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BERNSTEIN–DOETSCH TYPE THEOREMS FOR DECREASING JENSEN m–CONVEX FUNCTIONS

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Abstract. The main purpose of this paper is to generalize the Bernstein-Doetsch theorem to the setting of the decreasing Jensen m-convex functions for $c_m \in \mathbb{N} \cap [2,\infty)$, where $c_m = \frac{m+1}{m}$. Also we ask: is this theorem still true for $c_m \in [2,\infty)$?

1. Introduction

In what follows, we recall the definition of an m-convex function [18]. Let b be a positive real number and $m \in [0,1]$. A function $f : [0,b] \to \mathbb{R}$ is said to be m-convex if the inequality

$$f(tx + m(1-t)y) \leqslant t f(x) + m(1-t) f(y) \qquad (x,y \in [0,b],\ t \in [0,1]) \tag{1}$$

holds. A function $f : [0,b] \to \mathbb{R}$ is said to be a Jensen convex if the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \qquad x,y \in [0,b] \tag{2}$$

holds.

In 2016, Lara, Quintero, Rosales and Sánchez [10] introduced a class of functions. This class can be defined as follows: a function $f : [0,b] \to \mathbb{R}$ is said to be a Jensen m-convex if the inequality

$$f\left(\frac{x+y}{c_m}\right) \leqslant \frac{f(x)+f(y)}{c_m} \tag{3}$$

holds for all $x,y \in [0,b]$ and $c_m = \frac{m+1}{m}$, where $m \in (0,1]$. This shows that $c_m \geqslant 2$. It is easy to see that this class coincides to the functions that satisfy inequality (2) with $m = 1$. Furthermore, the functions that satisfy inequality (3) are a subclass of the functions that satisfy inequality (1).

In 1915, Bernstein and Doetsch [1] proved that if a Jensen convex function is bounded at any point of its domain then it is continuous. Many authors have generalized this theorem to different settings. (cf. Kominek and Kuczma [9], Páles [11], Gilányi,

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Nikodem and Páles [4], Burai, Hazy and Juhasz [2], Házy [6], Házy [7], Házy [8], González, Nikodem, Páles, and Roa [5], Gilányi, González, Nikodem and Páles [3]). Recently, Páles and Shihab [13] have generalized Bernstein–Doetsch theorem to the setting of convexity with respect to the Chebyshev system and Shihab and Kluza [16] generalized this theorem to the geometric convexity setting. Note that the class of additive functions is a Jensen convex, therefore the Bernstein–Doetsch holds for additive functions for more characteristics of additive functions see the papers [14] and [17]. For more details see the papers mentioned and their references. We ask a question: does the Bernstein–Doetsch theorem hold in context of higher order convexity? For definitions and properties of this concept we refer to the paper [12] and [15]. This paper is divided into four sections: the first section provides the basic definitions and preliminaries, the second discusses the Jensen m-convex functions with $t \in [0,1] \cap \mathbb{Q}$, the third explores the boundedness of Jensen m-convex functions and the fourth aims to generalize the Bernstein-Doetsch theorem to the Jensen m-convex functions setting.

2. Jensen m-convex function with $t \in [0,1] \cap \mathbb{Q}$

In this section we show that the Jensen m-convex functions satisfy some inequalities. Let $\mathbb{R}_+$ and $\mathbb{N}$ be the nonnegative and natural numbers, respectively. The following lemma shows that the Jensen m-convex function satisfies such inequality with $c_m \in [2,\infty) \cap \mathbb{N}$, that will be useful in the sequel.

LEMMA 2.1. *Let $b > 0$, $m \in (0,1]$ and $c_m \in [2,\infty) \cap \mathbb{N}$. If $f : [0,b] \to \mathbb{R}_+$ is a decreasing Jensen m-convex function then the inequality*

$$f\left(\frac{x_1 + \ldots + x_{c_m^n}}{c_m^n}\right) \leqslant \frac{f(x_1) + \ldots + f(x_{c_m^n})}{c_m^n} \tag{4}$$

holds for all $x_1, \ldots, x_{c_m^n} \in [0,b]$ and $n \in \mathbb{N}$.

Proof. At the beginning we need to show that for every $p \in \mathbb{N}$ with $x_1, \ldots, c_{2^p}$ are in the domain of f the sum $\frac{1}{c_m^p} \sum_{i=1}^{2^p} x_i$ is in the domain of f. Using the fact that $c_m \in [2,\infty) \cap \mathbb{N}$, it yields that

$$\frac{1}{c_m^p} \sum_{i=1}^{2^p} x_i \leqslant \frac{1}{2^p} \sum_{i=1}^{2^p} x_i.$$

Since $\frac{1}{2^p} \sum_{i=1}^{2^p} x_i \in [0,b]$ this implies that $\frac{1}{2^p} \sum_{i=1}^{2^p} x_i \leqslant b$, therefore the above inequality yields that $\frac{1}{c_m^p} \sum_{i=1}^{2^p} x_i \leqslant b$ and so the sum $\frac{1}{c_m^p} \sum_{i=1}^{2^p} x_i$ is in the domain of f.

Now from inequality (3), for all $p \in \mathbb{N}$ and $x_1, \ldots, x_{2^p} \in [0,b]$, it follows by induction that

$$f\left(\frac{1}{c_m^p} \sum_{i=1}^{2^p} x_i\right) \leqslant \frac{1}{c_m^p} \sum_{i=1}^{2^p} f(x_i). \tag{5}$$

Indeed, assume that (5) is true for p, we show that (5) is also true for $p+1$. Using the inequality (3), it follows that

$$f\left(\frac{1}{c_m^{p+1}}\sum_{i=1}^{2^{p+1}} x_i\right) = f\left(\frac{\frac{1}{c_m^p}\sum_{i=1}^{2^p} x_i + \frac{1}{c_m^p}\sum_{i=2^p+1}^{2^{p+1}} x_i}{c_m}\right)$$
$$\leqslant \frac{f\left(\frac{1}{c_m^p}\sum_{i=1}^{2^p} x_i\right) + f\left(\frac{1}{c_m^p}\sum_{i=2^p+1}^{2^{p+1}} x_i\right)}{c_m}$$
$$\leqslant \frac{\frac{1}{c_m^p}\sum_{i=1}^{2^p} f(x_i) + \frac{1}{c_m^p}\sum_{i=2^p+1}^{2^{p+1}} f(x_i)}{c_m}$$
$$= \frac{1}{c_m^{p+1}}\sum_{i=1}^{2^{p+1}} f(x_i).$$

Now fix $n \in \mathbb{N}$ and choose $p \in \mathbb{N}$ such that $c_m^n < 2^p$. Let $x_1, \ldots, x_{c_m^n} \in [0,b]$ be arbitrary. Since $c_m \in [2,\infty) \cap \mathbb{N}$, therefore, we can define x_k as:

$$x_k := \frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i, \qquad k = c_m^n + 1, \ldots, 2^p.$$

Using the definition of x_k, it follows that

$$\frac{1}{c_m^p}\sum_{i=1}^{2^p} x_i \leqslant \frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i. \tag{6}$$

Indeed, since $c_m^n \in [2,\infty) \cap \mathbb{N}$, therefore

$$\frac{1}{c_m^p}\sum_{i=1}^{2^p} x_i = \frac{1}{c_m^p}\left(\sum_{i=1}^{c_m^n} x_i + (2^p - c_m^n)x_k\right)$$
$$= \frac{1}{c_m^p}\left(\sum_{i=1}^{c_m^n} x_i + \left(2^p - c_m^n\right)\frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i\right)$$
$$= \left(\frac{2}{c_m}\right)^p \frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i \leqslant \frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i.$$

Therefore, the inequality (6), the decreasingness of f and (5) yield that

$$f\left(\frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i\right) \leqslant f\left(\frac{1}{c_m^p}\sum_{i=1}^{2^p} x_i\right) \leqslant \frac{1}{c_m^p}\sum_{i=1}^{2^p} f(x_i)$$
$$= \frac{1}{c_m^p}\Big(f(x_1) + \ldots + f(x_{c_m^n}) + (2^p - c_m^n)f(x_k)\Big)$$
$$= \frac{1}{c_m^p}\left(\sum_{i=1}^{c_m^n} f(x_i) + (2^p - c_m^n)f\left(\frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i\right)\right)$$
$$= \frac{1}{c_m^p}\left(\sum_{i=1}^{c_m^n} f(x_i) + 2^p f\left(\frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i\right) - c_m^n f\left(\frac{1}{c_m^n}\sum_{i=1}^{c_m^n} x_i\right)\right).$$

Multiply both sides of the above inequality by c_m^p we deduce that

$$(c_m^p - 2^p) f\left(\frac{1}{c_m^n} \sum_{i=1}^{c_m^n} x_i\right) \leqslant \sum_{i=1}^{c_m^n} f(x_i) - c_m^n f\left(\frac{1}{c_m^n} \sum_{i=1}^{c_m^n} x_i\right). \tag{7}$$

Note that $c_m^p - 2^p \geqslant 0$ therefore, the nonnegativity of f implies that

$$0 \leqslant (c_m^p - 2^p) f\left(\frac{1}{c_m^n} \sum_{i=1}^{c_m^n} x_i\right).$$

This inequality shows that the right hand side of (7) is non-negative. Thus

$$0 \leqslant \sum_{i=1}^{c_m^n} f(x_i) - c_m^n f\left(\frac{1}{c_m^n} \sum_{i=1}^{c_m^n} x_i\right).$$

So that

$$c_m^n f\left(\frac{1}{c_m^n} \sum_{i=1}^{c_m^n} x_i\right) \leqslant \sum_{i=1}^{c_m^n} f(x_i).$$

This inequality is equivalent to (4). The proof of lemma is done. □

In the next theorem we show that the Jensen m-convex functions satisfy such inequality similar to (1), for rational t and $m \in (0,1]$.

THEOREM 2.2. *Let $b > 0$, $m \in (0,1]$ and $c_m \in [2,\infty) \cap \mathbb{N}$. If $f : [0,b] \to \mathbb{R}_+$ is a decreasing Jensen m-convex function, then the inequality*

$$f\big(tx + m(1-t)y\big) \leqslant t f(x) + m(1-t) f(y) \qquad (x,y \in [0,b],\ t \in [0,1] \cap \mathbb{Q} \tag{8}$$

holds.

Proof. Assume that $t \in [0,1] \cap \mathbb{Q}$ and $m \in (0,1]$. Let $t = \frac{k}{c_m^n}$, $n,k \in \mathbb{N}$ such that $0 < k < c_m^n$. Since f is decreasing Jensen m-convex, therefore Lemma 2.1, with $x_1 = \ldots = x_k = x$ and $x_{k+1} = \ldots = x_{c_m^n} = my$, implies that the inequality

$$f\left(\frac{kx + m(c_m^n - k)y}{c_m^n}\right) \leqslant \frac{kf(x) + (c_m^n - k) f(my)}{c_m^n} \qquad (x,y \in [0,b],\ m \in (0,1]) \tag{9}$$

holds.

[10, Proposition 3.1] says that if a function $f : [0,b] \to \mathbb{R}$ is Jensen m-convex then

$$f(qx) \leqslant q f(x) \qquad (q \in (0,1], x \in [0,b]). \tag{10}$$

Therefore, the fact that $\frac{k}{c_m^n} = t$, $\frac{c_m^n - k}{c_m^n} = 1 - t$ and $0 < m \leqslant 1$ implies that (8) is true for all $x,y \in [0,b]$, $t \in (0,1) \cap \mathbb{Q}$ and $m \in (0,1]$. For $t = 1$ and $t = 0$ the inequality (8) is obvious. □

3. Boundedness of Jensen m-convex functions

In this section we show that if the Jensen m-convex functions locally bounded at point on its domain then it is locally bounded in the whole domain.

THEOREM 3.1. *Let $b>0$, $m\in(0,1]$, $c_m\in[2,\infty)\cap\mathbb{N}$ and $f:(0,b)\to\mathbb{R}_+$ be a decreasing Jensen m-convex function. If f is a locally bounded from above at a point $x_0\in(0,b)$ then it is locally bounded from above at every point $x\in(0,b)$.*

Proof. Let $x\in(0,b)$ be arbitrary different from x_0. There exists a rational number $r\in\mathbb{Q}$ such that the point

$$y=\frac{x-r(x_0-x)}{m} \tag{11}$$

belongs to $(0,b)$. Let $t=\frac{r}{r+1}\in(0,1)\cap\mathbb{Q}$. This means that $r=\frac{t}{1-t}$. Therefore, by (11) we have that

$$x=tx_0+m(1-t)y. \tag{12}$$

Since f is bounded from above at x_0 therefore there exists $\rho>0$ such that the open ball $B=B(x_0,\rho)$ centered at x_0 with radius ρ, is contained in $(0,b)$, and f bounded from above on B i.e

$$f(w)\leqslant M \qquad w\in B, \tag{13}$$

where $M\in\mathbb{R}$ is a constant. Let $V=B(x,t\rho)$ be an open ball centered at x with radius $t\rho$. Let $v\in V$ be arbitrary and put

$$z=\frac{v-m(1-t)y}{t}, \qquad v\in V,\ y\in(0,b),\ t\in(0,1)\cap\mathbb{Q},\ m\in(0,1]. \tag{14}$$

Since $v\in V$, therefore $|v-x|\leqslant t\rho$, but by (12) we have that $\Big|v-tx_0-m(1-t)y\Big|\leqslant t\rho$, therefore

$$\left|\frac{v-m(1-t)y}{t}-x_0\right|\leqslant\rho,$$

which in other words we have that $|z-x_0|\leqslant\rho$, this means that $z\in B$. Now by (14) we have that

$$v=tz+m(1-t)y. \tag{15}$$

Since $z\in B\subseteq(0,b)$ and $y\in(0,b)$, m-convexity of $(0,b)$ implies that $v\in(0,b)$. But v was chosen arbitrarily from V, so $V\subseteq(0,b)$. Now by (15), Theorem 2.2 and (13) we have that

$$f(v)\leqslant tf(z)+m(1-t)f(y)\leqslant tM+m(1-t)f(y)\leqslant tM+m(1-t)f(y)\leqslant\max(M,f(y)).$$

Thus f is bounded from above on $V\subset(0,b)$. $\square$

THEOREM 3.2. *Let $b>0$, $m\in(0,1]$, $c_m\in[2,\infty)\cap\mathbb{N}$ and $f:(0,b)\to\mathbb{R}_+$ be a decreasing Jensen m-convex function. If f is locally bounded from below at a point $x_0\in(0,b)$ then it is locally bounded from below at every point $x\in(0,b)$.*

Proof. The proof is similar to the proof of Theorem 3.1. □

THEOREM 3.3. *Let $b>0$, $c_m \in [2,\infty)\cap\mathbb{N}$ and $f:(0,b)\to\mathbb{R}_+$ be a decreasing Jensen m-convex function. If f is locally bounded at a point $x_0\in(0,b)$ then it is locally bounded at every point $x\in(0,b)$.*

Proof. By Theorem 3.1 f is locally bounded above and by Theorem 3.2 f is locally bounded below. Therefore f is locally bounded at every point $x\in(0,b)$. □

4. Generalization of Bernstein-Doetsch theorem

In this section we give a proof of Bernstein-Doetsch theorem in the setting of Jensen m-convexity.

THEOREM 4.1. *Let $b>0$, $m\in(0,1]$, $c_m\in[2,\infty)\cap\mathbb{N}$ and $f:(0,b)\to\mathbb{R}_+$ be a decreasing Jensen m-convex function. If f is locally bounded from above at a point of $(0,b)$ then it is continuous in $(0,b)$.*

Proof. Let f be a locally bounded from above at a point of $(0,b)$. According to Theorem 3.1 the function f is locally bounded on $(0,b)$. Define two functions m_f and M_f as

$$m_f(x)=\lim_{r\to 0+}\inf_{N(x,r)}$$

and

$$M_f(x)=\lim_{r\to 0+}\sup_{N(x,r)},$$

where $N(x,r)$ is a neighborhood of x with radius r. It is easy to see that

$$m_f(x)\leqslant f(x)\leqslant M_f(x) \qquad (x\in(0,b)). \tag{16}$$

Let $x\in(0,b)$ be arbitrary. There exists a sequence $\{x_q\}\in(0,b)$, $q\in\mathbb{N}$ such that

$$\lim_{q\to\infty}x_q=x \qquad \text{and} \qquad \lim_{q\to\infty}f(x_q)=m_f(x) \tag{17}$$

and a sequence $\{z_q\}$ in $(0,b)$, $q\in\mathbb{N}$ such that

$$\lim_{q\to\infty}z_q=x \qquad \text{and} \qquad \lim_{q\to\infty}f(z_q)=M_f(x). \tag{18}$$

Consider the sequence

$$y_q:=\frac{c_m^n}{m(1-t)}z_q-\frac{t}{m(1-t)}x_q,$$

for some $t\in[0,1]\cap\mathbb{Q}$ and $n\in\mathbb{N}$. Using (17) and (18), it follows that

$$\lim_{q\to\infty}y_q=\frac{c_m^n-t}{m(1-t)}x. \tag{19}$$

By definition of the sequence of y_q we have that $z_q = \frac{tx_q+m(1-t)y_q}{c_m^n}$. Now according to (8), it follows that

$$f(z_q) \leqslant \frac{t}{c_m^n} f(x_q) + m\frac{1-t}{c_m^n} f(y_q).$$

This inequality yields that

$$f(y_q) \geqslant \frac{c_m^n}{m(1-t)} f(z_q) - \frac{t}{m(1-t)} f(x_q)$$

Letting $q \to \infty$, it follows that

$$\liminf_{q\to\infty} f(y_q) \geqslant \frac{c_m^n}{m(1-t)} M_f(x) - \frac{t}{m(1-t)} m_f(x).$$

But by (19) we have that

$$\liminf_{q\to\infty} f(y_q) \leqslant \frac{c_m^n - t}{m(1-t)} M_f(x).$$

The above two inequalities imply that

$$\frac{c_m^n}{m(1-t)} M_f(x) - \frac{t}{m(1-t)} m_f(x) \leqslant \liminf_{q\to\infty} f(y_q) \leqslant \frac{c_m^n - t}{m(1-t)} M_f(x).$$

Rewrite these inequalities as follows

$$\begin{aligned}\frac{c_m^n}{m(1-t)} M_f(x) \leqslant \liminf_{q\to\infty} f(y_q) + \frac{t}{m(1-t)} m_f(x) &\leqslant \frac{c_m^n - t}{m(1-t)} M_f(x) + \frac{t}{m(1-t)} m_f(x) \\ &\leqslant \frac{c_m^n}{m(1-t)} M_f(x).\end{aligned}$$

These two inequalities yield that

$$\liminf_{q\to\infty} f(y_q) + \frac{t}{m(1-t)} m_f(x) = \frac{c_m^n}{m(1-t)} M_f(x).$$

Therefore, (19) yields that

$$\frac{t}{m(1-t)} M_f(x) \leqslant \frac{t}{m(1-t)} m_f(x).$$

This shows that

$$M_f(x) \leqslant m_f(x).$$

This inequality together with (16) imply that

$$M_f(x) = m_f(x).$$

One can easy see this condition is necessary and sufficient for continuity of f at every $x \in (0,b)$. Therefore, f is continuous on $(0,b)$. □

Open problem

In Theorem 4.1 we have proved the Bernstein–Doetsch theorem for $c_m \in \mathbb{N} \cap [2,\infty)$, it is natural to ask does this theorem hold for all $c_m \in [2,\infty)$?

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