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$GSDD_1^+$ MATRICES AND ERROR BOUNDS FOR LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. A new subclass of H-matrices named $GSDD_1^+$ matrices is introduced and some properties of $GSDD_1^+$ matrices are presented. The relationships between $GSDD_1^+$ matrices and other subclasses of H-matrices is studied. Moreover, the infinity norm bound for the inverse of $GSDD_1^+$ matrices is provided. As applications, error bound of the linear complementarity problems for $GSDD_1^+$ matrices is also presented, which improves some existing results. Numerical examples are given to illustrate the validity of new results.

1. Introduction

Throughout this paper, $\mathbb{C}^{n\times n}(\mathbb{R}^{n\times n})$ denotes the set of all complex (real) matrices. For a positive integer $n \geqslant 2$, we set the index set $N = \{1,2,\cdots,n\}$. Given a matrix $M = (m_{ij}) \in \mathbb{C}^{n\times n}$, denote

$$N_1 = \{i \in N | 0 < |m_{ii}| \leqslant r_i(M)\},\ \ N_2 = \{i \in N | |m_{ii}| > r_i(M)\},\ \ r_i(M) = \sum_{j=1, j\neq i}^{n} |m_{ij}|.$$

A matrix $M \in \mathbb{R}^{n\times n}$ is said an M-matrix if its inverse is nonnegative and all its off-diagonal entries are nonpositive. Matrix $M = (m_{ij}) \in \mathbb{C}^{n\times n}$ is called an H-matrix if its comparison matrix denoted by $\mu(M) = (\tilde{m}_{ij})$ with

$$\tilde{m}_{ij} = \begin{cases} |m_{ij}|, & \text{if } i = j, \\ -|m_{ij}|, & \text{if } i \neq j. \end{cases}$$

is an M-matrix [1]. We also know that matrix $M \in \mathbb{C}^{n\times n}$ is an H-matrix if and only if there exists a positive diagonal matrix D such that MD is a strictly diagonally dominant matrix [18]. Here, matrix $M = (m_{ij}) \in \mathbb{C}^{n\times n}$ is called a strictly diagonally dominant (SDD) matrix if

$$|m_{ii}| > r_i(M), \quad \forall i \in N. \tag{1}$$

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H-matrices play an important role in a lot of science fields such as computational mathematics, mathematical physics and control theory, see [1, 6, 12, 18]. Moreover, infinity norm bounds of the inverse for H-matrices can be used in convergence analysis of matrix splitting and matrix multi-splitting iterative methods for solving large sparse systems of linear equations [1], as well as bounding errors of the linear complementarity problems [3, 4]. In recent years, the research for infinity norm bound for the inverse of special H-matrices has aroused many scholars' interest, such as $GSDD_1$ matrices [10], CKV-type matrices [5], S-$SDDS$ matrices [13], S-Nekrasov matrices [7], S-SDD matrices [8], SDD_1 matrices [2], SDD_k matrices [20], strong SDD_1 matrices [21], Nekrasov matrices [16], et al. In this paper, we introduce a new subclass of H-matrices named $GSDD_1^+$ matrices, and show several benefits from the obtained new class.

Next, some definitions and notations are recalled, which will be used in the sequel.

DEFINITION 1. [18] Matrix $M=(m_{ij})\in\mathbb{C}^{n\times n}$ is called an SDD_1 matrix if

$$|m_{ii}|>p_i(M),\quad \forall i\in N_1,$$

where

$$p_i(M)=\sum_{j\in N_1\setminus\{i\}}|m_{ij}|+\sum_{j\in N_2\setminus\{i\}}\frac{r_j(M)}{|m_{jj}|}|m_{ij}|.$$

DEFINITION 2. [10] Matrix $M=(m_{ij})\in\mathbb{C}^{n\times n}$ is said a generalized SDD_1 ($GSDD_1$) matrix if for $i\in N_2$, $j\in N_1$,

$$\begin{cases} r_i(M)-p_i^{N_2}(M)>0, \\ (r_i(M)-p_i^{N_2}(M))(|m_{jj}|-p_j^{N_1}(M))>p_i^{N_1}(M)p_j^{N_2}(M), \end{cases} \tag{2}$$

where $p_i^{N_2}(M)=\sum\limits_{j\in N_2\setminus\{i\}}|m_{ij}|\frac{r_j(M)}{|m_{jj}|}$, $p_i^{N_1}(M)=\sum\limits_{j\in N_1\setminus\{i\}}|m_{ij}|$, $\forall i\in N$.

DEFINITION 3. [5] Assume that S is a nonempty subset of N and $\overline{S}$ is the complement set of S in N. Matrix $M=(m_{ij})\in\mathbb{C}^{n\times n}$ is named a CKV-type matrix if for all $i\in N$, the set $S_i^\star(M)$ is not empty, where
$S_i^\star(M)=\{S\in\Sigma(i):|m_{ii}|>r_i^S(M)$, and for all $j\in\overline{S}$

$$\left(|m_{ii}|-r_i^S(M)\right)\left(|m_{jj}|-r_j^{\overline{S}}(M)\right)>r_i^{\overline{S}}(M)\,r_j^S(M)\},$$

with $\Sigma(i)=\{S\subsetneq N: i\in S\}$ and $r_i^S(M):=\sum\limits_{j\in S\setminus\{i\}}|m_{ij}|$.

The paper is organized as follows: In Sect. 2, we propose a new subclass of H-matrices named $GSDD_1^+$ matrix, discuss some of its properties and consider its relationship with other subclasses of H-matrices involving $GSDD_1$ matrices and CKV-type matrices. Meanwhile, a scaling matrix D is constructed to verify that a $GSDD_1^+$ matrix is an H-matrix. In Sect. 3, we give new infinity norm upper bound for the inverse of $GSDD_1^+$ matrices, which only depends on the elements of the matrix. In Sect.

4, a new error bound for the linear complementarity problems of $GSDD_1^+$ matrices is presented by using the infinity norm bound. Numerical examples show the validity of the theoretical results. Finally, some conclusions are summarized in Sect. 5.

2. $GSDD_1^+$ matrices and scaling matrices

In this section, we present the definition of $GSDD_1^+$ matrix and prove that it is an H-matrix. Moreover, we give some properties of $GSDD_1^+$ matrices and discuss the relationships between $GSDD_1^+$ matrices and some existing matrices.

DEFINITION 4. Matrix $M=(m_{ij})\in\mathbb{C}^{n\times n}$ is called a $GSDD_1^+$ matrix if

$$\left(R_i(M)-\sum_{t\in N_2\backslash\{i\}}|m_{it}|\frac{R_t(M)}{|m_{tt}|}\right)\left(|m_{jj}|-\sum_{t\in N_1\backslash\{j\}}|m_{jt}|\right)$$
$$>\sum_{t\in N_1}|m_{it}|\sum_{t\in N_2}|m_{jt}|\frac{R_t(M)}{|m_{tt}|},\quad \forall i\in N_2,\ \forall j\in N_1, \tag{3}$$

where

$$k=\max_{i\in N_2}\left(\frac{\sum\limits_{t\in N_1}|m_{it}|}{|m_{ii}|-\sum\limits_{t\in N_2\backslash\{i\}}|m_{it}|}\right),\quad R_i(M)=k\sum_{t\in N_2\backslash\{i\}}|m_{it}|+\sum_{t\in N_1}|m_{it}|,\ \ i\in N_2. \tag{4}$$

REMARK 1. If $N_1=\{i||m_{ii}|\leqslant r_i(M)\}=\emptyset$, then M is an SDD matrix and $R_i(M)=0(\forall i\in N)$. Thus, an SDD matrix is not a $GSDD_1^+$ matrix.

THEOREM 1. *Let $M=(m_{ij})\in\mathbb{C}^{n\times n}$ be a $GSDD_1^+$ matrix. Then there exists at least one entry $m_{it}\neq 0$, $i\neq t$, $i\in N_2$, $t\in N$.*

Proof. Suppose that $m_{it}=0$, for any $i\in N_2$, $t\in N$, $t\neq i$, then for any $i\in N_2$, we have

$$k=\max_{i\in N_2}\left(\frac{\sum\limits_{t\in N_1}|m_{it}|}{|m_{ii}|-\sum\limits_{t\in N_2\backslash\{i\}}|m_{it}|}\right)=0,$$

and

$$R_i(M)=k\sum_{t\in N_2\backslash\{i\}}|m_{it}|+\sum_{t\in N_1}|m_{it}|=0,$$

which makes that both sides in (3) are zero simultaneously. This is a contradiction. □

THEOREM 2. *Let $M=(m_{ij})\in\mathbb{C}^{n\times n}$ be a $GSDD_1^+$ matrix. Then for $\forall i\in N_2$,*

$$0<\frac{R_i(M)}{|m_{ii}|}\leqslant k<1,$$

where k is defined as in (4).

Proof. For $\forall i \in N_2$, by (1), we have

$$|m_{ii}| > r_i(M) = \sum_{t\in N_1} |m_{it}| + \sum_{t\in N_2\setminus\{i\}} |m_{it}|,$$

that is,

$$\frac{\sum\limits_{t\in N_1} |m_{it}|}{|m_{ii}| - \sum\limits_{t\in N_2\setminus\{i\}} |m_{it}|} < 1.$$

So, it holds that

$$0 < k = \max_{i\in N_2} \left(\frac{\sum\limits_{t\in N_1} |m_{it}|}{|m_{ii}| - \sum\limits_{t\in N_2\setminus\{i\}} |m_{it}|} \right) < 1.$$

Furthermore, by the definition of k, we have that

$$k|m_{ii}| \geqslant k \sum_{t\in N_2\setminus\{i\}} |m_{it}| + \sum_{t\in N_1} |m_{it}|,$$

i.e., $R_i(M) \leqslant k|m_{ii}|$, the conclusion follows. □

In the following, we construct a positive diagonal matrix D with a parameter that scales a $GSDD_1^+$ matrix to transform it into an SDD matrix.

THEOREM 3. *Let $M = (m_{ij}) \in \mathbb{C}^{n\times n}$ be a $GSDD_1^+$ matrix. Then there exists a positive diagonal matrix $D = diag(d_1, d_2, \cdots, d_n)$ with*

$$d_j = \begin{cases} \dfrac{pR_j(M)}{|m_{jj}|} + \varepsilon, & j \in N_2, \\ 1, & j \in N_1, \end{cases}$$

where $p = \max\limits_{i\in N_2} \{Q_i(M)\}$,

$$Q_i(M) = \frac{\sum\limits_{t\in N_1} |m_{it}|}{R_i(M) - \sum\limits_{t\in N_2\setminus\{i\}} |m_{it}| \frac{R_t(M)}{|m_{tt}|}},$$

$$0 < \varepsilon < \min_{j\in N_1} \left(\frac{|m_{jj}| - \sum\limits_{t\in N_2} |m_{jt}| \frac{pR_t(M)}{|m_{tt}|} - \sum\limits_{t\in N_1\setminus\{j\}} |m_{jt}|}{\sum\limits_{t\in N_2} |m_{jt}|} \right),$$

such that MD is an SDD matrix.

Proof. For $\forall i \in N_2$, by Theorem 2, we have

$$Q_i(M) = \frac{R_i(M) - k \sum\limits_{t \in N_2 \setminus \{i\}} |m_{it}|}{R_i(M) - \sum\limits_{t \in N_2 \setminus \{i\}} |m_{it}| \frac{R_t(M)}{|m_{tt}|}} \leqslant \frac{R_i(M) - \sum\limits_{t \in N_2 \setminus \{i\}} |m_{it}| \frac{R_t(M)}{|m_{tt}|}}{R_i(M) - \sum\limits_{t \in N_2 \setminus \{i\}} |m_{it}| \frac{R_t(M)}{|m_{tt}|}} = 1,$$

which implies that $0 < p \leqslant 1$. Hence, for $\forall j \in N_1$, we get

$$|m_{jj}| - p \sum_{t \in N_2} |m_{jt}| \frac{R_t(M)}{|m_{tt}|} - \sum_{t \in N_1 \setminus \{j\}} |m_{jt}| > 0.$$

So, there exists a sufficiently small positive number $\varepsilon > 0$ such that

$$\varepsilon \sum_{t \in N_2} |m_{jt}| < |m_{jj}| - \sum_{t \in N_2} |m_{jt}| \frac{pR_t(M)}{|m_{tt}|} - \sum_{t \in N_1 \setminus \{j\}} |m_{jt}|, \quad \forall j \in N_1. \tag{5}$$

Construct a positive diagonal matrix $D = (d_1, d_2, \cdots, d_n)$, where

$$d_j = \begin{cases} \dfrac{pR_j(M)}{|m_{jj}|} + \varepsilon, & j \in N_2, \\ 1, & j \in N_1. \end{cases}$$

Next, we will prove that MD is a strictly diagonally dominant matrix. Let us consider the following two cases:

Case 1. For any $j \in N_1$, from (5), we get

$$\begin{aligned} r_j(MD) &= \sum_{t \in N_1 \setminus \{j\}} d_t |m_{jt}| + \sum_{t \in N_2} d_t |m_{jt}| \\ &= \sum_{t \in N_1 \setminus \{j\}} |m_{jt}| + \sum_{t \in N_2} |m_{jt}| \left(\frac{pR_t(M)}{|m_{tt}|} + \varepsilon \right) \\ &= \sum_{t \in N_1 \setminus \{j\}} |m_{jt}| + \varepsilon \sum_{t \in N_2} |m_{jt}| + \sum_{t \in N_2} |m_{jt}| \frac{pR_t(M)}{|m_{tt}|} \\ &< \sum_{t \in N_1 \setminus \{j\}} |m_{jt}| + \left(|m_{jj}| - \sum_{t \in N_2} |m_{jt}| \frac{pR_t(M)}{|m_{tt}|} - \sum_{t \in N_1 \setminus \{j\}} |m_{jt}| \right) \\ &\quad + \sum_{t \in N_2} |m_{jt}| \frac{pR_t(M)}{|m_{tt}|} \\ &= |m_{jj}| = |(MD)_{jj}|. \end{aligned}$$

Case 2. For any $j \in N_2$, we have

$$|(MD)_{jj}| = |m_{jj}| \left(\frac{pR_j(M)}{|m_{jj}|} + \varepsilon \right) = \varepsilon |m_{jj}| + pR_j(M),$$

and

$$
\begin{aligned}
r_j(MD) &= \sum_{t\in N_1} d_t \left|m_{jt}\right| + \sum_{t\in N_2\setminus\{j\}} d_t \left|m_{jt}\right| \\
&= \sum_{t\in N_1} \left|m_{jt}\right| + \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| \left(\frac{pR_t(M)}{|m_{tt}|} + \varepsilon\right) \\
&= \sum_{t\in N_1} \left|m_{jt}\right| + \varepsilon \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| + \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|} \\
&\leqslant pR_j(M) + \varepsilon \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| \\
&< pR_j(M) + \varepsilon \left|m_{jj}\right| = \left|(MD)_{jj}\right|.
\end{aligned}
$$

Hence, for $j \in N$, we obtain that $|(MD)_{jj}| > r_j(MD)$, i.e., MD is a strictly diagonal dominant matrix. □

COROLLARY 1. *Let $M = (m_{ij}) \in \mathbb{C}^{n\times n}$ be a $GSDD_1^+$ matrix. Then M is also an H-matrix. In addition, If M has positive diagonal entries, then $det(M) > 0$.*

The following examples will illustrate the relationships between $GSDD_1^+$ matrices and other subclasses of H-matrices.

EXAMPLE 1. Consider the matrix:

$$
M_1 = \begin{pmatrix} 8 & -5 & 2 & -3 \\ -3 & 9 & -6 & 2 \\ -4 & 2 & 9 & 0 \\ 3 & -1 & 2 & 10 \end{pmatrix}.
$$

Obviously, $N_1 = \{1,2\}$ and $N_2 = \{3,4\}$. By calculations, we have

$$
\frac{|m_{31}|+|m_{32}|}{|m_{33}|-|m_{34}|} = \frac{4+2}{9-0} = \frac{2}{3}, \qquad \frac{|m_{41}|+|m_{42}|}{|m_{44}|-|m_{43}|} = \frac{3+1}{10-2} = \frac{1}{2},
$$

so $k = \max\limits_{i\in N_2}\left\{\frac{2}{3}, \frac{1}{2}\right\} = \frac{2}{3}$. Then

$$
\begin{aligned}
R_3(M_1) &= \frac{2}{3}\times|m_{34}| + |m_{31}| + |m_{32}| = \frac{2}{3}\times 0 + 4 + 2 = 6, \\
R_4(M_1) &= \frac{2}{3}\times|m_{43}| + |m_{41}| + |m_{42}| = \frac{2}{3}\times 2 + 3 + 1 = \frac{16}{3}.
\end{aligned}
$$

When $i = 3$ and $j = 1$, we get

$$
\begin{aligned}
&\left(R_3(M_1) - |m_{34}|\frac{R_4(M_1)}{|m_{44}|}\right)(|m_{11}| - |m_{12}|) = 18 \\
&> (|m_{31}| + |m_{32}|)\left(|m_{13}|\frac{R_3(M_1)}{|m_{33}|} + |m_{14}|\frac{R_4(M_1)}{|m_{44}|}\right) = \frac{51}{5}.
\end{aligned}
$$

When $i=4$ and $j=1$, we have

$$\left(R_4(M_1)-|m_{43}|\frac{R_3(M_1)}{|m_{33}|}\right)(|m_{11}|-|m_{12}|)=12$$
$$>(|m_{41}|+|m_{42}|)\left(|m_{13}|\frac{R_3(M_1)}{|m_{33}|}+|m_{14}|\frac{R_4(M_1)}{|m_{44}|}\right)=\frac{176}{15}.$$

When $i=3$ and $j=2$, we have

$$\left(R_3(M_1)-|m_{34}|\frac{R_4(M_1)}{|m_{44}|}\right)(|m_{22}|-|m_{21}|)=36$$
$$>(|m_{31}|+|m_{32}|)\left(|m_{23}|\frac{R_3(M_1)}{|m_{33}|}+|m_{24}|\frac{R_4(M_1)}{|m_{44}|}\right)=\frac{152}{5}.$$

when $i=4$ and $j=2$, we get

$$\left(R_4(M_1)-|m_{43}|\frac{R_3(M_1)}{|m_{33}|}\right)(|m_{22}|-|m_{21}|)=24$$
$$>(|m_{41}|+|m_{42}|)\left(|m_{23}|\frac{R_3(M_1)}{|m_{33}|}+|m_{24}|\frac{R_4(M_1)}{|m_{44}|}\right)=\frac{304}{15}.$$

Thus, M_1 is a $GSDD_1^+$ matrix. However, when $i=3$ and $j=1$, we have

$$\left(r_3(M_1)-p_3^{N_2}(M_1)\right)\left(|m_{11}|-p_1^{N_1}(M_1)\right)=18<p_3^{N_1}(M_1)\,p_1^{N_2}(M_1)=\frac{92}{5},$$

so, M_1 is not a $GSDD_1$ matrix.

EXAMPLE 2. Consider the following tri-diagonal matrix $M_2\in\mathbb{R}^{n\times n}$ arising from the finite difference method for free boundary problems, where

$$M_2=\begin{pmatrix} b+\alpha\sin\left(\frac{1}{n}\right) & c & 0 & \cdots & 0\\ a & b+\alpha\sin\left(\frac{2}{n}\right) & c & \cdots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & a & b+\alpha\sin\left(\frac{n-1}{n}\right) & c\\ 0 & \cdots & 0 & a & b+\alpha\sin(1) \end{pmatrix}.$$

Take that $n=10000$, $\alpha=10$, $a=7.5$, $b=15$ and $c=6$. It is easy to verify that M_2 is a $GSDD_1$ matrix, but not a $GSDD_1^+$ matrix.

EXAMPLE 3. Consider the matrix:

$$M_3=\begin{pmatrix} 24.54 & 0.75 & 5.25 & 2.25 & 4.5 & 6 & 5.25 & 3.75\\ 3.75 & 25.29 & 3.75 & 0.75 & 3.75 & 3.75 & 2.25 & 5.25\\ 5.25 & 5.25 & 29.04 & 2.25 & 6 & 0.75 & 3 & 1.5\\ 2.25 & 0.75 & 2.25 & 26.04 & 5.25 & 3.75 & 3.75 & 0.75\\ 2.25 & 3 & 0.75 & 2.25 & 28.29 & 1.5 & 0.75 & 1.5\\ 0.75 & 5.25 & 4.5 & 6 & 6 & 26.79 & 6 & 0.75\\ 3 & 6 & 4.5 & 4.5 & 0.75 & 0.75 & 26.79 & 1.5\\ 5.25 & 3.75 & 3 & 3 & 3.75 & 1.5 & 6 & 29.04 \end{pmatrix}.$$

It is easy to verify that that matrix M_3 is a $GSDD_1$ matrix and a $GSDD_1^+$ matrix.

From Examples 1–3, we can notice that $GSDD_1^+$ matrices and $GSDD_1$ matrices have the following relationship:

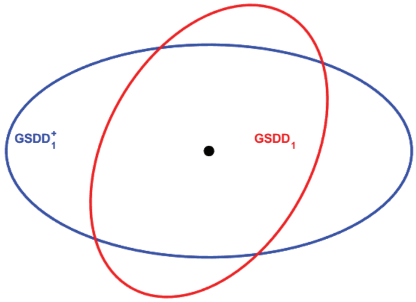

Figure 1: $\{GSDD_1^+\} \not\subseteq \{GSDD_1\}$ *and* $\{GSDD_1\} \not\subseteq \{GSDD_1^+\}$.

Next, we explore the relations between $GSDD_1^+$ matrices and CKV-type matrices.

EXAMPLE 4. Consider the matrix:

$$M_4 = \begin{pmatrix} 39.8 & 1.3 & -2.1 & -1 & 2.2 \\ 0 & 10.02 & 4.1 & 3.9 & -6 \\ 19.9 & -2 & 33 & -4 & 8 \\ 0 & 4 & -6 & 20 & -2 \\ -30 & -4 & 2 & 0 & 39.8 \end{pmatrix}.$$

Then $N_1 = \{2,3\}$ and $N_2 = \{1,4,5\}$. By computations, we have

$$k = 0.6122, \quad R_1(M_4) = 5.3592, \quad R_4(M_4) = 11.2245, \quad R_5(M_4) = 24.3673.$$

For $i = 1,\ j = 2$,

$$\left(R_1(M_4) - |m_{14}|\frac{R_4(M_4)}{|m_{44}|} - |m_{15}|\frac{R_5(M_4)}{|m_{55}|}\right)(|m_{22}| - |m_{23}|) = 20.4305$$
$$> (|m_{12}| + |m_{13}|)\left(|m_{21}|\frac{R_1(M_4)}{|m_{11}|} + |m_{24}|\frac{R_4(M_4)}{|m_{44}|} + |m_{25}|\frac{R_5(M_4)}{|m_{55}|}\right)$$
$$= 19.9352.$$

For $i = 1,\ j = 3$,

$$\left(R_1(M_4) - |m_{14}|\frac{R_4(M_4)}{|m_{44}|} - |m_{15}|\frac{R_5(M_4)}{|m_{55}|}\right)(|m_{33}| - |m_{32}|) = 106.9841$$
$$> (|m_{12}| + |m_{13}|)\left(|m_{21}|\frac{R_1(M_4)}{|m_{11}|} + |m_{24}|\frac{R_4(M_4)}{|m_{44}|} + |m_{25}|\frac{R_5(M_4)}{|m_{55}|}\right)$$
$$= 33.3962.$$

For $i=4,\ j=2$,

$$\begin{aligned}&\left(R_4(M_4)-|m_{41}|\frac{R_1(M_4)}{|m_{11}|}-|m_{45}|\frac{R_5(M_4)}{|m_{55}|}\right)(|m_{22}|-|m_{23}|)=59.2\\&>(|m_{42}|+|m_{43}|)\left(|m_{21}|\frac{R_1(M_4)}{|m_{11}|}+|m_{24}|\frac{R_4(M_4)}{|m_{44}|}+|m_{25}|\frac{R_5(M_4)}{|m_{55}|}\right)\\&=58.633.\end{aligned}$$

For $i=4,\ j=3$,

$$\begin{aligned}&\left(R_4(M_4)-|m_{41}|\frac{R_1(M_4)}{|m_{11}|}-|m_{45}|\frac{R_5(M_4)}{|m_{55}|}\right)(|m_{33}|-|m_{32}|)=310\\&>(|m_{42}|+|m_{43}|)\left(|m_{31}|\frac{R_1(M_4)}{|m_{11}|}+|m_{34}|\frac{R_4(M_4)}{|m_{44}|}+|m_{35}|\frac{R_5(M_4)}{|m_{55}|}\right)\\&=98.224.,\end{aligned}$$

For $i=5,\ j=2$

$$\begin{aligned}&\left(R_5(M_4)-|m_{51}|\frac{R_1(M_4)}{|m_{11}|}-|m_{54}|\frac{R_4(M_4)}{|m_{55}|}\right)(|m_{22}|-|m_{23}|)=120.34\\&>(|m_{52}|+|m_{53}|)\left(|m_{21}|\frac{R_1(M_4)}{|m_{11}|}+|m_{24}|\frac{R_4(M_4)}{|m_{44}|}+|m_{25}|\frac{R_5(M_4)}{|m_{55}|}\right)\\&=35.1798.\end{aligned}$$

For $i=5,\ j=2$

$$\begin{aligned}&\left(R_5(M_4)-|m_{51}|\frac{R_1(M_4)}{|m_{11}|}-|m_{54}|\frac{R_4(M_4)}{|m_{55}|}\right)(|m_{33}|-|m_{32}|)=630.1587\\&>(|m_{52}|+|m_{53}|)\left(|m_{31}|\frac{R_1(M_4)}{|m_{11}|}+|m_{34}|\frac{R_4(M_4)}{|m_{44}|}+|m_{35}|\frac{R_5(M_4)}{|m_{55}|}\right)\\&=58.9344.\end{aligned}$$

So, M_4 is a $GSDD_1^+$ matrix. However, there does not exist any subset S of $N=\{1,2,3,4,5\}$ such that M_4 is a CKV-type matrix.

EXAMPLE 5. Consider the matrix:

$$M_5=\begin{pmatrix}-1.4 & 0.7 & 0 & 0.15\\ 0.6 & 1.4 & 0 & 0\\ -1.1 & 1.5 & 1.3 & -0.6\\ 1.2 & -1.95 & 0.3 & 1.3\end{pmatrix}.$$

When $S=\{3,4\}$, $\overline{S}=\{1,2\}$, M_5 is a CKV-type matrix. But, M_5 is not a $GSDD_1^+$ matrix.

EXAMPLE 6. Consider the following tri-diagonal matrix $M_5 \in \mathbb{R}^{n\times n}$ arising from the finite difference method for free boundary problems, where

$$M_6 = \begin{pmatrix} b+\alpha\sin\left(\frac{1}{n}\right) & c & 0 & \cdots & 0 \\ a & b+\alpha\sin\left(\frac{2}{n}\right) & c & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & a & b+\alpha\sin\left(\frac{n-1}{n}\right) & c \\ 0 & \cdots & 0 & a & b+\alpha\sin(1) \end{pmatrix}.$$

Take $n = 10000$, $\alpha = 14.3417$, $a = -5.5887$, $b = 16.5159$, $c = -10.9312$. Then $N_1 = \{2\}$, $N_2 = \{1,3,4,\cdots,10000\}$, we can verify that M_6 is a $GSDD_1^+$ matrix. Meanwhile, when $S = \{2,3\}$, $\overline{S} = \{1,4,\cdots,10000\}$, M_6 is a CKV-type matrix.

As a result for Examples 4–6, $GSDD_1^+$ matrices and CKV-type matrices have the following relationship:

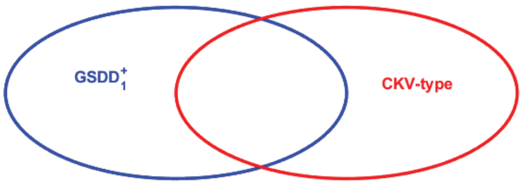

Figure 2: $\{GSDD_1^+\} \not\subseteq \{CKV\text{-}type\}$ *and* $\{CKV\text{-}type\} \not\subseteq \{GSDD_1^+\}$.

3. Infinity norm bound for the inverse of $GSDD_1^+$ matrices

In this section, new upper bound of the infinity norm for the inverse of $GSDD_1^+$ matrices. We first recall some results.

THEOREM 4. [19] *Let $M = (m_{ij}) \in \mathbb{C}^{n\times n}$ be an SDD matrix. Then*

$$\|M^{-1}\|_\infty \leqslant \frac{1}{\min\limits_{i\in N}\{|m_{ii}| - r_i(M)\}}. \tag{6}$$

THEOREM 5. [10] *Let $M = (m_{ij}) \in \mathbb{C}^{n\times n}$ be a $GSDD_1$ matrix. Then*

$$\|M^{-1}\|_\infty \leqslant \frac{\max\left\{\varepsilon, \max\limits_{i\in N_2}\frac{r_i(M)}{|m_{ii}|}\right\}}{\min\left\{\min\limits_{i\in N_2}\phi_i, \min\limits_{i\in N_1}\psi_i\right\}}, \tag{7}$$

where

$$\phi_i = r_i(M) - \sum_{j\in N_2\backslash\{i\}} |m_{ij}|\frac{r_j(M)}{|m_{jj}|} - \sum_{j\in N_1}|m_{ij}|\varepsilon, \quad i \in N_2, \tag{8}$$

$$\psi_i = |m_{ii}|\varepsilon - \sum_{j\in N_1\setminus\{i\}} |m_{ij}|\varepsilon - \sum_{j\in N_2} |m_{ij}| \frac{r_j(M)}{|m_{jj}|}, \quad i \in N_1, \tag{9}$$

and

$$\varepsilon \in \left\{ \max_{i\in N_1} \frac{p_i^{N_2}(M)}{|m_{ii}| - p_i^{N_1}(M)},\ \min_{j\in N_2} \frac{r_j(M) - p_j^{N_2}(M)}{p_j^{N_1}(M)} \right\}. \tag{10}$$

Next, we use the scaling matrix D in the proof of Theorem 3 to obtain the infinity norm bound for the inverse of $GSDD_1^+$ matrices.

THEOREM 6. *Let $M = (m_{ij}) \in \mathbb{C}^{n\times n}$ be an $GSDD_1^+$ matrix. Then*

$$\left\|M^{-1}\right\|_\infty \leqslant \frac{\max\left\{1, \max\limits_{j\in N_2}\left(\frac{pR_j(M)}{|m_{jj}|} + \varepsilon\right)\right\}}{\min\left\{\min\limits_{j\in N_1} G_j, \min\limits_{j\in N_2} Y_j\right\}}, \tag{11}$$

where

$$G_j = \left|m_{jj}\right| - \sum_{t\in N_1\setminus\{j\}} \left|m_{jt}\right| - \varepsilon \sum_{t\in N_2} \left|m_{jt}\right| - \sum_{t\in N_2} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|}, \ \forall j \in N_1, \tag{12}$$

$$\begin{aligned} Y_j = pR_j(M) &- \sum_{t\in N_1} \left|m_{jt}\right| - \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|} \\ &+\varepsilon\left(\left|m_{jj}\right| - \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right|\right), \ \forall j \in N_2, \end{aligned} \tag{13}$$

and

$$0 < \varepsilon < \min_{j\in N_1} \left(\frac{\left|m_{jj}\right| - \sum\limits_{t\in N_2} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|} - \sum\limits_{t\in N_1\setminus\{j\}} \left|m_{jt}\right|}{\sum\limits_{t\in N_2} \left|m_{jt}\right|} \right). \tag{14}$$

Proof. By Theorem 3, we know that there exists a positive diagonal matrix D such that MD is an SDD matrix. Hence, it holds that

$$\left\|M^{-1}\right\|_\infty = \left\|D\left(D^{-1}M^{-1}\right)\right\|_\infty = \left\|D(MD)^{-1}\right\|_\infty \leqslant \|D\|_\infty \left\|(MD)^{-1}\right\|_\infty. \tag{15}$$

Since D is a diagonal matrix, then

$$\|D\|_\infty = \max_{1\leqslant j\leqslant n} d_j = \max\left\{1, \max_{j\in N_2}\left(\varepsilon + \frac{pR_j(M)}{|m_{jj}|}\right)\right\}, \tag{16}$$

where ε is given by (14). Note that MD is an SDD matrix, by Theorem 4, we have

$$\left\|(MD)^{-1}\right\|_\infty \leqslant \frac{1}{\min\limits_{i\in N}\{|(MD)_{ii}| - r_i(MD)\}}.$$

Let us consider the following two cases:

Case 1. If $j \in N_1$, then we have

$$\begin{aligned}
&\left|(MD)_{jj}\right| - r_j(MD) \\
&= \left|m_{jj}\right| - \left(\sum_{t\in N_1\setminus\{j\}} \left|m_{jt}\right| + \varepsilon \sum_{t\in N_2} \left|m_{jt}\right| + \sum_{t\in N_2} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|}\right) \\
&= \left|m_{jj}\right| - \sum_{t\in N_1\setminus\{j\}} \left|m_{jt}\right| - \varepsilon \sum_{t\in N_2} \left|m_{jt}\right| - \sum_{t\in N_2} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|} \\
&= G_j.
\end{aligned}$$

Case 2. If $j \in N_2$, then we get

$$\begin{aligned}
&\left|(MD)_{jj}\right| - r_j(MD) \\
&= \left|m_{jj}\right| \left(\frac{pR_j(M)}{\left|m_{jj}\right|} + \varepsilon\right) - \left(\sum_{t\in N_1} \left|m_{jt}\right| + \varepsilon \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| + \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|}\right) \\
&= pR_j(M) + \varepsilon \left|m_{jj}\right| - \sum_{t\in N_1} \left|m_{jt}\right| - \varepsilon \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| - \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|} \\
&= pR_j(M) - \sum_{t\in N_1} \left|m_{jt}\right| - \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|} + \varepsilon\left(\left|m_{jj}\right| - \sum_{t\in N_2\setminus\{j\}} \left|m_{jt}\right|\right) \\
&= Y_j.
\end{aligned}$$

Hence, by (15) and (16), (11) holds. □

Next, we give some numerical examples to illustrate the effectiveness of new results.

EXAMPLE 7. Consider the following matrix:

$$M_7 = \begin{pmatrix} 7.8 & -4.8 & 2 & -3 \\ -3 & 8.9 & -6 & 2 \\ -4 & -1 & 8 & 0 \\ -3 & 1 & -2 & 10 \end{pmatrix}.$$

It is easy to verify that M_7 is a $GSDD_1^+$ matrix. However, M_7 is not a $GSDD_1$ matrix nor a CKV-type matrix. By Theorem 6, we have

$$\left\|M_7^{-1}\right\|_\infty \leqslant \frac{\max\{1, 0.6385\}}{\min\{0.1075, 0.1080\}} = 9.3023 \quad (\varepsilon = 0.0135), \quad \varepsilon \in (0, 0.0350).$$

In Example 7, the range of values for the infinity norm bound and its optimal solution can be seen from the Figure 3.

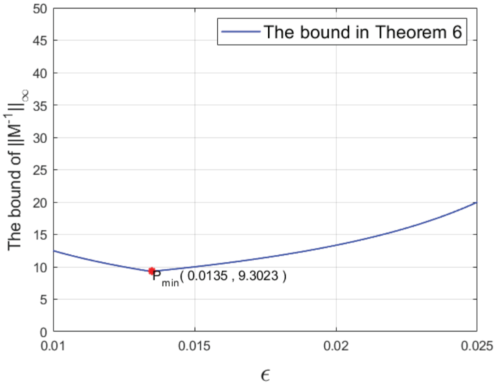

Figure 3: *The bound in Theorem 6*

EXAMPLE 8. Consider the following matrix:

$$M_8 = \begin{pmatrix} 32.54 & 5.25 & 5.25 & 2.25 & 7.5 & 5.25 & 0.75 & 2.25 \\ 3.75 & 31.79 & 4.5 & 1.5 & 7.5 & 6 & 2.25 & 3.75 \\ 3.75 & 6.75 & 31.04 & 2.25 & 3.75 & 2.25 & 6.75 & 7.5 \\ 3 & 4.5 & 3 & 32.54 & 1.5 & 1.5 & 0.75 & 4.5 \\ 4.5 & 3 & 3.75 & 3 & 31.04 & 2.25 & 7.5 & 4.5 \\ 4.5 & 7.5 & 2.25 & 7.5 & 3.75 & 31.79 & 6 & 2.25 \\ 6.75 & 6.75 & 6.75 & 3.75 & 4.5 & 3.75 & 32.54 & 3.75 \\ 6 & 4.5 & 1.5 & 1.5 & 2.25 & 4.5 & 4.5 & 34.04 \end{pmatrix}.$$

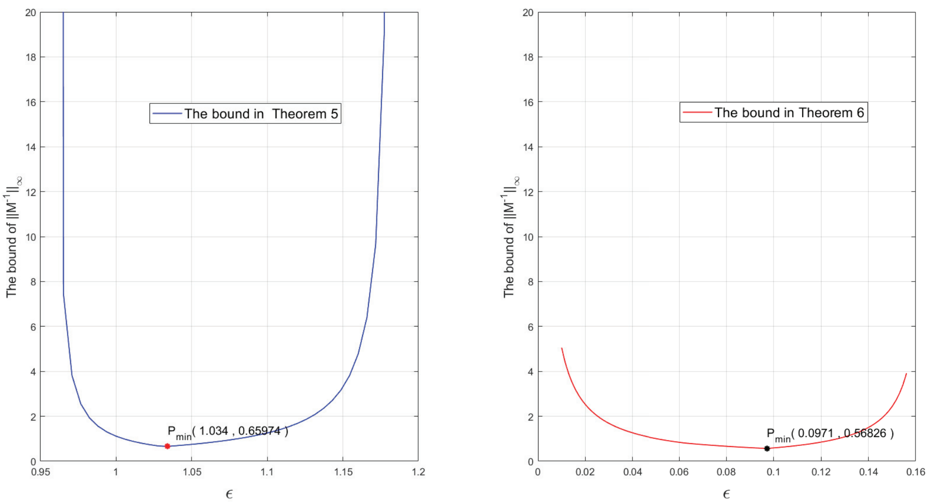

Figure 4: *The bound in Theorem 5 and The bound in Theorem 6*

Then $N_1 = \{3,6,7\}$ and $N_2 = \{1,2,4,5,8\}$. It is easy to verify that M_8 is a $GSDD_1^+$ matrix and a $GSDD_1$ matrix. By Theorem 5, we get

$$\left\|M_8^{-1}\right\|_\infty \leqslant 0.65947 \quad (\varepsilon = 1.0340), \quad \varepsilon \in (0.9592, 1.1833).$$

However, by Theorem 6, we have

$$\left\|M_8^{-1}\right\|_\infty \leqslant 0.56826 \quad (\varepsilon = 0.0971), \quad \varepsilon \in (0, 0.1661).$$

It is also shown by Fig. 4 that the optimal bound of $||M^{-1}||_\infty$ in Theorem 6 on $\varepsilon \in (0, 0.1661)$ is attained at $\varepsilon = 0.0971$.

4. Error bound of the LCP associated with $GSDD_1^+$ matrices

The linear complementarity problem (LCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$(Mx+z)^T x = 0, \qquad Mx+z \geqslant 0, \qquad x \geqslant 0,$$

or to show that no such vector x exists, where $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ and $z \in \mathbb{R}^n$. It is well known that the LCP has a unique solution for any $z \in \mathbb{R}^n$ if and only if M is a P-matrix. Here, a matrix $M \in \mathbb{R}^{n\times n}$ is called a P-matrix if all its principal minors are positive [16]. In 2006, Chen et al. [3] gave the following result for the LCP when M is a P-matrix:

$$\|x - x^*\|_\infty \leqslant \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty \quad for\ any\ x \in \mathbb{R}^n,$$

where x^* is the solution of the LCP, $r(x) = min\{x, Mx+z\}$, $D = diag(d_i)$ with $0 \leqslant d_i \leqslant 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. When the matrix M for the LCP belongs to P-matrices or some subclass of P-matrices, various bounds are established [2, 5, 9–11, 14, 15, 21].

THEOREM 7. [10] *Suppose that $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ is a $GSDD_1$ matrix with positive diagonal entries, then*

$$\max_{d\in[0,1]^n} \left\|(I - D + DM)^{-1}\right\|_\infty \leqslant \max\left\{ \frac{\max\left\{\max\limits_{i\in N_2} \frac{r_i(M)}{|m_{ii}|}, \varepsilon\right\}}{\min\left\{\min\limits_{i\in N_2} \phi_i, \min\limits_{i\in N_1} \psi_i, \right\}}, \frac{\max\left\{\max\limits_{i\in N_2} \frac{r_i(M)}{|m_{ii}|}, \varepsilon\right\}}{\min\left\{\min\limits_{i\in N_2} \frac{r_i(M)}{|m_{ii}|}, \varepsilon\right\}} \right\},$$

where ϕ_i, ψ_i and ε are defined as in (8), (9), (10), *respectively.*

THEOREM 8. [11] *Assume that $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ is an H-matrix with positive diagonal entries. Let $D = diag(d_j)$, $d_j > 0$, for all $i \in N$, be a diagonal matrix such* *that MD is strictly diagonally dominant by rows. For any $j \in N$, let $\beta_j := m_{jj}d_j - \sum_{j\neq t} |m_{jt}|d_t$. Then*

$$\max_{d\in[0,1]^n} \left\|(I-D+DM)^{-1}\right\|_\infty \leqslant \max\left\{\frac{\max\limits_{j\in N}\{d_j\}}{\min\limits_{j\in N}\{\beta_j\}}, \frac{\max\limits_{j\in N}\{d_j\}}{\min\limits_{j\in N}\{d_j\}}\right\}.$$

Next, error bound for the linear complementarity problem of $GSDD_1^+$ matrices is given by using the positive diagonal matrix D in Theorem 3.

THEOREM 9. *Assume that $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ is a $GSDD_1^+$ matrix with positive diagonal entries, then*

$$\max_{d\in[0,1]^n} \left\|(I-D+DM)^{-1}\right\|_\infty$$
$$\leqslant \max\left\{\frac{\max\left\{1, \max\limits_{j\in N_2}\left(\frac{pR_j(M)}{|m_{jj}|}+\varepsilon\right)\right\}}{\min\left\{\min\limits_{j\in N_1} G_j, \min\limits_{j\in N_2} Y_j\right\}}, \frac{\max\left\{1, \max\limits_{j\in N_2}\left(\frac{pR_j(M)}{|m_{jj}|}+\varepsilon\right)\right\}}{\min\left\{1, \min\limits_{j\in N_2}\frac{pR_j(M)}{|m_{jj}|}\right\}}\right\},$$

where G_j, Y_j and ε are defined as in (12), (13), (14), *respectively.*

Proof. Since matrix M is a $GSDD_1^+$ matrix with positive diagonal elements, then by Theorem 3, we know that there exists a positive diagonal matrix D such that MD is an SDD matrix. For $j \in N$, by Theorem 8, we get

$$\begin{aligned}\beta_j &= \left|(MD)_{jj}\right| - \sum_{t\in N\setminus\{j\}} \left|(MD)_{jt}\right| \\ &= \left|(MD)_{jj}\right| - \left(\sum_{t\in N_1\setminus\{j\}} \left|(MD)_{jt}\right| + \sum_{t\in N_2\setminus\{j\}} \left|(MD)_{jt}\right|\right) \\ &= \left|(MD)_{jj}\right| - \sum_{t\in N_1\setminus\{j\}} \left|(MD)_{jt}\right| - \sum_{t\in N_2\setminus\{j\}} \left|(MD)_{jt}\right|.\end{aligned}$$

Let us consider the following two cases:

Case 1. For any $j \in N_1$, we obtain that

$$\begin{aligned}\left|(MD)_{jj}\right| - r_j(MD) &= \left|m_{jj}\right| - \left(\sum_{t\in N_1\setminus\{j\}} \left|m_{jt}\right| + \varepsilon\sum_{t\in N_2} \left|m_{jt}\right| + \sum_{t\in N_2} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|}\right) \\ &= \left|m_{jj}\right| - \sum_{t\in N_1\setminus\{j\}} \left|m_{jt}\right| - \varepsilon\sum_{t\in N_2} \left|m_{jt}\right| - \sum_{t\in N_2} \left|m_{jt}\right| \frac{pR_t(M)}{|m_{tt}|}.\end{aligned}$$

Case 2. For any $j \in N_2$, we have

$$\begin{aligned}
&\left|(MD)_{jj}\right| - r_j(MD)\\
=&\left|m_{jj}\right|\left(\frac{pR_j(M)}{\left|m_{jj}\right|}+\varepsilon\right)-\left(\sum_{t\in N_1}\left|m_{jt}\right|+\varepsilon\sum_{t\in N_2\setminus\{j\}}\left|m_{jt}\right|+\sum_{t\in N_2\setminus\{j\}}\left|m_{jt}\right|\frac{pR_t(M)}{|m_{tt}|}\right)\\
=&pR_j(M)+\varepsilon\left|m_{jj}\right|-\sum_{t\in N_1}\left|m_{jt}\right|-\varepsilon\sum_{t\in N_2\setminus\{j\}}\left|m_{jt}\right|-\sum_{t\in N_2\setminus\{j\}}\left|m_{jt}\right|\frac{pR_t(M)}{|m_{tt}|}\\
=&pR_j(M)-\sum_{t\in N_1}\left|m_{jt}\right|-\sum_{t\in N_2\setminus\{j\}}\left|m_{jt}\right|\frac{pR_t(M)}{|m_{tt}|}+\varepsilon\left(\left|m_{jj}\right|-\sum_{t\in N_2\setminus\{j\}}\left|m_{jt}\right|\right).
\end{aligned}$$

To sum up, it can be seen that

$$\beta_j = \begin{cases} G_j, & j \in N_1, \\ Y_j, & j \in N_2. \end{cases}$$

Thus, by Theorem 8, the conclusion follows. □

The following examples show that new bound in Theorem 9 is better than that in Theorem 7 in some cases.

EXAMPLE 9. Consider the matrix:

$$M_9 = \begin{pmatrix} 39.8 & 1.3 & -2.1 & -1 & 2.2 \\ 0 & 10.02 & 4.1 & 3.9 & -6 \\ 19.9 & -2 & 33 & -4 & 8 \\ 0 & 4 & -6 & 20 & -2 \\ -30 & -4 & 2 & 0 & 39.8 \end{pmatrix}.$$

It is easy to verify that M_9 is a $GSDD_1^+$ matrix but not a $GSDD_1$ matrix. By Theorem 9, we have

$$\max_{d\in[0,1]^5}\left\|(I-D+DM_9)^{-1}\right\|_\infty \leqslant 27.0525 \quad (\varepsilon = 0.0021), \quad \varepsilon \in (0, 0.0058).$$

EXAMPLE 10. Consider the tri-diagonal matrix $M \in \mathbb{R}^{n\times n}$ arising from the finite difference method for free boundary problems , where

$$M_{10} = \begin{pmatrix} b+\alpha\sin\left(\frac{1}{n}\right) & c & 0 & \cdots & 0 \\ a & b+\alpha\sin\left(\frac{2}{n}\right) & c & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & a & b+\alpha\sin\left(\frac{n-1}{n}\right) & c \\ 0 & \cdots & 0 & a & b+\alpha\sin(1) \end{pmatrix}.$$

Take that $n=60$, $\alpha=15.3416$, $a=5.5893$, $b=15.61$, $c=10.89$. It is easy to verify that M_{10} is not a $GSDD_1$ matrix. But, M_{10} is a $GSDD_1^+$ matrix. By Theorem 9, we get

$$\max_{d\in[0,1]^{60}} \left\|(I-D+DM_{10})^{-1}\right\|_\infty \leqslant 9.2009 \quad (\varepsilon=0.0072), \quad \varepsilon\in(0,0.0172).$$

In the following Figure 5, we get the minimum value 9.2009 when the minimum point $\varepsilon=0.0072$.

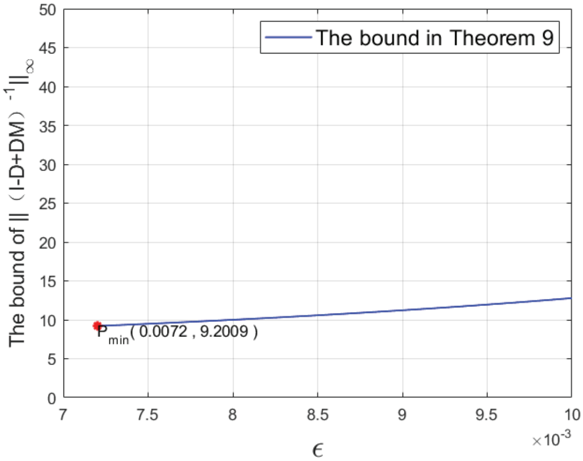

Figure 5: *The bound in Theorem 9*

EXAMPLE 11. Consider the matrix:

$$M_{11}=\begin{pmatrix} 36.75 & 5.25 & 4.5 & 1.5 & 2.25 & 4.5 & 5.25 & 3 & 0.75 & 3.75 \\ 6.75 & 31.5 & 6 & 7.5 & 3 & 3.75 & 2.25 & 6 & 2.25 & 3.75 \\ 1.5 & 2.25 & 36 & 2.25 & 0.75 & 4.5 & 4.5 & 3.75 & 4.5 & 6 \\ 0.75 & 2.25 & 7.5 & 37.5 & 4.5 & 6.75 & 1.5 & 0.75 & 3.75 & 3.75 \\ 0.75 & 1.5 & 1.5 & 5.25 & 37.5 & 1.5 & 3.75 & 3 & 1.5 & 0.75 \\ 1.5 & 3 & 6 & 6.75 & 6.75 & 33.75 & 0.75 & 3.75 & 2.25 & 0.75 \\ 1.5 & 6.75 & 1.5 & 3 & 0.75 & 0.75 & 32.25 & 6.75 & 6 & 4.5 \\ 5.25 & 2.25 & 0.75 & 2.25 & 5.25 & 3 & 2.25 & 33 & 5.25 & 3.75 \\ 0.75 & 6 & 1.5 & 1.5 & 3 & 6 & 4.5 & 6.75 & 31.5 & 7.5 \\ 3.75 & 4.5 & 1.5 & 4.5 & 1.5 & 0.75 & 2.25 & 4.5 & 6 & 30.75 \end{pmatrix}.$$

It is easy to verify that M_{11} is a $GSDD_1$ matrix and a $GSDD_1^+$ matrix. By Theorem 7, we get

$$\max_{d\in[0,1]^{10}} \left\|(I-D+DM_{11})^{-1}\right\|_\infty \leqslant 3.7637 \quad (\varepsilon=1.15), \quad \varepsilon\in(1.1360,1.1740).$$

However, by Theorem 9, we have

$$\max_{d\in[0,1]^{10}} \left\|(I-D+DM_{11})^{-1}\right\|_\infty = 2.6267 \quad (\varepsilon = 0.0282), \quad \varepsilon \in (0, 0.0380).$$

It is shown by Figure 6, in which the first 10000 matrices are given by the following MATLAB codes, that 2.6267 is better than 3.7637 for $\max\|(I-D+DM_{11})^{-1}\|_\infty$. Blue stars in Figure 6 represent the $\|(I-D+DM_{11})^{-1}\|_\infty$ when matrices D come from 10000 different random matrices in $[0,1]$.

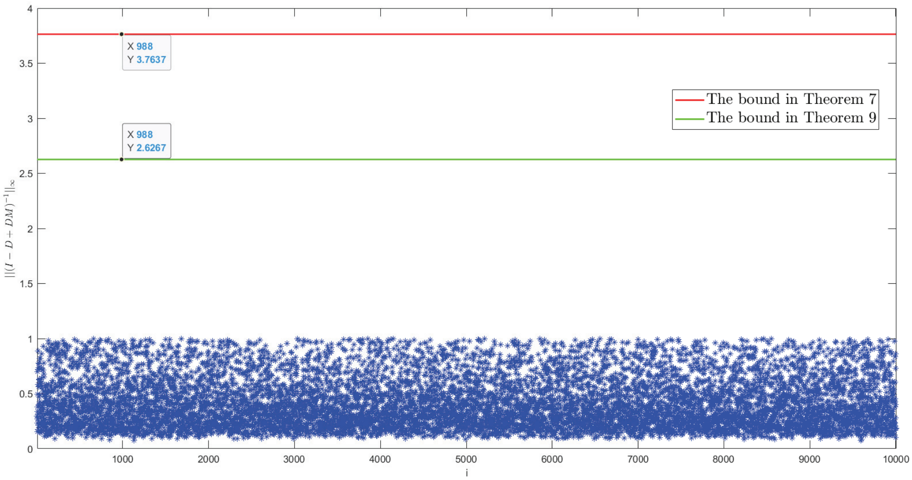

Figure 6: $\|(I-D+DM)^{-1}\|_\infty$ *for the first* 10000 *matrices* D *generated by diag(rand (10,1)). MATLAB codes: for* $i = 1:10000$*;* $D =$*diag(rand (10,1)); end.*

Conclusions

In this paper, a new subclass of H-matrices named $GSDD_1^+$ matrices is introduced. Some properties of $GSDD_1^+$ matrices are discussed, and the relationships between $GSDD_1^+$ matrices, $GSDD_1$ matrices and CKV-type matrices are analyzed. A scaling matrix D is constructed to transform $GSDD_1^+$ matrices into SDD matrices, which ensure the nonsingularity of $GSDD_1^+$ matrices. Moreover, we provide the infinity norm upper bound of the inverse for $GSDD_1^+$ matrices, and the error bounds of LCP of $GSDD_1^+$ matrices are also given. The validity of the obtained results is demonstrated by numerical examples. In the future, based on the proposed infinity norm bounds, we will explore the computable global error bounds of the extended vertical linear complementarity problems for $GSDD_1^+$ matrices.

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