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A NEW VIEW ON SOME TWO–SIDED INEQUALITIES FOR FRAMES IN HILBERT SPACES

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Abstract. In this paper, we present a new approach to the proofs of several two-sided inequalities for frames in Hilbert spaces obtained recently by Xiang et al. from the point of view of function theory, which greatly simplifies the proving process. We also provide an improvement to two results of them on this topic. Finally, we establish a new two-sided inequality for frames in Hilbert spaces.

1. Introduction

Let $\mathbb{J}$ be an index set and $\mathscr{N}$ be a Hilbert space. A family $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ in $\mathscr{N}$ is said to be a *frame* for $\mathscr{N}$, if there are $0 < C_\Phi \leqslant D_\Phi < \infty$ such that

$$C_\Phi \|x\|^2 \leqslant \sum_{j\in\mathbb{J}} |\langle x, \phi_j\rangle|^2 \leqslant D_\Phi \|x\|^2, \quad \forall x \in \mathscr{N}. \tag{1}$$

In 1950s, Duffin and Schaeffer [6] offered us a useful tool, namely frames, to process some profound problems deriving in nonharmonic Fourier series. Daubechies et al. in 1980s brought frames back to people's vision, thanks to their groundbreaking work on wavelets [5]. Owing to their key properties, such as redundancy and the flexibility offered by non-unique decompositions, frames have already been applied to many research fields (see [2, 4, 11, 13] for example). We refer also to [3] for more details about frame theory.

Given a frame $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ for $\mathscr{N}$, we can define a linear bounded operator U_Φ, called the *analysis operator* of Φ, in the following way

$$U_\Phi : \mathscr{N} \to \ell^2(\mathbb{J}), \quad U_\Phi x = \{\langle x, \phi_j\rangle\}_{j\in\mathbb{J}}. \tag{2}$$

Further, the so-called *frame operator* S_Φ of Φ, a self-adjoint and invertible operator, can be obtained if we take a compositional operation on U_Φ^* and U_Φ:

$$S_\Phi : \mathscr{N} \to \mathscr{N}, \quad S_\Phi x = U_\Phi^* U_\Phi x = \sum_{j\in\mathbb{J}} \langle x, \phi_j\rangle \phi_j, \tag{3}$$

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which thereby leads to the famous *reconstruction formula*:

$$x = \sum_{j\in\mathbb{J}} \langle S_\Phi^{-1}x, \phi_j\rangle \phi_j = \sum_{j\in\mathbb{J}} \langle x, \phi_j\rangle S_\Phi^{-1}\phi_j, \quad \forall x \in \mathscr{N}. \tag{4}$$

For any $\sigma \subset \mathbb{J}$, there are also two self-adjoint operators related to σ, σ^c (where $\sigma^c = \mathbb{J}\backslash\sigma$ denotes the complement of σ) and Φ, given below

$$S_\Phi^\sigma, S_\Phi^{\sigma^c} : \mathscr{N} \to \mathscr{N}, \quad S_\Phi^\sigma x = \sum_{j\in\sigma} \langle x, \phi_j\rangle \phi_j, \quad S_\Phi^{\sigma^c} x = \sum_{j\in\sigma^c} \langle x, \phi_j\rangle \phi_j. \tag{5}$$

It is clear that $S_\Phi^\sigma + S_\Phi^{\sigma^c} = S_\Phi$.

Let $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ and $\Psi = \{\psi_j\}_{j\in\mathbb{J}}$ be frames for $\mathscr{N}$. One calls Ψ an *alternate dual frame* of Φ (and vice versa), if for each $x \in \mathscr{N}$ we have

$$x = \sum_{j\in\mathbb{J}} \langle x, \psi_j\rangle \phi_j = \sum_{j\in\mathbb{J}} \langle x, \phi_j\rangle \psi_j, \quad \forall x \in \mathscr{N}. \tag{6}$$

Let $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ be a frame for $\mathscr{N}$ and $\Psi = \{\psi_j\}_{j\in\mathbb{J}}$ be an alternate dual frame of Φ. Then associated with any $\sigma \subset \mathbb{J}$ and the pair (Φ, Ψ), there are two linear bounded operators $W^\sigma, W^{\sigma^c} : \mathscr{N} \to \mathscr{N}$ defined by

$$W^\sigma x = \sum_{j\in\sigma} \langle x, \psi_j\rangle \phi_j, \quad W^{\sigma^c} x = \sum_{j\in\sigma^c} \langle x, \psi_j\rangle \phi_j, \quad \forall x \in \mathscr{N}. \tag{7}$$

When further exploring the well-known identity induced by Parseval frames, Balan et al. [1] showed us an interesting inequality for this type of frames, which was later extended to general frames [8]. In recent years, many scholars devoted themselves to the study of inequalities for frames and generalized frames and particularly, Poria in [12] offered us a novel class of one-sided inequalities for Hilbert-Schmidt frames with a parameter involved. By extracting the idea of Poria, Li and Leng in [9] and Xiang in [14] presented some two-sided inequalities associated with a parameter for fusion frames. Though two-sided inequalities for some other frame versions do subsequently emerge (see [7, 10, 16] for example), they are of the same form as those in [9]. Given this, the authors in [15] devoted their efforts to investigating inequalities with new forms for frames, and several two-sided inequalities admitting novel structures compared to existing ones for frames and generalized frames were established.

The following results were stated in [15] respectively as Theorems 2.2, 2.5, 2.6, 2.7 and 2.8.

THEOREM 1. *Suppose that $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathscr{N}$. Then for every $\lambda \geqslant 1$, for any $x \in \mathscr{N}$ and any $\sigma \subset \mathbb{J}$, we get*

$$\begin{aligned}\sum_{j\in\sigma} |\langle x, \phi_j\rangle|^2 - \lambda \sum_{j\in\sigma^c} |\langle x, \phi_j\rangle|^2 &\leqslant \sum_{j\in\mathbb{J}} |\langle S_\Phi^{-1} S_\Phi^\sigma x, \phi_j\rangle|^2 - \lambda \sum_{j\in\mathbb{J}} |\langle S_\Phi^{-1} S_\Phi^{\sigma^c} x, \phi_j\rangle|^2 \\ &\leqslant (\lambda^3 - \lambda^2 + 1) \sum_{j\in\sigma} |\langle x, \phi_j\rangle|^2 \\ &\quad + (\lambda^3 - 3\lambda^2 + 2\lambda - 1) \sum_{j\in\sigma^c} |\langle x, \phi_j\rangle|^2.\end{aligned} \tag{8}$$

THEOREM 2. *Suppose that $\Phi=\{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{N}$. Then for each $\lambda\geqslant\frac{1}{2}$, for any $x\in\mathcal{N}$ and any $\sigma\subset\mathbb{J}$, we have*

$$\begin{aligned}(4\lambda-1)\sum_{j\in\mathbb{J}}|\langle S_\Phi^{-1}S_\Phi^{\sigma^c}x,\phi_j\rangle|^2+(1-\lambda^2)\sum_{j\in\mathbb{J}}|\langle x,\phi_j\rangle|^2\\ \leqslant\sum_{j\in\sigma}|\langle x,\phi_j\rangle|^2+(1+2\lambda)\sum_{j\in\sigma^c}|\langle x,\phi_j\rangle|^2\\ \leqslant\sum_{j\in\mathbb{J}}|\langle S_\Phi^{-1}S_\Phi^{\sigma^c}x,\phi_j\rangle|^2+(1+\lambda^2)\sum_{j\in\mathbb{J}}|\langle x,\phi_j\rangle|^2.\end{aligned}\tag{9}$$

THEOREM 3. *Suppose that $\Phi=\{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{N}$. Then for each $\lambda\in\mathbb{R}$, for any $x\in\mathcal{N}$ and any $\sigma\subset\mathbb{J}$, we obtain*

$$\begin{aligned}(1+2\lambda)\sum_{j\in\sigma^c}|\langle x,\phi_j\rangle|^2-(1+\lambda^2)\sum_{j\in\mathbb{J}}|\langle x,\phi_j\rangle|^2\\ \leqslant\sum_{j\in\mathbb{J}}|\langle S_\Phi^{-1}S_\Phi^{\sigma^c}x,\phi_j\rangle|^2-\sum_{j\in\sigma}|\langle x,\phi_j\rangle|^2\\ \leqslant(3-2\lambda)\sum_{j\in\sigma^c}|\langle x,\phi_j\rangle|^2+(\lambda^2-1)\sum_{j\in\mathbb{J}}|\langle x,\phi_j\rangle|^2.\end{aligned}\tag{10}$$

THEOREM 4. *Suppose that $\Phi=\{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{N}$ and that $\Psi=\{\psi_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of Φ. Then for every $\lambda\in[0,1]$, for any $x\in\mathcal{N}$ and any $\sigma\subset\mathbb{J}$, we have*

$$\begin{aligned}(\lambda-\lambda^2)\Big\|\sum_{j\in\mathbb{J}}\langle x,\psi_j\rangle\phi_j\Big\|^2-\lambda\operatorname{Re}\sum_{j\in\sigma^c}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\ \leqslant\Big\|\sum_{j\in\sigma}\langle x,\psi_j\rangle\phi_j\Big\|^2-\lambda\operatorname{Re}\sum_{j\in\sigma}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\ \leqslant\frac{\lambda(\|W^\sigma-W^{\sigma^c}\|^2-1)+4(1-\lambda)\|W^\sigma\|^2}{4}\|x\|^2.\end{aligned}$$

THEOREM 5. *Suppose that $\Phi=\{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{N}$ and that $\Psi=\{\psi_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of Φ. Then for every $\lambda\in[0,\frac{1}{2}]$, for any $x\in\mathcal{N}$ and any $\sigma\subset\mathbb{J}$, we have*

$$\begin{aligned}(2\lambda-\lambda^2)\Big\|\sum_{j\in\mathbb{J}}\langle x,\psi_j\rangle\phi_j\Big\|^2\leqslant\Big\|\sum_{j\in\sigma}\langle x,\psi_j\rangle\phi_j\Big\|^2+2\lambda\operatorname{Re}\sum_{j\in\sigma^c}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\ \leqslant\frac{3\lambda+2(1-2\lambda)\|W^\sigma\|^2+\lambda\|W^\sigma-W^{\sigma^c}\|^2}{2}\|x\|^2.\end{aligned}$$

We observe that the proofs of Theorems 1–3 depend entirely on a result (namely, Lemma 2.1 in [15]) on linear bounded operators and the relationship of the operators related to the frames, making the process lengthy. In this paper, we present a new approach to the proofs of Theorems 1–3 from the perspective of function theory. Our new proofs are more direct and avoid the technical operator-theoretic lemma in [15], thereby providing a clearer and more unified methodology. As for Theorems 4 and 5, we found that there are two areas that need improvement. First, the two inequalities on the left hold for any parameter belonging to the set of real numbers, meaning that the involved intervals $[0,1]$ and $[0,\frac{1}{2}]$ are redundant conditions for them. So the left-hand terms in Theorems 4 and 5 should be replaced by new ones in order that the left-hand inequalities can be truly determined by the intervals where the parameter is taken from. Second, the intervals $[0,1]$ and $[0,\frac{1}{2}]$ respectively in Theorems 4 and 5 can be extended to larger ones. In later section we provide new expressions for those two theorems so that the above-mentioned two defects can be repaired. Moreover, a new two-sided inequality for frames in Hilbert spaces is established at the end of the paper.

2. New proofs, improved results and a new two-sided inequality

2.1. New proofs of Theorems 1, 2 and 3

2.1.1. Proof of Theorem 1

Let us denote $Q = U_\Phi S_\Phi^{-1} U_\Phi^*$. Then, Q is the orthogonal projection from $\ell^2(\mathbb{J})$ onto Range(U_Φ). For any $\sigma \subset \mathbb{J}$, we denote by P_σ the orthogonal projection on $\ell^2(\mathbb{J})$ given by

$$P_\sigma(\{y_i\}_{i\in\mathbb{J}}) = \{z_i\}_{i\in\mathbb{J}}, \quad \text{where} \quad \begin{cases} z_i = y_i & \text{if } i \in \sigma, \\ z_i = 0 & \text{if } i \in \sigma^c. \end{cases}$$

Since $S_\Phi^\sigma = U_\Phi^* P_\sigma U_\Phi$ and $S_\Phi^{\sigma^c} = U_\Phi^* P_\sigma^\perp U_\Phi$, the inequalities stated in (8) can be rewritten as

$$\begin{aligned} &\|P_\sigma U_\Phi x\|^2 - \lambda\|P_\sigma^\perp U_\Phi x\|^2 \\ &\quad \leqslant \|QP_\sigma U_\Phi x\|^2 - \lambda\|QP_\sigma^\perp U_\Phi x\|^2 \\ &\quad \leqslant (\lambda^3 - \lambda^2 + 1)\|P_\sigma U_\Phi x\|^2 + (\lambda^3 - 3\lambda^2 + 2\lambda - 1)\|P_\sigma^\perp U_\Phi x\|^2 \end{aligned} \tag{11}$$

for each $x \in \mathcal{N}$. Denote by $y = U_\Phi x$ and normalize it to $\|y\| = 1$. Then $Qy = y$. Since

$$\|P_\sigma y\|^2 - \|QP_\sigma y\|^2 = \|Q^\perp P_\sigma y\|^2 = \|Q^\perp P_\sigma^\perp y\|^2 = \|P_\sigma^\perp y\|^2 - \|QP_\sigma^\perp y\|^2, \tag{12}$$

it follows that

$$\begin{aligned} \|P_\sigma U_\Phi x\|^2 - \|QP_\sigma U_\Phi x\|^2 = \|P_\sigma y\|^2 - \|QP_\sigma y\|^2 &\leqslant \lambda(\|P_\sigma^\perp y\|^2 - \|QP_\sigma^\perp y\|^2) \\ &= \lambda(\|P_\sigma^\perp U_\Phi x\|^2 - \|QP_\sigma^\perp U_\Phi x\|^2) \end{aligned}$$

for each $\lambda \geqslant 1$, equivalently,

$$\|P_\sigma U_\Phi x\|^2 - \lambda\|P_\sigma^\perp U_\Phi x\|^2 \leqslant \|QP_\sigma U_\Phi x\|^2 - \lambda\|QP_\sigma^\perp U_\Phi x\|^2.$$

For the inequality on the right in (11), we know, by combining (11) with (12), that it is equivalent to

$$(1-\lambda)\|QP_\sigma^\perp y\|^2 \leqslant (1-\lambda)[2\lambda\|P_\sigma^\perp y\|^2 - \lambda^2]. \tag{13}$$

To show (13), it is equivalent to prove that, for $\lambda \geqslant 1$, the quadratic function $f(\lambda) = \lambda^2 - 2\lambda\|P_\sigma^\perp y\|^2 + \|QP_\sigma^\perp y\|^2 \geqslant 0$. Since the minimizer for $f(\lambda)$ is in $\lambda_0 = \|P_\sigma^\perp y\|^2$, and $\|P_\sigma^\perp y\|^2 = \langle QP_\sigma^\perp y, y\rangle \leqslant \|QP_\sigma^\perp y\|$, we arrive at

$$f(\lambda) \geqslant f(\lambda_0) = (\|QP_\sigma^\perp y\| - \|P_\sigma^\perp y\|^2)(\|QP_\sigma^\perp y\| + \|P_\sigma^\perp y\|^2) \geqslant 0,$$

as desired. □

With the help of the notations introduced in the proof of Theorem 1, we also offer completely new proofs for Theorems 2 and 3.

2.1.2. Proof of Theorem 2

Letting $y = U_\Phi x$ and normalize it to $\|y\| = 1$ for each $x \in \mathscr{N}$. Then we can rewrite the inequalities in (9) as

$$\begin{aligned}(4\lambda - 1)\|QP_\sigma^\perp y\|^2 + (1-\lambda^2) &\leqslant \|P_\sigma y\|^2 + (1+2\lambda)\|P_\sigma^\perp y\|^2 \\ &\leqslant \|QP_\sigma^\perp y\|^2 + (1+\lambda^2)\end{aligned} \tag{14}$$

for every $\lambda \geqslant \frac{1}{2}$. It is easy to see that the inequality on the left is equivalent to prove that, for $\lambda \geqslant \frac{1}{2}$,

$$g(\lambda) = \lambda^2 + 2\lambda(\|P_\sigma^\perp y\|^2 - 2\|QP_\sigma^\perp y\|^2) + \|QP_\sigma^\perp y\|^2 \geqslant 0,$$

which, actually, follows from the fact that

$$\begin{aligned}g(\lambda) &\geqslant \lambda^2 - 2\lambda\|P_\sigma^\perp y\|^2 + \|QP_\sigma^\perp y\|^2 \\ &= (\lambda - \|P_\sigma^\perp y\|^2)^2 + (\|QP_\sigma^\perp y\|^2 - \|P_\sigma^\perp y\|^4),\end{aligned}$$

and that $\|P_\sigma^\perp y\|^2 \leqslant \|QP_\sigma^\perp y\|$ as demonstrated in the proof of Theorem 1.

As for the right-hand inequality in (14), it is equivalent to show that

$$\begin{aligned}0 \leqslant h(\lambda) &= (1+\lambda^2) + \|QP_\sigma^\perp y\|^2 - \|P_\sigma y\|^2 - (1+2\lambda)\|P_\sigma^\perp y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\sigma^\perp y\|^2 + (1 - \|P_\sigma y\|^2 - \|P_\sigma^\perp y\|^2) + \|QP_\sigma^\perp y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\sigma^\perp y\|^2 + \|QP_\sigma^\perp y\|^2,\end{aligned}$$

which is the case as shown in the proof of Theorem 1, and we are done. □

2.1.3. Proof of Theorem 3

For any $x \in \mathscr{N}$, taking $y = U_\Phi x$ and normalize it to $\|y\| = 1$, then the inequalities in (10) can be rewritten as

$$\begin{aligned}(1+2\lambda)\|P_\sigma^\perp y\|^2 - (1+\lambda^2) &\leqslant \|QP_\sigma^\perp y\|^2 - \|P_\sigma y\|^2 \\ &\leqslant (3-2\lambda)\|P_\sigma^\perp y\|^2 + (\lambda^2 - 1).\end{aligned} \tag{15}$$

The left-hand inequality in (15) follows from the following calculation

$$\begin{aligned}(1+\lambda^2)&+\|QP_\sigma^\perp y\|^2-\|P_\sigma y\|^2-(1+2\lambda)\|P_\sigma^\perp y\|^2\\&=\lambda^2+\|QP_\sigma^\perp y\|^2+\|P_\sigma^\perp y\|^2-(1+2\lambda)\|P_\sigma^\perp y\|^2\\&=\lambda^2-2\lambda\|P_\sigma^\perp y\|^2+\|QP_\sigma^\perp y\|^2\\&=(\lambda-\|P_\sigma^\perp y\|^2)^2+\|QP_\sigma^\perp y\|^2-\|P_\sigma^\perp y\|^4\geqslant 0\end{aligned}$$

for any $\lambda\in\mathbb{R}$.

For the inequality on the right in (15), it is equivalent to show that

$$h(\lambda)=(3-2\lambda)\|P_\sigma^\perp y\|^2+(\lambda^2-1)+\|P_\sigma y\|^2-\|QP_\sigma^\perp y\|^2\geqslant 0,$$

which is indeed true, since

$$\begin{aligned}h(\lambda)&=\lambda^2+(3-2\lambda)\|P_\sigma^\perp y\|^2-\|P_\sigma^\perp y\|^2-\|QP_\sigma^\perp y\|^2\\&=\lambda^2+2(1-\lambda)\|P_\sigma^\perp y\|^2-\|QP_\sigma^\perp y\|^2\\&=(\lambda-\|P_\sigma^\perp y\|^2)^2+(\|P_\sigma^\perp y\|^2-\|P_\sigma^\perp y\|^4)+(\|P_\sigma^\perp y\|^2-\|QP_\sigma^\perp y\|^2).\quad\square\end{aligned}$$

2.2. Two improved results and a new two-sided inequality

The next two results provide an improvement to Theorems 4 and 5 respectively.

THEOREM 6. *Suppose that $\Phi=\{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathscr{N}$ and that $\Psi=\{\psi_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of Φ. Then for every $\lambda\in[0,2]$, for any $x\in\mathscr{N}$ and any $\sigma\subset\mathbb{J}$, we have*

$$\begin{aligned}(1-\lambda-(2-\lambda)\|W^{\sigma^c}\|)\|x\|^2&\leqslant\Big\|\sum_{j\in\sigma}\langle x,\psi_j\rangle\phi_j\Big\|^2-\lambda\mathrm{Re}\sum_{j\in\sigma}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\&\leqslant\frac{(2-\lambda)\|W^\sigma\|^2+\lambda(\|W^{\sigma^c}\|^2-1)}{2}\|x\|^2.\end{aligned}$$

Proof. Since $W^\sigma+W^{\sigma^c}=\mathrm{Id}_{\mathscr{N}}$, the identity operator on $\mathscr{N}$, we have, for each $\lambda\in[0,2]$, for any $x\in\mathscr{N}$ and any $\sigma\subset\mathbb{J}$, that

$$\begin{aligned}\Big\|\sum_{j\in\sigma}&\langle x,\psi_j\rangle\phi_j\Big\|^2-\lambda\mathrm{Re}\sum_{j\in\sigma}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\&=\|W^\sigma x\|^2-\lambda\mathrm{Re}\langle W^\sigma x,x\rangle\\&=\|W^\sigma x\|^2+\lambda\frac{\|x-W^\sigma x\|^2-\|W^\sigma x\|^2-\|x\|^2}{2}\\&=\frac{2\|W^\sigma x\|^2+\lambda\|W^{\sigma^c}x\|^2-\lambda\|W^\sigma x\|^2-\lambda\|x\|^2}{2}\\&\leqslant\frac{(2-\lambda)\|W^\sigma\|^2+\lambda(\|W^{\sigma^c}\|^2-1)}{2}\|x\|^2.\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&\Big\|\sum_{j\in\sigma}\langle x,\psi_j\rangle\phi_j\Big\|^2-\lambda\mathrm{Re}\sum_{j\in\sigma}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\
&\quad=\|W^{\sigma}x\|^2-\lambda\mathrm{Re}\langle W^{\sigma}x,x\rangle\\
&\quad=\|x\|^2+\|W^{\sigma^c}x\|^2-2\mathrm{Re}\langle W^{\sigma^c}x,x\rangle-\lambda\left(\|x\|^2-\mathrm{Re}\langle W^{\sigma^c}x,x\rangle\right)\\
&\quad=(1-\lambda)\|x\|^2+\|W^{\sigma^c}x\|^2-(2-\lambda)\mathrm{Re}\langle W^{\sigma^c}x,x\rangle\\
&\quad\geqslant(1-\lambda)\|x\|^2-(2-\lambda)\|W^{\sigma^c}\|\|x\|^2\\
&\quad=(1-\lambda-(2-\lambda)\|W^{\sigma^c}\|)\|x\|^2
\end{aligned}$$

for each $\lambda\in[0,2]$, for any $x\in\mathscr{N}$ and any $\sigma\subset\mathbb{J}$, and the proof is finished. □

THEOREM 7. *Suppose that $\Phi=\{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathscr{N}$ and that $\Psi=\{\psi_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of Φ. Then for every $\lambda\in[0,1]$, for any $x\in\mathscr{N}$ and any $\sigma\subset\mathbb{J}$, we have*

$$\begin{aligned}
(1-2(1-\lambda)\|W^{\sigma^c}\|)\|x\|^2&\leqslant\Big\|\sum_{j\in\sigma}\langle x,\psi_j\rangle\phi_j\Big\|^2+2\lambda\mathrm{Re}\sum_{j\in\sigma^c}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\
&\leqslant((1-\lambda)\|W^{\sigma}\|^2+\lambda(1+\|W^{\sigma^c}\|^2))\|x\|^2.
\end{aligned}\tag{16}$$

Proof. For each $\lambda\in[0,1]$, for any $x\in\mathscr{N}$ and any $\sigma\subset\mathbb{J}$, we can obtain the left-hand inequality in (16) by the following computation

$$\begin{aligned}
&\Big\|\sum_{j\in\sigma}\langle x,\psi_j\rangle\phi_j\Big\|^2+2\lambda\mathrm{Re}\sum_{j\in\sigma^c}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\
&\quad=\|W^{\sigma}x\|^2+2\lambda\mathrm{Re}\langle W^{\sigma^c}x,x\rangle\\
&\quad=\|x\|^2+\|W^{\sigma^c}x\|^2-2\mathrm{Re}\langle W^{\sigma^c}x,x\rangle+2\lambda\mathrm{Re}\langle W^{\sigma^c}x,x\rangle\\
&\quad=\|x\|^2+\|W^{\sigma^c}x\|^2-2(1-\lambda)\mathrm{Re}\langle W^{\sigma^c}x,x\rangle\\
&\quad\geqslant\|x\|^2-2(1-\lambda)\|W^{\sigma^c}\|\|x\|^2\\
&\quad=(1-2(1-\lambda)\|W^{\sigma^c}\|)\|x\|^2.
\end{aligned}$$

For the inequality on the right hand side, we compute that

$$\begin{aligned}
&\Big\|\sum_{j\in\sigma}\langle x,\psi_j\rangle\phi_j\Big\|^2+2\lambda\mathrm{Re}\sum_{j\in\sigma^c}\langle x,\psi_j\rangle\langle\phi_j,x\rangle\\
&\quad=\|W^{\sigma}x\|^2+2\lambda\mathrm{Re}\langle W^{\sigma^c}x,x\rangle\\
&\quad=\|W^{\sigma}x\|^2+2\lambda\|x\|^2-2\lambda\mathrm{Re}\langle W^{\sigma}x,x\rangle\\
&\quad=\|W^{\sigma}x\|^2+2\lambda\|x\|^2-\lambda\left(\|x\|^2+\|W^{\sigma}x\|^2-\|x-W^{\sigma}x\|^2\right)\\
&\quad=(1-\lambda)\|W^{\sigma}x\|^2+\lambda\|x\|^2+\lambda\|W^{\sigma^c}x\|^2\\
&\quad\leqslant((1-\lambda)\|W^{\sigma}\|^2+\lambda(1+\|W^{\sigma^c}\|^2))\|x\|^2
\end{aligned}$$

for each $\lambda \in [0,1]$, for any $x \in \mathcal{N}$ and any $\sigma \subset \mathbb{J}$, and we arrive at the conclusion. □

We conclude the paper with a new two-sided inequality for frames.

THEOREM 8. *Suppose that $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{N}$. Then for each $\lambda \in \mathbb{R}$, for any $x \in \mathcal{N}$ and any $\sigma \subset \mathbb{J}$, we get*

$$\begin{aligned}\sum_{j\in\sigma^c} |\langle x,\phi_j\rangle|^2 - 2\sum_{j\in\sigma} |\langle x,\phi_j\rangle|^2 &\leqslant \sum_{j\in\mathbb{J}} |\langle S_\Phi^{-1} S_\Phi^{\sigma^c} x,\phi_j\rangle|^2 - 2\sum_{j\in\mathbb{J}} |\langle S_\Phi^{-1} S_\Phi^{\sigma} x,\phi_j\rangle|^2 \\ &\leqslant (1+\lambda^2)\sum_{j\in\sigma^c} |\langle x,\phi_j\rangle|^2 + (\lambda^2-2\lambda-1)\sum_{j\in\sigma} |\langle x,\phi_j\rangle|^2.\end{aligned} \tag{17}$$

Proof. For each $x \in \mathcal{N}$, we let $y = U_\Phi x$ and normalize it to $\|y\| = 1$. By means of the above-mentioned notations, we can rewrite the inequalities in (17) as follows:

$$\begin{aligned}\|P_\sigma^\perp y\|^2 - 2\|P_\sigma y\|^2 &\leqslant \|QP_\sigma^\perp y\|^2 - 2\|QP_\sigma y\|^2 \\ &\leqslant (1+\lambda^2)\|P_\sigma^\perp y\|^2 + (\lambda^2-2\lambda-1)\|P_\sigma y\|^2.\end{aligned} \tag{18}$$

Noting that $Qy = y$ and that $\langle QP_\sigma y, Qy\rangle = \|P_\sigma y\|^2$, we obtain

$$\begin{aligned}\|QP_\sigma^\perp y\|^2 &= \langle Q(y-P_\sigma y), Q(y-P_\sigma y)\rangle \\ &= \langle Qy,Qy\rangle + \langle QP_\sigma y, QP_\sigma y\rangle - \langle QP_\sigma y, Qy\rangle - \langle Qy, QP_\sigma y\rangle \\ &= \|y\|^2 + \|QP_\sigma y\|^2 - 2\|P_\sigma y\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|QP_\sigma^\perp y\|^2 - 2\|QP_\sigma y\|^2 &= \|P_\sigma y\|^2 + \|P_\sigma^\perp y\|^2 + \|QP_\sigma y\|^2 - 2\|P_\sigma y\|^2 - 2\|QP_\sigma y\|^2 \\ &= \|P_\sigma^\perp y\|^2 - \|QP_\sigma y\|^2 - \|P_\sigma y\|^2 \\ &= \|P_\sigma^\perp y\|^2 - 2\|P_\sigma y\|^2 + (\|P_\sigma y\|^2 - \|QP_\sigma y\|^2) \\ &\geqslant \|P_\sigma^\perp y\|^2 - 2\|P_\sigma y\|^2.\end{aligned}$$

This proves the inequality on the left of (18).

We define the function $s : \mathbb{R} \to \mathbb{R}$ by

$$s(\lambda) = (1+\lambda^2)\|P_\sigma^\perp y\|^2 + (\lambda^2-2\lambda-1)\|P_\sigma y\|^2 - \|QP_\sigma^\perp y\|^2 + 2\|QP_\sigma y\|^2, \quad \forall \lambda \in \mathbb{R}.$$

Since $\|P_\sigma y\|^2 \leqslant \|QP_\sigma y\|$ and $\|QP_\sigma^\perp y\|^2 = \|P_\sigma^\perp y\|^2 + \|QP_\sigma y\|^2 - \|P_\sigma y\|^2$, it follows that

$$\begin{aligned}s(\lambda) &= \lambda^2 + \|P_\sigma^\perp y\|^2 - 2\lambda\|P_\sigma y\|^2 - \|P_\sigma y\|^2 \\ &\quad + 2\|QP_\sigma y\|^2 - \|P_\sigma^\perp y\|^2 - \|QP_\sigma y\|^2 + \|P_\sigma y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\sigma y\|^2 + \|QP_\sigma y\|^2 \\ &= (\lambda - \|P_\sigma y\|^2)^2 + (\|QP_\sigma y\|^2 - \|P_\sigma y\|^4) \geqslant 0,\end{aligned}$$

meaning that the right-hand inequality in (18) holds, and we have the result. □

3. Conclusion

In this paper, we have introduced a novel function-theoretic approach to reprove several two-sided inequalities for frames in Hilbert spaces, which significantly simplifies the original proofs. We have also improved two existing theorems by extending the parameter ranges and refining the inequalities to eliminate redundancies. Furthermore, a new two-sided inequality for frames has been established. These contributions enhance the theoretical framework of frame inequalities and suggest potential for further generalizations in related settings.

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