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THREE EQUIVALENT DISTANCES ABOUT SINGULAR SUBSPACES

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Abstract. Cai and Zhang proved that three distances between singular subspaces under spectral and Frobenius norms are equivalent (T. T. Cai and A. Zhang, *The Annals of Statistics*, 2018, **46** (1), 60–89). We show that these distances remain equivalent under any unitarily invariant norm. The constants in Theorems 1–2 (Cai and Zhang's bounds) can be made sharper in our results (Theorems 3–4). Furthermore, our proof methods for Theorems 4 which extend Cai and Zhang's results (Theorem 2) are significantly simpler.

1. Introduction

The distances about singular subspaces have been widely applied in various statistical and machine learning problems, including low-rank matrix denoising ([1]), high-dimensional clustering ([1]), covariance estimation ([2], [11]), canonical correlation analysis ([3], [4], [7]), and singular-space estimation ([9], [10]), among others. In this paper, we demonstrate that three distances are equivalent under any unitarily invariant norm.

1.1. Notation

Let $X \in \mathbb{C}^{p_1 \times p_2}$. A unitarily invariant norm $\|\cdot\|$ satisfies

$$\|UXV^{\mathrm{H}}\| = \|X\|$$

for all unitary matrices $U \in \mathbb{O}_{p_1}$ (the set of p_1-dimensional unitary matrices) and $V \in \mathbb{O}_{p_2}$. Throughout this paper, X^{H} denotes the conjugate transpose of X. Important examples of unitarily invariant norms include the spectral norm, Frobenius norm, nuclear norm, and Ky Fan norms. We denote the spectral norm and Frobenius norm by $\|\cdot\|_{\mathrm{sp}}$ and $\|\cdot\|_{\mathrm{F}}$, respectively.

REMARK 1. While the standard definition of unitarily invariant norms requires the condition $\|UXV^{\mathrm{H}}\| = \|X\|$ to hold for all unitary matrices $U \in \mathbb{O}_{p_1}$ and $V \in \mathbb{O}_{p_2}$, this equality in fact extends to matrices with orthonormal columns. Specifically, it holds

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for $U \in \mathbb{O}_{m,p_1}$ (the set of $m \times p_1$ matrices with orthonormal columns) and $V \in \mathbb{O}_{n,p_2}$, since

$$(UX)^{\mathrm{H}}(UX) = X^{\mathrm{H}}X,$$
$$(XV^{\mathrm{H}})(XV^{\mathrm{H}})^{\mathrm{H}} = XX^{\mathrm{H}}.$$

Let the singular values of $V^T\hat{V}$ be denoted by $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r \geqslant 0$. The principle angles ([1]) between these subspaces are then defined as

$$\Theta(V,\hat{V}) := \mathrm{diag}\left(\cos^{-1}(\sigma_1), \cos^{-1}(\sigma_2), \ldots, \cos^{-1}(\sigma_r)\right).$$

Moreover, $R(V)$ denotes the column space of matrix $V \in \mathbb{O}_{p,r}$.

1.2. Cai and Zhang's work

Cai and Zhang's results can be restated as in Theorems 1–2.

THEOREM 1. ([1]) *Let* $V, \hat{V} \in \mathbb{O}_{p,r}$. *Then*

$$\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}} \leqslant \inf_{O\in\mathbb{O}_r} \|\hat{V} - VO\|_{\mathrm{sp}} \leqslant \sqrt{2}\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}},$$
$$\|\sin\Theta(\hat{V},V)\|_{\mathrm{F}} \leqslant \inf_{O\in\mathbb{O}_r} \|\hat{V} - VO\|_{\mathrm{F}} \leqslant \sqrt{2}\|\sin\Theta(\hat{V},V)\|_{\mathrm{F}}.$$

THEOREM 2. ([1]) *Let* $V, \hat{V} \in \mathbb{O}_{p,r}$. *Then*

$$\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}} \leqslant \|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\|_{\mathrm{sp}} \leqslant 2\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}}.$$

Especially,

$$\|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\|_F = \sqrt{2}\|\sin\Theta(\hat{V},V)\|_{\mathrm{F}}.$$

Liu and Ren ([6]) gived the following proposition about $\sin\Theta(\hat{V},V)$.

PROPOSITION 1. ([6]) *Let* $\|\cdot\|$ *be a unitarily invariant norm,* $V, \hat{V} \in \mathbb{O}_{p,r}$ *with* $[V\ V_\perp] \in \mathbb{O}_p$ *and* $[\hat{V}\ \hat{V}_\perp] \in \mathbb{O}_p$. *Then*

$$\|\sin\Theta(V,\hat{V})\| = \|V_\perp^{\mathrm{H}}\hat{V}\| = \|V^{\mathrm{H}}\hat{V}_\perp\| = \|\sin\Theta(\hat{V},V)\|.$$

For $V_1, V_2, V_3 \in \mathbb{O}_{p,r}$,

$$\|\sin\Theta(V_2,V_3)\| \leqslant \|\sin\Theta(V_1,V_2)\| + \|\sin\Theta(V_1,V_3)\|.$$

In fact, $\|\sin\Theta(\hat{V},V)\| = 0 \Longleftrightarrow R(V) = R(\hat{V})$.

1.3. Main work

Let $V, \hat{V} \in \mathbb{O}_{p,r}$ $(p \geqslant r)$,

$$D(\hat{V},V) := \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\|, \quad d(\hat{V},V) := \|\hat{V}\hat{V}^T - VV^T\|.$$

In addition, the infimum is achieved over $\mathbb{O}_r$ which is a compact set. Hence, $D(\hat{V},V) = \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\| = \min_{O \in \mathbb{O}_r} \|\hat{V} - VO\|$.

PROPOSITION 2. *Let $\|\cdot\|$ be a unitarily invariant norm and $V, \hat{V} \in \mathbb{O}_{p,r}$. Then*
(i) $D(\hat{V},V) = D(V,\hat{V})$;
(ii) $D(\hat{V},V) = 0 \Longleftrightarrow R(V) = R(\hat{V})$;
(iii) If $V_1, V_2, V_3 \in \mathbb{O}_{p,r}$,

$$D(V_1,V_2) \leqslant D(V_1,V_3) + D(V_2,V_3).$$

PROPOSITION 3. *Let $\|\cdot\|$ be a unitarily invariant norm and $V, \hat{V} \in \mathbb{O}_{p,r}$. Then*
(i) $d(\hat{V},V) = d(V,\hat{V})$;
(ii) $d(\hat{V},V) = 0 \Longleftrightarrow R(V) = R(\hat{V})$;
(iii) If $V_1, V_2, V_3 \in \mathbb{O}_{p,r}$,

$$d(V_1,V_2) \leqslant d(V_1,V_3) + d(V_2,V_3).$$

Propositions 1–3 show that $\|\sin\Theta(\hat{V},V)\|$, $D(\hat{V},V)$ and $d(\hat{V},V)$ are all metrics about singular subspaces. The following two theorems present that all the three different metrics are equivalent under any unitarily invariant norm.

THEOREM 3. *Let $\|\cdot\|$ be a unitarily invariant norm and $V, \hat{V} \in \mathbb{O}_{p,r}$. Then*

$$\|\sin\Theta(\hat{V},V)\| \leqslant \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\| \leqslant (1 + \|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}})\|\sin\Theta(\hat{V},V)\|.$$

REMARK 2. Since $\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}} \leqslant 1$,

$$\|\sin\Theta(\hat{V},V)\| \leqslant \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\| \leqslant 2\|\sin\Theta(\hat{V},V)\|.$$

REMARK 3. By Theorem 1 and Theorem 3, we have

$$\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}} \leqslant \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\|_{\mathrm{sp}} \leqslant \min\left\{1 + \|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}}, \sqrt{2}\right\}\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}}.$$

Compered with Theorem 1, our estimation becomes sharper when

$$\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}} < \sqrt{2} - 1.$$

THEOREM 4. *Let $\|\cdot\|$ be a unitarily invariant norm and $V, \hat{V} \in \mathbb{O}_{p,r}$. Then*

$$\|\sin\Theta(\hat{V},V)\| \leqslant \|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\| \leqslant 2\|\sin\Theta(\hat{V},V)\|.$$

Especially,

$$\|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\|_{\mathrm{sp}} \leqslant \sqrt{2}\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}}.$$

Hence, the constant 2 in Theorem 2 can be sharper in our estimation.

2. Proofs of Propositions 2–3

We begin with a useful lemma, which plays a key role for our later discussions.

LEMMA 1. *Let* $V, \hat{V} \in \mathbb{O}_{p,r}$. *Then* $R(V) = R(\hat{V})$ *if and only if there exists* $O \in \mathbb{O}_r$ *such that* $V = \hat{V}O$.

Proof. Denote α_i, β_i as the i-th columns of $V, \hat{V}$ respectively. Then

$$R(V) = \text{span}\{\alpha_1, \cdots, \alpha_r\}, \quad R(\hat{V}) = \text{span}\{\beta_1, \cdots, \beta_r\}.$$

($\Longleftarrow$) Since $V = \hat{V}O$ and $O \in \mathbb{O}_r$, $VO^{\mathrm{H}} = \hat{V}$. Then $\alpha_1, \cdots, \alpha_r$ and $\beta_1, \cdots, \beta_r$ can be mutually lineally expressed. Furthermore,

$$\text{span}\{\alpha_1, \cdots, \alpha_r\} = \text{span}\{\beta_1, \cdots, \beta_r\}.$$

($\Longrightarrow$) Two sets of column vectors $\alpha_1, \cdots, \alpha_r$ and $\beta_1, \cdots, \beta_r$ can be regarded as two sets of standard orthogonal bases of $R(V) = R(\hat{V})$, then there must exist invertible transition matrix $O \in \mathbb{C}^{r \times r}$ such that $V = \hat{V}O$. This with

$$V^{\mathrm{H}}V = O^{\mathrm{H}}\hat{V}^{\mathrm{H}}\hat{V}O = O^{\mathrm{H}}O$$

and $V^{\mathrm{H}}V = I_r$ implies $O^{\mathrm{H}}O = I_r$. Hence, $O \in \mathbb{O}_r$. $\square$

Now, we can prove Proposition 2.

Proof. (i) Note that $\|\cdot\|$ is a unitarily invariant norm and $O, O^{\mathrm{H}} \in \mathbb{O}_r$, then

$$\begin{aligned} D(\hat{V}, V) &= \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\| = \inf_{O \in \mathbb{O}_r} \|\hat{V}O^{\mathrm{H}} - VOO^{\mathrm{H}}\| = \inf_{O \in \mathbb{O}_r} \|\hat{V}O^{\mathrm{H}} - V\| \\ &= \inf_{O^{\mathrm{H}} \in \mathbb{O}_r} \|V - \hat{V}O^{\mathrm{H}}\| = D(V, \hat{V}). \end{aligned}$$

(ii) By $D(\hat{V}, V) = \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\| = \min_{O \in \mathbb{O}_r} \|\hat{V} - VO\| = 0$, one knows that there exists $O \in \mathbb{O}_r$ such that $\hat{V} = VO$. This concludes that

$$R(V) = R(\hat{V})$$

which thanks to Lemma 1.

On the other hand, if $R(V) = R(\hat{V})$, Lemma 1 tells that there exists $O \in \mathbb{O}_r$ such that $\hat{V} = VO$. This leads to

$$D(\hat{V}, V) = \inf_{O \in \mathbb{O}_r} \|\hat{V} - VO\| = 0.$$

(iii) Let $O, O' \in \mathbb{O}_r$. Then

$$\|V_1 - V_2O\| = \|V_1 - V_3O' + V_3O' - V_2O\| \leqslant \|V_1 - V_3O'\| + \|V_3O' - V_2O\|.$$

For any fixed $O' \in \mathbb{O}_r$, finding infimum about O in the above inequality, one obtains

$$D(V_1,V_2) \leqslant \|V_1 - V_3O'\| + D(V_3O',V_2) = \|V_1 - V_3O'\| + D(V_3,V_2)$$

which thanks to $R(V_3O') = R(V_3)$. Furthermore, finding infimum about O' in the above inequality,

$$D(V_1,V_2) \leqslant D(V_1,V_3) + D(V_3,V_2). \quad \square$$

Next, we prove Proposition 3.

Proof. (i) Note that $\|\cdot\|$ is a unitarily invariant norm, then

$$d(\hat{V},V) = \|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\| = \|VV^{\mathrm{H}} - \hat{V}\hat{V}^{\mathrm{H}}\| = d(V,\hat{V})$$

which dues to the homogeneity of a norm.

(ii) By $d(\hat{V},V) = \|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\| = 0$, it has $\hat{V}\hat{V}^{\mathrm{H}} = VV^{\mathrm{H}}$. Since $\hat{V}^{\mathrm{H}}\hat{V} = I_r$, it has $\hat{V} = VV^{\mathrm{H}}\hat{V}$. Then

$$R(\hat{V}) \subseteq R(V).$$

Similarly, $R(V) \subseteq R(\hat{V})$. Hence,

$$R(V) = R(\hat{V}).$$

On the other hand, if $R(V) = R(\hat{V})$, Lemma 1 tells that there must exist $O \in \mathbb{O}_r$ such that $\hat{V} = VO$. Then

$$d(\hat{V},V) = \|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\| = 0.$$

(iii) Let $V_1,V_2,V_3 \in \mathbb{O}_{p,r}$. Then

$$\begin{aligned} d(V_1,V_2) &= \|V_1V_1^{\mathrm{H}} - V_2V_2^{\mathrm{H}}\| = \|V_1V_1^{\mathrm{H}} - V_3V_3^{\mathrm{H}} + V_3V_3^{\mathrm{H}} - V_2V_2^{\mathrm{H}}\| \\ &\leqslant d(V_1,V_3) + d(V_2,V_3). \quad \square \end{aligned}$$

3. Proof of Theorem 3

We introduce a fundamental lemma, which is important for the following proofs.

LEMMA 2. ([5], [8]) *Let $\|\cdot\|$ be a unitarily invariant norm, and $A,B \in \mathbb{C}^{p_1\times p_2}$. Then*

$$\|AB^{\mathrm{H}}\| \leqslant \|A\|_{\mathrm{sp}}\|B\|,$$

where $\|A\|_{\mathrm{sp}}$ stands for spectral norm. Moreover, $\|A\| \leqslant \|B\|$ holds if $\|A\|_{(k)} \leqslant \|B\|_{(k)}$ for $k = 1,2,\cdots,\min\{p_1,\ p_2\}$, where $\|A\|_{(k)}$ stands for Ky Fan norm.

We now proceed to prove Theorem 3.

Proof. By Proposition 1, $\|\sin\Theta(\hat{V},V)\| = \|V_\perp^{\mathrm{H}}\hat{V}\|$. Note that $V_\perp^{\mathrm{H}}V=0$ and $\|V_\perp^{\mathrm{H}}\|_{\mathrm{sp}} = 1$, these with Lemma 2 show that

$$\|\sin\Theta(\hat{V},V)\| = \|V_\perp^{\mathrm{H}}\hat{V} - V_\perp^{\mathrm{H}}VO\| \leqslant \|V_\perp^{\mathrm{H}}\|_{\mathrm{sp}}\|\hat{V}-VO\| \leqslant \|\hat{V}-VO\|.$$

Furthermore,

$$\|\sin\Theta(\hat{V},V)\| \leqslant \inf_{O\in\mathbb{O}_r}\|\hat{V}-VO\| = D(\hat{V},V).$$

Denote the SVD of $V^{\mathrm{H}}\hat{V}$ as

$$V^{\mathrm{H}}\hat{V} := P\Sigma Q^{\mathrm{H}},\ \ P,Q\in\mathbb{O}_r.$$

Let $O := PQ^{\mathrm{H}}$. Then $\hat{V}-VO = I_r(\hat{V}-VPQ^{\mathrm{H}}) = (VV^{\mathrm{H}}+V_\perp V_\perp^{\mathrm{H}})(\hat{V}-VPQ^{\mathrm{H}}) = VV^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}}) + V_\perp V_\perp^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}})$. Furthermore,

$$\begin{aligned}
D(\hat{V},V) &= \inf_{O\in\mathbb{O}_r}\|\hat{V}-VO\| \\
&\leqslant \|VV^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}})\| + \|V_\perp V_\perp^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}})\| \\
&\leqslant \|V\|_{\mathrm{sp}}\|V^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}})\| + \|V_\perp\|_{\mathrm{sp}}\|V_\perp^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}})\| \\
&= \|V^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}})\| + \|V_\perp^{\mathrm{H}}(\hat{V}-VPQ^{\mathrm{H}})\| \\
&= \|V^{\mathrm{H}}\hat{V}-PQ^{\mathrm{H}}\| + \|V_\perp^{\mathrm{H}}\hat{V}\| = \|P\Sigma Q^{\mathrm{H}}-PQ^{\mathrm{H}}\| + \|\sin\Theta(\hat{V},V)\| \\
&= \|\mathrm{diag}\,\{1-\sigma_1(V^{\mathrm{H}}\hat{V}),\cdots,1-\sigma_r(V^{\mathrm{H}}\hat{V})\}\| + \|\sin\Theta(\hat{V},V)\|. \qquad (1)
\end{aligned}$$

By $\sigma_i(V^{\mathrm{H}}\hat{V})\in[0,1]$, $1-\sigma_i(V^{\mathrm{H}}\hat{V}) \leqslant 1-\sigma_i^2(V^{\mathrm{H}}\hat{V})$ $(i=1,\cdots,r)$. Then Lemma 2 tells

$$\begin{aligned}
&\left\|\mathrm{diag}\,\{1-\sigma_1(V^{\mathrm{H}}\hat{V}),\cdots,1-\sigma_r(V^{\mathrm{H}}\hat{V})\}\right\| \\
\leqslant &\left\|\mathrm{diag}\left\{1-\sigma_1^2(V^{\mathrm{H}}\hat{V}),\cdots,1-\sigma_r^2(V^{\mathrm{H}}\hat{V})\right\}\right\|.
\end{aligned}$$

Moveover, by the definition of $\Theta(\hat{V},V)$,

$$\sin\Theta(\hat{V},V) = \mathrm{diag}\left\{\sqrt{1-\sigma_1^2(V^{\mathrm{H}}\hat{V})},\cdots,\sqrt{1-\sigma_r^2(V^{\mathrm{H}}\hat{V})}\right\}.$$

This with Proposition 1 leads to

$$\begin{aligned}
\|\sin\Theta(\hat{V},V)\| &= \left\|\mathrm{diag}\{\sigma_r(V_\perp^{\mathrm{H}}\hat{V}),\ \cdots,\ \sigma_1(V_\perp^{\mathrm{H}}\hat{V})\}\right\| \\
&= \left\|\mathrm{diag}\left\{\sqrt{1-\sigma_1^2(V^{\mathrm{H}}\hat{V})},\cdots,\sqrt{1-\sigma_r^2(V^{\mathrm{H}}\hat{V})}\right\}\right\|.
\end{aligned}$$

Then Lemma 2 implies that

$$\begin{aligned}
&\left\|\mathrm{diag}\left\{1-\sigma_1^2(V^{\mathrm{H}}\hat{V}),\cdots,1-\sigma_r^2(V^{\mathrm{H}}\hat{V})\right\}\right\| = \left\|\mathrm{diag}\{\sigma_r^2(V_\perp^{\mathrm{H}}\hat{V}),\ \cdots,\ \sigma_1^2(V_\perp^{\mathrm{H}}\hat{V})\}\right\| \\
= &\left\|\left(\mathrm{diag}\{\sigma_r(V_\perp^{\mathrm{H}}\hat{V}),\ \cdots,\ \sigma_1(V_\perp^{\mathrm{H}}\hat{V})\}\right)^2\right\| \leqslant \|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}}\|\sin\Theta(\hat{V},V)\|.
\end{aligned}$$

The above inequality and (1) tell that

$$D(\hat{V},V) \leqslant (1+\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}})\|\sin\Theta(\hat{V},V)\|. \quad \square$$

4. Proof of Theorem 4

We first present a key lemma that will be useful for subsequent proofs.

LEMMA 3. ([1]) *Suppose* $A,\ B \in \mathbb{C}^{p\times n}$, $A^{\mathrm{H}}B = 0$ *or* $AB^{\mathrm{H}} = 0$. *Then*

$$\sigma_1^2(A+B) \leqslant \sigma_1^2(A) + \sigma_1^2(B).$$

The proof of Theorem 4 is presented as follows.

Proof. Note that $V_\perp^{\mathrm{H}}V = 0$. Then by Proposition 1 and Remark 1,

$$\|\sin\Theta(\hat{V},V)\| = \|V_\perp^{\mathrm{H}}\hat{V}\| = \|V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}}\| = \|V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}} - V_\perp^{\mathrm{H}}VV^{\mathrm{H}}\|.$$

Using Lemma 2,

$$\|\sin\Theta(\hat{V},V)\| \leqslant \|V_\perp^{\mathrm{H}}\|_{\mathrm{sp}}\|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\| \leqslant \|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\|.$$

On the other hand, $\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}} = (VV^{\mathrm{H}} + V_\perp V_\perp^{\mathrm{H}})\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}(\hat{V}\hat{V}^{\mathrm{H}} + \hat{V}_\perp\hat{V}_\perp^{\mathrm{H}}) = V_\perp V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\hat{V}_\perp\hat{V}_\perp^{\mathrm{H}}$. Then

$$\begin{aligned}\|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\| &= \|VV^{\mathrm{H}}\hat{V}_\perp\hat{V}_\perp^{\mathrm{H}} - V_\perp V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}}\| \\ &\leqslant \|VV^{\mathrm{H}}\hat{V}_\perp\hat{V}_\perp^{\mathrm{H}}\| + \|V_\perp V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}}\| \\ &\leqslant \|V^{\mathrm{H}}\hat{V}_\perp\| + \|V_\perp^{\mathrm{H}}\hat{V}\| = 2\|\sin\Theta(\hat{V},V)\|.\end{aligned} \tag{2}$$

Specifically, when the unitarily invariant norm is the spectral norm, we have

$$\begin{aligned}\|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\|_{\mathrm{sp}} &= \|VV^{\mathrm{H}}\hat{V}_\perp\hat{V}_\perp^{\mathrm{H}} - V_\perp V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}}\|_{\mathrm{sp}} \\ &\leqslant \sqrt{\|VV^{\mathrm{H}}\hat{V}_\perp\hat{V}_\perp^{\mathrm{H}}\|_2^2 + \|V_\perp V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}}\|_2^2} \\ &\leqslant \sqrt{\|V^{\mathrm{H}}\hat{V}_\perp\|_2^2 + \|V_\perp^{\mathrm{H}}\hat{V}\|_2^2}\end{aligned} \tag{3}$$

where (3) follows from the orthogonality condition $(VV^{\mathrm{H}}\hat{V}_\perp\hat{V}_\perp^{\mathrm{H}})^{\mathrm{H}}V_\perp V_\perp^{\mathrm{H}}\hat{V}\hat{V}^{\mathrm{H}} = 0$, equation (2), and Lemma 3. Combining this inequality with Proposition 1, we conclude that

$$\|\hat{V}\hat{V}^{\mathrm{H}} - VV^{\mathrm{H}}\|_{\mathrm{sp}} \leqslant \sqrt{2}\|\sin\Theta(\hat{V},V)\|_{\mathrm{sp}}. \quad \square$$

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