aligned}$$

If we replace a and b by $(1-\nu)a+\nu b$ and a, respectively, in the second inequality of Lemma 2.2, we get

$$2f\left(\frac{(2-\nu)a+\nu b}{2}\right)-f((1-\nu)a+\nu b)\leqslant f(a) \tag{2.6}$$

Again, if we replace a by $(1-\nu)a+\nu b$, in the second inequality of Lemma 2.2, we get

$$2f\left(\frac{(1-\nu)a+(1+\nu)b}{2}\right)-f((1-\nu)a+\nu b)\leqslant f(b). \tag{2.7}$$

We deduce the second inequality by multiplying $1-\nu$ and ν to (2.6) and (2.7), respectively, and adding two inequalities. $\square$

By weakening the condition imposed on f, i.e., f to be twice differentiable and $m\leqslant f''\leqslant M$, we improve (1.5), as follows.

THEOREM 2.2. *If $f:[a,b]\to\mathbb{R}$ is a twice differentiable function such that there exist real constants m and M so that $m\leqslant f''\leqslant M$, then*

$$\frac{m}{2}(a-b)^2\leqslant f(b)-f(a)-(b-a)f'(a)\leqslant\frac{M}{2}(a-b)^2.$$

Proof. Applying relation (1.1), we deduce

$$\begin{aligned} m\frac{\nu(1-\nu)}{2}(a-b)^2+f((1-\nu)a+tb) &\leqslant (1-\nu)f(a)+\nu f(b) \\ &\leqslant M\frac{\nu(1-\nu)}{2}(a-b)^2+f((1-\nu)a+\nu b) \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{m(1-\nu)}{2}(a-b)^2+\frac{f((1-\nu)a+\nu b)-f(a)}{\nu} \\ &\leqslant f(b)-f(a) \\ &\leqslant \frac{M(1-\nu)}{2}(a-b)^2+\frac{f((1-\nu)a+\nu b)-f(a)}{\nu}. \end{aligned}$$

Consequently, we get the statement's relation if we take $\nu\to 0^+$ and add $f(a)$. $\square$

REMARK 2.3. If we replace b by $\frac{a+b}{2}$ in Theorem 2.2, we obtain the following inequality:

$$\begin{aligned}
&\frac{m(a-b)^2}{4}+(b-a)f'(a)+f(a)\\
&\leqslant 2f\left(\frac{a+b}{2}\right)-f(a)\\
&\leqslant \frac{M(a-b)^2}{4}+(b-a)f'(a)+f(a).
\end{aligned}$$

This represents an improvement in the first part of relation from Lemma 2.2 when f is a twice differentiable function such that there exist real constants m and M so that $m \leqslant f'' \leqslant M$.

3. Hermite-Hadamard type inequalities

The main goal of this section is to prove a delicate refinement of the H-H inequality. However, other related results will be shown too.

The following lemma presents the integral's upper and lower bounds in the H-H inequality. These bounds involve derivatives.

Although the following lemma is weaker than the original H-H inequality, we present it and use it in Theorem 3.2 to obtain an interesting reverse of the second inequality in the H-H inequality.

LEMMA 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function. Then*

$$f'(a)\left(\frac{b-a}{2}\right)+f(a)\leqslant\int_0^1 f((1-t)a+tb)\,dt\leqslant f(b)-f'(a)\left(\frac{b-a}{2}\right).$$

Proof. It has been shown in [28, Theorem 2.1] that

$$\begin{aligned}
f((1-\nu)a+\nu b) &\leqslant (1-\nu)\int_0^1 f(t\nu(b-a)+a)\,dt\\
&\qquad +\nu\int_0^1 f(t(1-\nu)(b-a)+(1-\nu)a+\nu b)\,dt\\
&\leqslant (1-\nu)f(a)+\nu f(b),
\end{aligned}$$

for any $0<\nu<1$. This can be written as

$$\begin{aligned}&\frac{f(a+\nu(b-a))-f(a)}{\nu}\\&\leqslant\frac{\int_0^1 f(t\nu(b-a)+a)\,dt-f(a)}{\nu}+\int\limits_0^1 f(t(1-\nu)(b-a)+(1-\nu)a+\nu b)\,dt\\&\quad-\int\limits_0^1 f(t\nu(b-a)+a)\,dt\leqslant f(b)-f(a).\end{aligned}$$

If we take $\nu\to 0^+$, we get

$$\begin{aligned}&f'(a)(b-a)\\&=\lim_{\nu\to 0}\left\{\frac{f(a+\nu(b-a))-f(a)}{\nu}\right\}\\&\leqslant\lim_{\nu\to 0}\left\{\frac{\int_0^1 f(t\nu(b-a)+a)\,dt-f(a)}{\nu}+\int\limits_0^1 f(t(1-\nu)(b-a)+(1-\nu)a+\nu b)\,dt\right.\\&\quad\left.-\int\limits_0^1 f(t\nu(b-a)+a)\,dt\right\}\\&=\int\limits_0^1 t(b-a)f'(a)\,dt+\int\limits_0^1 f(t(b-a)+a)\,dt-f(a)\\&=\left(\frac{b-a}{2}\right)f'(a)+\int\limits_0^1 f(t(b-a)+a)\,dt-f(a)\leqslant f(b)-f(a),\end{aligned}$$

i.e.,

$$f'(a)(b-a)+f(a)\leqslant f'(a)\left(\frac{b-a}{2}\right)+\int\limits_0^1 f((1-t)a+tb)\,dt\leqslant f(b),$$

as expected. □

Although the following result is weaker than the original H-H inequality, it will help obtain multiple refinements of this inequality.

PROPOSITION 3.1. *Let $f:J\to\mathbb{R}$ be a differentiable convex function and let $a,b\in J$. Then*

$$\begin{aligned}f'(a)\left(\frac{b-a}{2}\right)+f(a)\leqslant f\left(\frac{a+b}{2}\right)&\leqslant\int\limits_0^1 f((1-t)a+tb)\,dt\\&\leqslant\frac{f(a)+f(b)}{2}\leqslant f(b)-f'(a)\left(\frac{b-a}{2}\right).\end{aligned}$$

Proof. The first, second, and third inequalities come from Lemma 3.1 and 2.2. The last inequality is equivalent to $f'(a)(b-a)+f(a)\leqslant f(b)$. $\square$

In the following remark, we utilize Proposition 3.1 to present a refinement and a reverse for the Jensen inequality.

REMARK 3.1. By choosing $b=\sum_{i=1}^{n} w_i x_i$, in Proposition 3.1, we may write

$$\begin{aligned}
&\left(\frac{\left(\sum_{i=1}^{n} w_i x_i\right) f'(a)-af'(a)}{2}\right)+f(a)\\
&\leqslant f\left(\frac{a+\sum_{i=1}^{n} w_i x_i}{2}\right)\\
&\leqslant \int_0^1 f\left((1-t)a+t\sum_{i=1}^{n} w_i x_i\right)dt\\
&\leqslant \frac{1}{2}\left(f(a)+f\left(\sum_{i=1}^{n} w_i x_i\right)\right)\\
&\leqslant f\left(\sum_{i=1}^{n} w_i x_i\right)-\left(\frac{\left(\sum_{i=1}^{n} w_i x_i\right) f'(a)-f'(a)a}{2}\right).
\end{aligned}$$

Substituting a by x_i $(i=1,2,\ldots,n)$ and then multiplying by $w_i\geqslant 0$ $(i=1,2,\ldots,n)$ and summing over i from 1 to n we may deduce

$$\begin{aligned}
&\left(\frac{\left(\sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} w_i f'(x_i)\right)-\sum_{i=1}^{n} w_i x_i f'(x_i)}{2}\right)+\sum_{i=1}^{n} w_i f(x_i)\\
&\leqslant \sum_{i=1}^{n} w_i f\left(\frac{x_i+\sum_{j=1}^{n} w_j x_j}{2}\right)\\
&\leqslant \sum_{i=1}^{n} w_i\left(\int_0^1 f\left((1-t)x_i+t\sum_{j=1}^{n} w_j x_j\right)dt\right)\\
&\leqslant \frac{1}{2}\left(\sum_{i=1}^{n} w_i f(x_i)+f\left(\sum_{i=1}^{n} w_i x_i\right)\right)\\
&\leqslant f\left(\sum_{i=1}^{n} w_i x_i\right)-\left(\frac{\left(\sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} w_i f'(x_i)\right)-\sum_{i=1}^{n} w_i f'(x_i)x_i}{2}\right),
\end{aligned}$$

which provides a reverse for the Jensen inequality. Notice that this result improves an inequality in [8].

Now we present the promised refinement of H-H inequality.

THEOREM 3.1. *Let $f : J \to \mathbb{R}$ be a convex function and let $a, b \in J$. Then*

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \\
& \leqslant \int_0^1 f\left(\frac{1}{2}\left((1-v)a+vb+\frac{a+b}{2}\right)\right) dv \\
& \leqslant \int_0^1 \left(\int_0^1 f\left((1-t)\frac{a+b}{2}+t\left((1-v)a+vb\right)\right) dt\right) dv \\
& \leqslant \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\int_0^1 f\left((1-v)a+vb\right) dv\right) \\
& \leqslant \int_0^1 f\left((1-v)a+vb\right) dv \\
& \leqslant \int_0^1 f\left((1-v)f\left(\frac{(2-v)a+vb}{2}\right)+vf\left(\frac{(1-v)a+(1+v)b}{2}\right)\right) dv \\
& \leqslant \int_0^1 \left((1-v)\int_0^1 f\left((1-t)\left((1-v)a+vb\right)+ta\right) dt+v\int_0^1 f\left((1-t)\left((1-v)a+vb\right)+tb\right) dt\right) dv \\
& \leqslant \frac{1}{2}\left(\int_0^1 f\left((1-v)a+vb\right) dv+\frac{f(a)+f(b)}{2}\right) \\
& \leqslant \frac{f(a)+f(b)}{2}.
\end{aligned}$$

Proof. We prove the inequality for differentiable convex functions. Then, our conclusion follows for any convex function since it is a uniform limit of smooth convex functions [1, Theorem 1]. Replacing a and b by $\frac{a+b}{2}$ and $(1-v)a+vb$, in Proposition 3.1, we get

$$\begin{aligned}
& \frac{1}{2}f'\left(\frac{a+b}{2}\right)\left((1-v)a+vb-\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right) \\
& \leqslant f\left(\frac{1}{2}\left((1-v)a+vb+\frac{a+b}{2}\right)\right) \\
& \leqslant \int_0^1 f\left((1-t)\frac{a+b}{2}+t\left((1-v)a+vb\right)\right) dt \\
& \leqslant \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+f\left((1-v)a+vb\right)\right) \\
& \leqslant f\left((1-v)a+vb\right)-\frac{1}{2}f'\left(\frac{a+b}{2}\right)\left((1-v)a+vb-\frac{a+b}{2}\right).
\end{aligned}$$

Now, taking integral over $0 \leqslant v \leqslant 1$, we get

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leqslant \int_0^1 f\left(\frac{1}{2}\left((1-v)a+vb+\frac{a+b}{2}\right)\right)dv \\
&\leqslant \int_0^1\left(\int_0^1 f\left((1-t)\frac{a+b}{2}+t\left((1-v)a+vb\right)\right)dt\right)dv \\
&\leqslant \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\int_0^1 f\left((1-v)a+vb\right)dv\right) \\
&\leqslant \int_0^1 f\left((1-v)a+vb\right)dv.
\end{aligned}$$

Thus the first, second, third, and fourth inequalities are proven. Replacing b and a by a and $(1-v)a+vb$, in Proposition 3.1, we obtain

$$\begin{aligned}
&f'\left((1-v)a+vb\right)\left(\frac{a-\left((1-v)a+vb\right)}{2}\right)+f\left((1-v)a+vb\right) \\
&\leqslant f\left(\frac{(1-v)a+vb+a}{2}\right) \\
&\leqslant \int_0^1 f\left((1-t)\left((1-v)a+vb\right)+ta\right)dt \\
&\leqslant \frac{1}{2}\left(f\left((1-v)a+vb\right)+f(a)\right) \\
&\leqslant f(a)-f'\left((1-v)a+vb\right)\left(\frac{a-\left((1-v)a+vb\right)}{2}\right).
\end{aligned}$$

Multiplying the above inequality by $1-v$, we have

$$\begin{aligned}
&f'\left((1-v)a+vb\right)\left(\frac{(1-v)a-(1-v)\left((1-v)a+vb\right)}{2}\right)+(1-v)f\left((1-v)a+vb\right) \\
&\leqslant (1-v)f\left(\frac{(1-v)a+vb+a}{2}\right) \\
&\leqslant (1-v)\int_0^1 f\left((1-t)\left((1-v)a+vb\right)+ta\right)dt \\
&\leqslant \frac{1-v}{2}\left(f\left((1-v)a+vb\right)+f(a)\right) \\
&\leqslant (1-v)f(a)-f'\left((1-v)a+vb\right)\left(\frac{(1-v)a-(1-v)\left((1-v)a+vb\right)}{2}\right).
\end{aligned} \tag{3.1}$$

Replacing a by $(1-\nu)a+\nu b$, in Proposition 3.1, we get

$$\begin{aligned}
&f'((1-\nu)a+\nu b)\left(\frac{b-((1-\nu)a+\nu b)}{2}\right)+f((1-\nu)a+\nu b)\\
&\leqslant f\left(\frac{(1-\nu)a+\nu b+b}{2}\right)\\
&\leqslant \int_0^1 f((1-t)((1-\nu)a+\nu b)+tb)\,dt\\
&\leqslant \frac{1}{2}\left(f((1-\nu)a+\nu b)+f(b)\right)\\
&\leqslant f(b)-f'((1-\nu)a+\nu b)\left(\frac{b-((1-\nu)a+\nu b)}{2}\right).
\end{aligned}$$

Multiplying the above inequality by ν, we obtain

$$\begin{aligned}
&f'((1-\nu)a+\nu b)\left(\frac{\nu b-\nu((1-\nu)a+\nu b)}{2}\right)+\nu f((1-\nu)a+\nu b)\\
&\leqslant \nu f\left(\frac{(1-\nu)a+\nu b+b}{2}\right)\\
&\leqslant \nu\int_0^1 f((1-t)((1-\nu)a+\nu b)+tb)\,dt\\
&\leqslant \frac{\nu}{2}\left(f((1-\nu)a+\nu b)+f(b)\right)\\
&\leqslant \nu f(b)-f'((1-\nu)a+\nu b)\left(\frac{\nu y-\nu((1-\nu)a+\nu b)}{2}\right).
\end{aligned}\tag{3.2}$$

Adding (3.1) and (3.2), we infer that

$$\begin{aligned}
&f((1-\nu)a+\nu b)\\
&\leqslant (1-\nu)f\left(a+\nu\frac{b-a}{2}\right)+\nu f\left(\frac{(1-\nu)a+(1+\nu)b}{2}\right)\\
&\leqslant (1-\nu)\int_0^1 f((1-t)((1-\nu)a+\nu b)+ta)\,dt+\nu\int_0^1 f((1-t)((1-\nu)a+\nu b)+tb)\,dt\\
&\leqslant \frac{1}{2}\left(f((1-\nu)a+\nu b)+(1-\nu)f(a)+\nu f(b)\right)\\
&\leqslant (1-\nu)f(a)+\nu f(b).
\end{aligned}$$

Now, taking integral over $0 \leqslant v \leqslant 1$, we get

$$
\begin{aligned}
&\int_0^1 f\left((1-v)a+vb\right)dv \\
&\leqslant \int_0^1 \left((1-v)f\left(a+v\frac{b-a}{2}\right)+vf\left(\frac{(1-v)a+(1+v)b}{2}\right)\right)dv \\
&\leqslant \int_0^1 \left((1-v)\int_0^1 f\left((1-t)\left((1-v)a+vb\right)+ta\right)dt+v\int_0^1 f\left((1-t)\left((1-v)a+vb\right)+tb\right)dt\right)dv \\
&\leqslant \frac{1}{2}\left(\int_0^1 f\left((1-v)a+vb\right)dv+\frac{f(a)+f(b)}{2}\right) \\
&\leqslant \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

Thus the fifth, sixth, seventh, and eighth inequalities are proven. This completes the proof. □

In the following result, we present a reverse for the second inequality in H-H inequality.

THEOREM 3.2. *Let $f : J \to \mathbb{R}$ be a differentiable convex function and let $a, b \in J$. Then*

$$\frac{f(a)+f(b)}{2}-\left(\frac{a-b}{4}\right)\left(f'(a)-f'(b)\right) \leqslant \int_0^1 f\left((1-v)a+vb\right)dv.$$

Proof. It follows from Lemma 3.1 that

$$f'(a)\left(\frac{b-a}{2}\right)+f(a) \leqslant \int_0^1 f\left((1-v)a+vb\right)dv. \tag{3.3}$$

From the above inequality, we also have

$$f'(b)\left(\frac{a-b}{2}\right)+f(b) \leqslant \int_0^1 f\left((1-v)b+va\right)dv. \tag{3.4}$$

Now adding (3.3) and (3.4) and using the fact that

$$\int_0^1 f\left((1-v)b+va\right)dv = \int_0^1 f\left((1-v)a+vb\right)dv,$$

we reach

$$\frac{f(a)+f(b)}{2}+\left(\frac{a-b}{4}\right)\left(f'(b)-f'(a)\right)\leqslant\int_0^1 f((1-\nu)a+\nu b)d\nu,$$

as desired. □

4. Applications

We give some applications of inequalities for several means in this section. To state the following results, we use the standard notations on symmetric homogeneous means for $a,b>0$. The arithmetic mean, geometric mean, logarithmic mean, and identric mean are defined by $A(a,b):=\frac{a+b}{2}$, $G(a,b):=\sqrt{ab}$, $L(a,b):=\frac{b-a}{\log b-\log a}$, $(a\neq b)$, with, $L(a,a):=a$ and $I(a,b):=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, $(a\neq b)$, with $I(a,a):=a$. It is known that the ordering among these means is

$$G(a,b)\leqslant L(a,b)\leqslant I(a,b)\leqslant A(a,b).$$

We also use the notations for the weighted geometric mean, logarithmic mean, identric mean, and arithmetic mean as

$$\begin{aligned}&G_\nu(a,b):=a^{1-\nu}b^\nu\\&L_\nu(a,b):=\frac{1}{\log a-\log b}\left\{\frac{1-\nu}{\nu}\left(a-a^{1-\nu}b^\nu\right)+\frac{\nu}{1-\nu}\left(a^{1-\nu}b^\nu-b\right)\right\}\\&I_\nu(a,b):=\frac{1}{e}\left(a\nabla_\nu b\right)^{\frac{(1-2\nu)(a\nabla_\nu b)}{\nu(1-\nu)(b-a)}}\left(\frac{b^{\frac{\nu b}{1-\nu}}}{a^{\frac{(1-\nu)a}{\nu}}}\right)^{\frac{1}{b-a}},\quad A_\nu(a,b):=(1-\nu)a+\nu b\end{aligned}$$

for $\nu\in(0,1)$, respectively.

EXAMPLE 4.1. Let $a,b>0$. Taking $f(x):=e^x$ in Theorem 3.1, we have

$$\begin{aligned}e^{\frac{a+b}{2}}&\leqslant\frac{2\left(e^{\frac{a+3b}{4}}-e^{\frac{3a-b}{4}}\right)}{b-a}\\&\leqslant\int_0^1 L\left(e^{A(a,b)},e^{A_\nu(a,b)}\right)d\nu\\&\leqslant\frac{1}{2}\left(e^{A(a,b)}+L(e^a,e^b)\right)\\&\leqslant L(e^a,e^b)\\&\leqslant\int_0^1\exp\left[A_\nu\left(e^{A_{\frac{\nu}{2}}(a,b)},e^{A_{\frac{1+\nu}{2}}(a,b)}\right)\right]d\nu\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^1 A_\nu\left(L\left(e^{A_\nu(a,b)}, e^a\right), L\left(e^{A_\nu(a,b)}, e^b\right)\right) d\nu \\
&\leqslant \frac{1}{2}\left(L(e^a,e^b) + \frac{e^a+e^b}{2}\right) \\
&\leqslant \frac{e^a+e^b}{2}.
\end{aligned}
$$

Replacing e^a and e^b by a and b in the above inequalities, we have the following inequalities by simple calculations.

$$
\begin{aligned}
G(a,b) &\leqslant L(a^{\frac{1}{4}}b^{\frac{3}{4}}, a^{\frac{3}{4}}b^{\frac{1}{4}}) \\
&\leqslant \int_0^1 L(G(a,b), G_\nu(a,b))\, d\nu \\
&\leqslant \frac{G(a,b)+L(a,b)}{2} \\
&\leqslant L(a,b) \\
&\leqslant \int_0^1 \exp\left\{A_\nu\left(G_{\frac{\nu}{2}(a,b)}, G_{\frac{1+\nu}{2}}(a,b)\right)\right\} d\nu \\
&\leqslant \int_0^1 L_\nu(a,b) d\nu \\
&\leqslant \frac{A(a,b)+L(a,b)}{2} \\
&\leqslant A(a,b).
\end{aligned}
\tag{4.1}
$$

In addition, if we take $f(x) := -\log x$ for $x > 0$ in Theorem 3.1, then we have

$$
\begin{aligned}
\log\sqrt{ab} &\leqslant \frac{1}{2}\left(\log I(a,b) + \log\sqrt{ab}\right) \\
&\leqslant \int_0^1 \log I(A_\nu(a,b),a)^{1-\nu} I(A_\nu(a,b),b)^\nu d\nu \\
&\leqslant \int_0^1 \log A_\nu\left(\log\left(A_{\frac{\nu}{2}}(a,b)\right)^{-1}, \log\left(A_{\frac{1+\nu}{2}}(a,b)\right)^{-1}\right) d\nu \\
&\leqslant \log I(a,b) \\
&\leqslant \frac{1}{2}\left(\log\left(\frac{a+b}{2}\right) + \log I(a,b)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \int_0^1 \log I\left(\frac{a+b}{2}, A_v(a,b)\right) dv \\
&\leqslant \log \frac{1}{4e}\left(\frac{(a+3b)^{a+3b}}{(3a+b)^{3a+b}}\right)^{\frac{1}{2(b-a)}} \\
&\leqslant \log\left(\frac{a+b}{2}\right).
\end{aligned}
$$

Then we obtain the following inequalities by some simple calculations.

$$
\begin{aligned}
G(a,b) &\leqslant \sqrt{I(a,b)G(a,b)} \\
&\leqslant \exp\left(\int_0^1 \log I_v(a,b)dv\right) \\
&\leqslant \exp\left(\int_0^1 \log\left(\log G_{-v}\left(A_{\frac{v}{2}}, A_{\frac{1+v}{2}}\right)\right) dv\right) \\
&\leqslant I(a,b) \qquad\qquad (4.2)\\
&\leqslant \sqrt{A(a,b)I(a,b)} \\
&\leqslant \exp\left(\int_0^1 I\left(A(a,b), A_v(a,b)\right) dv\right) \\
&\leqslant I\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) \\
&\leqslant A(a,b).
\end{aligned}
$$

The inequalities (4.1) and (4.2) give refinements for $G(a,b) \leqslant L(a,b) \leqslant A(a,b)$ and $G(a,b) \leqslant I(a,b) \leqslant A(a,b)$, respectively.

EXAMPLE 4.2. If we take $f(x) := e^x$ and $f(x) := -\log x$, $(x > 0)$ in Proposition 2.2 with $G_v(a,b) \leqslant A_v(a,b)$, then we obtain the following inequalities for $a, b > 0$, respectively:

$$
G_v(a,b) \leqslant A_v\left(G_{\frac{v}{2}}(a,b), G_{\frac{1+v}{2}}(a,b)\right) \leqslant A\left(G_v(a,b), A_v(a,b)\right) \leqslant A_v(a,b)
$$

and

$$
G_v(a,b) \leqslant G\left(G_v(a,b), A_v(a,b)\right) \leqslant G_v\left(A_{\frac{v}{2}}(a,b), A_{\frac{1+v}{2}}(a,b)\right) \leqslant A_v(a,b).
$$

The inequalities for the nested means have been studied in [12, 31], and the results in Corollary 4.2 were wholly included in [31, Theorem 2.2].

LEMMA 4.1. *For $a,b > 0$, we have*

$$A(a,b) - \frac{1}{4}(a-b)(\log a - \log b) \leqslant L(a,b) \leqslant I(a,b) \leqslant e^{\frac{(a-b)^2}{4ab}} G(a,b). \tag{4.3}$$

Proof. Taking $f(x) := e^x$ and $f(x) := -\log x$, $(x > 0)$ in Theorem 3.2, we have $A(a,b) - \frac{1}{4}(a-b)(\log a - \log b) \leqslant L(a,b)$ and $I(a,b) \leqslant e^{\frac{(a-b)^2}{4ab}} G(a,b)$, respectively. Since $L(a,b) \leqslant I(a,b)$, we obtain the desired result. Indeed, it is not difficult to prove $\dfrac{x-1}{\log x} \leqslant \dfrac{1}{e} x^{\frac{x}{x-1}}$ for $x > 0$. See [12, Remark 2.6 (ii)] for example. □

The first and last inequality in (4.3) give reverses for $L(a,b) \leqslant A(a,b)$ and $G(a,b) \leqslant I(a,b)$, respectively.

We give operator inequalities as an application of the obtained result. To state an application of Lemma 4.1, we review the operator arithmetic mean, operator geometric mean, and the operator relative entropy [11, 13], which are defined respectively by

$$A \nabla_p B := (1-p)A + pB, \quad A \sharp_p B := A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^p A^{1/2}, \quad (0 \leqslant p \leqslant 1)$$

and

$$S(A|B) := A^{1/2}\left(\log A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

for two strictly positive operators A and B on a Hilbert space $\mathscr{H}$. For simplicity, we use ∇ and $\sharp$ instead of $\nabla_{1/2}$ and $\sharp_{1/2}$, respectively. Then we have the following theorem.

THEOREM 4.1. *Let $p \in [0,1]$ and let A and B be two strictly positive operators on a Hilbert space $\mathscr{H}$. Then*

$$A \nabla B - \frac{1}{4}(A \sharp B) A^{-1} S(A|B) A^{-1} (A \sharp B) + \frac{1}{4} S(A|B) \leqslant \int_0^1 A \sharp_p B \, dp, \tag{4.4}$$

and

$$\int_0^1 S(A|A \nabla_p B) \, dp \leqslant \frac{1}{2} S(A|B) + \frac{1}{4}(B - A) + \frac{1}{4}(AB^{-1}A - A). \tag{4.5}$$

Proof. From the first inequality in (4.3), we have

$$\frac{1+x}{2} - \frac{1}{4}\sqrt{x}(\log x)\sqrt{x} + \frac{1}{4}\log x \leqslant \int_0^1 x^p \, dp, \quad (x > 0).$$

By putting $x := A^{-1/2} B A^{-1/2}$ and multiplying $A^{1/2}$ from the both sides, we obtain (4.4).

Taking the logarithm of the last inequality in (4.3), we have

$$\int_0^1 \log\{(1-p) + xp\} \, dp \leqslant \frac{1}{2}\log x + \frac{1}{4}(x-1) + \frac{1}{4}\left(x^{-1} - 1\right), \quad (x > 0),$$

since we have

$$\log\left(\frac{x^{\frac{x}{x-1}}}{e}\right) = \int_0^1 \log\{(1-p)+xp\}\,dp.$$

Similarly to the above, we obtain (4.5). □

Note that $B \geqslant A \Longleftrightarrow AB^{-1}A \leqslant A$ in the left hand side of (4.5).

Finally, we give examples with some applications for Theorem 2.1.

EXAMPLE 4.3.

(i) Let $a,b>0$. Taking $f(x) := e^x$ in Theorem 2.1, we have

$$a\left(1+\log\frac{b}{G(w_i;x_i)}\right) \leqslant \frac{ab}{G(w_i;x_i)} \leqslant a+b-A(w_i;x_i),$$

where $A(w_i;x_i) := \sum_{i=1}^n w_i x_i$ and $G(w_i;x_i) := \prod_{i=1}^n x_i^{w_i}$. We obtain the equivalent inequalities above by taking $f(x) := -\log x$, $(x>0)$. The second inequality above gives the relation between $G(w_i;x_i)$ and $A(w_i;x_i)$, while the first inequality above is trivial by the inequality $\log t \leqslant t-1$, $(t>0)$.

(ii) Let $a<0$, $b=0$, $x_i := \log w_i$. Then the condition $a \leqslant x_i \leqslant b$ is satisfied. Taking $f(x) := e^x$ in Theorem 2.1, we have

$$H(w_1,\cdots,w_n) \leqslant e^{H(w_1,\cdots,w_n)} - 1 \leqslant e^{-a}\left(1-\sum_{i=1}^n w_i^2\right),$$

where $H(w_1,\cdots,w_n) := -\sum_{i=1}^n w_i \log w_i$ is Shannon entropy for a probability distribution $\{w_1,\cdots,w_n\}$. The first inequality above is trivial by the inequality $\log t \leqslant t-1$, $(t>0)$. From the second inequality above, we have an upper bound of the Shannon entropy:

$$H(w_1,\cdots,w_n) \leqslant \log\left\{1+e^{-a}\left(1-\sum_{i=1}^n w_i^2\right)\right\}. \tag{4.6}$$

It is known that $H(w_1,\cdots,w_n) \leqslant \log n$ with equality iff $w_i = \frac{1}{n}$ for all $i = 1,\cdots,n$. So we consider the special case such as $w_i = \frac{1}{n}$ for all $i = 1,\cdots,n$. Then the condition $e^a \leqslant \frac{1}{n} \leqslant e^b = 1$ implies $e^{-a} \geqslant n$. Thus we have $1+e^{-a}\left(1-\sum_{i=1}^n w_i^2\right) \geqslant 1+n(1-1/n) = n$ which implies $\log n \leqslant \log\left\{1+e^{-a}\left(1-\sum_{i=1}^n w_i^2\right)\right\}$. Therefore the upper bound in (4.6) is not better than the known bound $\log n$ for the special case such that $w_i = \frac{1}{n}$ for all $i = 1,\cdots,n$.

(iii) Let $0 < a \leqslant 1 \leqslant b$, $w_i > 0$, $v_i \geqslant 0$ and $a \leqslant \frac{v_i}{w_i} \leqslant b$ for $i = 1, \cdots, n$ with $\sum_{i=1}^n v_i = 1$. Taking $f(x) := -\log x$, $(x > 0)$ and $x_i := \frac{v_i}{w_i}$, we have

$$D(w_1, \cdots, w_n | v_1, \cdots, v_n) \leqslant \log\left(\frac{a+b-1}{ab}\right) \leqslant \frac{b-1}{a} - \log b,$$

where $D(w_1, \cdots, w_n | v_1, \cdots, v_n) := \sum_{i=1}^{n} w_i \log \frac{w_i}{v_i}$ is relative entropy (see [13, Chapter 7] for example) for two probability distributions $\{w_1, \cdots, w_n\}$ and $\{v_1, \cdots, v_n\}$. The case gives $D(w_1, \cdots, w_n | v_1, \cdots, v_n) = 0$ since $\{w_1, \cdots, w_n\} = \{v_1, \cdots, v_n\}$ from the condition $\frac{v_i}{w_i} \leqslant 1$ for $i = 1, \cdots, n$. The first inequality gives an upper bound of the relative entropy, while the second inequality above is trivial by the inequality $\log t \leqslant t - 1$, $(t > 0)$.

Declarations

Availability of data and materials. Not applicable.

Competing interests. The authors declare that they have no competing interests.

Funding. This research is supported by a grant (JSPS KAKENHI, Grant Number: 21K03341) awarded to the author, S. Furuichi.

Authors' contributions. Authors declare they have contributed equally to this paper. All authors have read and approved this version.

REFERENCES

[1] D. AZAGRA, *Global and fine approximation of convex functions*, Proc. Lond. Math. Soc. **107** (2013), 799–824.

[2] S.-P. BAI, F. QI, AND S.-H. WANG, *Some new integral inequalities of Hermite-Hadamard type for $(\alpha, m; p)$-convex functions on co-ordinates*, J. Appl. Anal. Comput. **6** (1) (2016), 171–178.

[3] S. BANIĆ, *Mappings Connected with Hermite-Hadamard Inequalities for Superquadratic Functions*, J. Math. Inequal. **3** (4) (2009), 577–589.

[4] S. BUTT, R. JAKŠIĆ, L. KVESIĆ, AND J. PEČARIĆ, *n-exponential convexity of weighted Hermite-Hadamard's inequality*, J. Math. Inequal. **8** (2) (2014), 299–311.

[5] P. CHEN, AND W.-S. CHEUNG, *Hermite-Hadamard inequality for semiconvex functions of rate (k_1, k_2) on the coordinates and optimal mass transportation*, Rocky mountain J. Math. **50** (6) (2020), 2011–2021.

[6] S. M. DAVARPANAH, H. R. MORADI, *A log-convex approach to Jensen-Mercer inequality*, J. Linear Topol. Algeb. **11** (3) (2022), 169–176.

[7] S. S. DRAGOMIR, *Bounds for the normalised Jensen functional*, Bull. Austral. Math. Soc. **74** (2006), 471–478.

[8] S. S. DRAGOMIR, N. M. IONESCU, *Some converse of Jensen's inequality and applications*, Anal. Num. Theor. Approx. **23** (1994), 71–78.

[9] Z. EKEN, S. KEMALI, G. TINAZTEPE, AND G. ADILOV, *The Hermite-Hadamard inequalities for p-convex functions*, Hacettepe J. Math. Stat. **50** (5) (2021), 1268–1279.

[10] A. FARISSI, *Simple proof and refinement of Hermite-Hadamard inequality*, J. Math. Inequal. **4** (3) (2010), 36–369.

[11] J. I. FUJII, E. KAMEI, *Relative operator entropy in non-commutative information theory*, Math. Japon. **34** (1989), 341–348.

[12] S. FURUICHI, M. E. AMLASHI, *More results on weighted means*, J. Math. Inequal. **16** (4) (2022), 1645–1659.

[13] S. FURUICHI, H. R. MORADI, *Advances in mathematical inequalities*, De Gruyter, 2020.

[14] S. FURUICHI, H. R. MORADI, AND A ZARDADI, *Some new Karamata type inequalities and their applications to some entropies*, Rep. Math. Phys. **84** (2) (2019), 201–214.

[15] W. HAQ AND D. KOTRYS, *On symmetrized stochastic convexity and the inequalities of Hermite–Hadamard type*, Aequ. Math. **95** (2021), 821–828.

[16] I. ISCAN, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe J. Math. Stat. **43** (6) (2014), 935–942.

[17] M. KADAKAL, I. ISCAN, P. AGRAWAL, AND M. JLELI, *Exponential trigonometric convex functions and Hermite-Hadamard type inequalities*, Math. Slovaca **71** (1) (2021), 43–56.

[18] M. LATIF, *Weighted Hermite-Hadamard type inequalities for differentiable GA–convex and geometrically quasiconvex mappings*, Rocky mountain J. Math. **51** (6) (2021), 1899–1908.

[19] M. LATIF, AND S. S. DRAGOMIR, *Generalization of Hermite-Hadamard type inequalities for n-times differentiable functions through preinvexity*, Georgian Math. J. **23** (1) (2016), 97–104.

[20] W. LIU, AND H. ZHUANG, *Some quantum estimates of Hermite–Hadamard inequalities for convex functions*, J. Appl. Anal. Comput. **7** (2) (2017), 201–522.

[21] A. MCD. MERCER, *A variant of Jensen's inequality*, J. Inequal. Pure Appl. Math. **4** (2) (2003), Article 73.

[22] B. MEFTAH, C. MARROUCHE, *Some new Hermite-Hadamard type inequalities for $n-$times log-convex functions*, Jordan J. Math. Stat. **14** (2021), 651–669.

[23] B. MEFTAH, M. MERAD, AND A. SOUAHI, *Some Hermite-Hadamard type inequalities for functions whose derivatives are quasi-convex*, Jordan J. Math. Stat. **12** (2019), 219–231.

[24] N. MINCULETE, F. C. MITROI, *Fejér type inequalities*, Aust. J. Math. Anal. Appl. **9** (1), (2012), Art. 12.

[25] H. R. MORADI, S. FURUICHI, *Improvement and generalization of some Jensen-Mercer-type inequalities*, J. Math. Inequal. **19** (2020), 377–383.

[26] H. R. MORADI, S. FURUICHI, AND M. SABABHEH, *Some operator inequalities via convexity*, Linear Multilinear Algebra **70** (22) (2022), 7740–7752.

[27] M. E. OMIDVAR, AND H. R. MORADI, *On inequalities of Bellman and Aczel type*, AIMS Mathematics **5** (4) (2020), 3357–3364.

[28] R. PAL, M. SINGH, M. S. MOSLEHIAN, AND J. S. AUJLA, *A new class of operator monotone functions via operator means*, Linear Multilinear Algebra **64** (12) (2016), 2463–2473.

[29] F. QI, Z.-L. WEI, AND Q. YANG, *Generalization and Refinements of Hermite-Hadamard's Inequality*, Rocky mountain J. Math. **35** (1) (2005), 235–251.

[30] M. RAÏSOULLI, *Refining the Hermite-Hadamard inequalities for operator convex maps in multiple operator variables*, Aequ. Math. **96** (2022), 803–814.

[31] M. RAÏSSOULI, S. FURUICHI, *On weighted means and their inequalities*, J. Inequal. Appl. **65** (2021), `https://doi.org/10.1186/s13660-021-02589-9`.

[32] J. ROOIN, A. ALIKHANI, AND M. SAL MOSLEHIAN, *Operator m-convex functions*, Georgian Math. J. **25** (1) (2018), 93–107.

[33] M. SABABHEH, S. FURUICHI, AND H. R. MORADI, *Composite convex functions*, J. Math. Inequal. **15** (3) (2021), 1267–1285.

[34] M. SABABHEH, H. R. MORADI, *Radical convex functions*, Mediterr. J. Math. **18**, 137 (2021), `https://doi.org/10.1007/s00009-021-01784-8`.

[35] M. SABABHEH, H. R. MORADI, AND S. FURUICHI, *Integrals refining convex inequalities*, Bull. Malays. Math. Sci. Soc. **43** (3) (2020), 2817–2833.

[36] M. SARIKAYA, AND H. BUDAK, *Generalized Hermite-Hadamard type integral inequalities for fractional integrals*, Filomat **30** (5) (2016), 1315–1326.

[37] T. TRIF, *Characterizations of convex functions of a vector variable via Hermite-Hadamard's inequality*, J. Math. Inequal. **2** (1) (2008), 37–44.

[38] K.-L. TSENG, S. R. HWANG, AND S. S. DRAGOMIR, *Fejér-type inequalities (II)*, Math. Slovaca **67** (1) (2017), 109–120.

[39] T. TUNÇ, *Hermite-Hadamard type inequalities for interval-valued fractional integrals with respect to another function*, Math. Slovaca **72** (6) (2022), 1501–1512.

[40] J. R. WANG, X. LI, AND Y. ZHOU, *Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals via s-convex functions and applications to special means*, Filomat **30** (5) (2016), 1143–1150.

(Received June 14, 2023)

Shigeru Furuichi
Department of Information Science
College of Humanities and Sciences, Nihon University
Tokyo, Japan
and
Department of Mathematics
Saveetha School of Engineering, SIMATS
Thandalam, Chennai-602105, Tamilnadu, India
e-mail: `furuichi.shigeru@nihon-u.ac.jp`

Nicuşor Minculete
Department of Mathematics and Computer Science
Transilvania University of Braşov
Braşov, 500091, Romania
e-mail: `minculeten@yahoo.com`

Hamid Reza Moradi
Department of Mathematics
Mashhad Branch, Islamic Azad University
Mashhad, Iran
e-mail: `hrmoradi@mshdiau.ac.ir`

Mohammad Sababheh
Department of basic sciences
Princess Sumaya University For Technology
Al Jubaiha, Amman 11941, Jordan
e-mail: `sababheh@yahoo.com; sababheh@psut.edu.jo`