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ON THE δ–NUMERICAL RADIUS AND THE δ–CRAWFORD NUMBER FUNCTIONS OF DIRECT SUM OF HILBERT SPACE OPERATORS

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Abstract. In this work, some important properties of the δ-numerical radius and the δ-Crawford number functions of bounded linear operators on Hilbert spaces are investigated. Then, the relationships between the δ-numerical radius and the δ-Crawford number functions of the direct sum of Hilbert space operators and those of their coordinate operators are explored. Furthermore, some generalization formulas for lower and upper bounds of the δ-numerical radius and the δ-Crawford number functions of such direct sums are obtained. Finally, some lower and upper bounds for the δ-numerical radius in the sectorial cases of coordinate operators are established, and these evaluations generalize some results known in the literature. Moreover, the obtained results offer a range of useful techniques that can facilitate further advancements in this field.

1. Introduction

The concepts of numerical range and numerical radius of an operator play a very significant role in pure and applied mathematics and have been studied extensively due to their applications in engineering, quantum computing, quantum mechanics, numerical analysis, differential equations, fluid dynamics, the geometry of Banach space, etc. (see, [2, 6, 13]).

Throughout this article, H denotes a complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. Let $\mathfrak{B}(H)$ and $\mathfrak{B}_\infty(H)$ stand for the C^*-algebra of all bounded linear operators and compact operators acting on H, respectively. The numerical radius of an operator T is given by

$$\omega(T)=\sup\{|\langle Tx,x\rangle| : \ x\in H,\ \|x\|=1\}.$$

The usual operator norm and the Crawford number of an operator T are, respectively, defined by

$$\|T\|=\sup\{\|Tx\| : \ x\in H,\ \|x\|=1\}$$

and

$$c(T)=\inf\{|\langle Tx,x\rangle| : x\in H,\ \|x\|=1\}$$

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[10].

In literature there is one important formula of Yamazaki [27] for calculation of numerical radius $\omega(T)$ of $T \in \mathfrak{B}(H)$ as

$$\omega(T) = \sup_{t \in \mathbb{R}} \|Re(e^{it}T)\|.$$

There is a another formula for computation of generalized numerical radius as

$$\omega_N(T) = \sup_{\theta \in \mathbb{R}} N(Re(e^{i\theta}T)),$$

where N is given norm of $T \in \mathfrak{B}(H)$ [1].

Finding the exact value of $\omega(T)$ for an arbitrary $T \in \mathfrak{B}(H)$ is not an easy task, except in some special cases. Therefore, researchers in this field have devoted a considerable amount of time and effort to find lower and upper bounds of $\omega(T)$.

Recall that for any $T \in \mathfrak{B}(H)$ the numerical radius $\omega(T)$ is a norm on $\mathfrak{B}(H)$ and among the most basic bounds of $\omega(T)$ is the following two-sided inequality

$$\frac{1}{2}\|T\| \leqslant \omega(T) \leqslant \|T\| \tag{1}$$

(see, e.g., [10]). The right hand side of (1) becomes an equality when T is normal, while the left side becomes an equality when $T^2 = 0$.

Sharpening (1) has received a considerable attention in the literature. We refer the reader to [3, 5, 15, 16, 17, 21, 22] as a sample of recent attempts to obtain sharper and better bounds of $\omega(T)$. Among those attempts, the following two-sided inequality will be of interest, see [16].

$$\frac{1}{4}\|T^*T + TT^*\| \leqslant \omega^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|. \tag{2}$$

The inequality (2) refine the left and right inequalities of (1).

The literature has several generalizations of the classical numerical radius and the classical Crawford number. We will focus on the δ-numerical radius and the δ-Crawford number functions of direct sum of Hilbert space operators.

DEFINITION 1. ([24]) For the operator $T \in \mathfrak{B}(H)$ and $0 \leqslant \delta \leqslant \|T\|$, the δ-numerical range is defined as

$$W_\delta(T) = cl\left(\{\langle Tx, x\rangle : x \in H,\ \|x\| = 1,\ \|Tx\| \geqslant \delta\}\right),$$

where $cl(\cdot)$ is denoted closure of a set in complex plane.

Clearly, $W_\delta(T)$ is a closed subset of the closure of the usual numerical range, and $W_0(T) = \bigcap_{\delta < \|T\|} W_\delta(T)$. By a slight modification of a theorem of Dekker [7], it is not hard to see that $W_\delta(T)$ is connected. It would be interesting to know if $W_\delta(T)$ is convex. It is, if T is normal, or if the underlying Hilbert space is two-dimensional.

DEFINITION 2. For the operator $T \in \mathfrak{B}(H)$ and $0 \leqslant \delta \leqslant \|T\|$, the δ-numerical radius and the δ-Crawford number functions are defined as

$$\omega_\delta(T) = \sup\{|\lambda| : \lambda \in W_\delta(T)\} \quad \text{and} \quad c_\delta(T) = \inf\{|\lambda| : \lambda \in W_\delta(T)\},$$

respectively.

From the definitions of the δ-numerical radius and the δ-Crawford number functions, it follows immediately that $\omega_0(T) = \omega(T)$ and $c_0(T) = c(T)$.

Logically, the concept of the δ-numerical radius and the δ-Crawford number functions can be generalized as follows:

For the operator $T \in \mathfrak{B}(H)$ and $-\infty < \delta < 0$,

$$\omega_\delta(T) = \omega(T) \quad \text{and} \quad c_\delta(T) = c(T).$$

It is easy to see that

$$c(T) \leqslant c_\delta(T) \leqslant \omega_\delta(T) \leqslant \omega(T).$$

Also, for $\tau \leqslant \delta \leqslant \|T\|$,

$$W_\delta(T) \subset W_\tau(T), \;\; \omega_\delta(T) \leqslant \omega_\tau(T), \quad \text{and} \quad c_\tau(T) \leqslant c_\delta(T).$$

Throughout this article, we will define for $T \in \mathfrak{B}(H)$ and $-\infty < \lambda \leqslant \|T\|$

$$\Delta_\lambda(T) := \{x \in S_1(H) : \; \|Tx\| \geqslant \lambda\},$$

where $S_1(H)$ is a unit sphere on H.

The infinite direct sum of Hilbert spaces and the infinite direct sum of operators have been studied in [9]. Namely, the infinite direct sum of Hilbert space H_n, $n \geqslant 1$ and infinite direct sum of operators A_n in H_n, $n \geqslant 1$ are defined as

$$H = \bigoplus_{n=1}^{\infty} H_n = \left\{ u = (u_n) : u_n \in H_n, \; n \geqslant 1, \; \|u\|_H^2 = \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < +\infty \right\}$$

and

$$A = \bigoplus_{n=1}^{\infty} A_n : D(A) \subset H \to H,$$

where $D(A) = \{u = (u_n) \in H : u_n \in D(A_n), \; n \geqslant 1, \; Au = (A_n u_n) \in H\}$, respectively. Recall that H is a Hilbert space with the norm induced by the inner product

$$\langle u, v \rangle_H = \sum_{n=1}^{\infty} \langle u_n, v_n \rangle_{H_n}, \;\; u = u_n, \;\; v = v_n \in H.$$

We present the following theorem, which is well known in the literature on the boundedness of direct sums of Hilbert space operators.

THEOREM 1. ([18]) *Let* $A_n \in \mathfrak{B}(H_n)$, $n \geqslant 1$ *and* $A = \bigoplus_{n=1}^{\infty} A_n : H \to H$. *In order to* $A \in \mathfrak{B}(H)$ *the necessary and sufficient condition is* $\sup_{n \geqslant 1} \|A_n\| < \infty$. *Moreover, in the case that* $A \in \mathfrak{B}(H)$, *the norm of* A *is in the form* $\|A\| = \sup_{n \geqslant 1} \|A_n\|$.

For clarification, we discuss an example.

EXAMPLE 1. In the Eucledian space $\mathbb{R}^2(\mathbb{R})$, consider the following operator in form

$$A = A_1 \oplus A_2 : \mathbb{R}^2 \to \mathbb{R}^2, \;\; A_1x_1 = x_1, \;\; A_1 : \mathbb{R} \to \mathbb{R}, \;\; A_2x_2 = 2x_2, \;\; A_2 : \mathbb{R} \to \mathbb{R}.$$

It is clear that $\|A\| = \|A_1 \oplus A_2\| = 2$. Then, for any $0 \leqslant \delta \leqslant 2$ and $x = (x_1, x_2) \in S_1(\mathbb{R}^2)$

$$\delta^2 \leqslant \|Ax\|^2 = \|A_1x_1\|^2 + \|A_2x_2\|^2 = x_1^2 + 4x_2^2 = 4 - 3x_1^2$$

and from this we have $|x_1| \leqslant \left(\frac{4-\delta^2}{3}\right)^{1/2}$. So

$$\Delta_\delta(A) = \begin{cases} S_1(\mathbb{R}^2), & \text{if } -\infty < \delta < 0, \\ \left\{ x = (x_1, x_2) : \; x_1^2 + x_2^2 = 1, \; |x_1| \leqslant \left(\frac{4-\delta^2}{3}\right)^{1/2} \right\}, & \text{if } 0 \leqslant \delta \leqslant 2. \end{cases}$$

Then, for $0 \leqslant \delta \leqslant 2$ we have

$$\begin{aligned} \omega_\delta(A) &= \sup\{|\langle Ax, x\rangle| : x \in \Delta_\delta(A)\} \\ &= \sup\{x_1^2 + 2x_2^2 : (x_1, x_2) \in \Delta_\delta(A)\} \\ &= \sup\left\{ 2 - x_1^2 : \; |x_1| \leqslant \left(\frac{4-\delta^2}{3}\right)^{1/2} \right\}. \end{aligned}$$

Therefore, for $\delta \leqslant 2$ we get

$$\begin{aligned} \omega_\delta(A) &= \begin{cases} \omega(A), & \text{if } -\infty < \delta < 0, \\ 2, & \text{if } 0 \leqslant \delta \leqslant 2 \end{cases} \\ &= \begin{cases} 2, & \text{if } -\infty < \delta < 0, \\ 2, & \text{if } 0 \leqslant \delta \leqslant 2 \end{cases} \\ &= 2. \end{aligned}$$

Similarly, for $0 \leqslant \delta \leqslant 2$

$$\begin{aligned} c_\delta(A) &= \inf\{|\langle Ax, x\rangle| : x \in \Delta_\delta(A)\} \\ &= \inf\{x_1^2 + 2x_2^2 : (x_1, x_2) \in \Delta_\delta(A)\} \\ &= \inf\left\{ 2 - x_1^2 : \; |x_1| \leqslant \left(\frac{4-\delta^2}{3}\right)^{1/2} \right\}. \end{aligned}$$

Then, for $\delta \leqslant 2$ we obtain

$$c_\delta(A) = \begin{cases} 1, & \text{if } -\infty < \delta < 0, \\ \dfrac{2+\delta^2}{3}, & \text{if } 0 \leqslant \delta \leqslant 2. \end{cases}$$

DEFINITION 3. For any fixed $-\dfrac{\pi}{2} < \gamma \leqslant \alpha < \dfrac{\pi}{2}$,

(i) $S(\gamma,\alpha) = \left\{re^{i\varphi} : r \geqslant 0,\ \gamma \leqslant \varphi \leqslant \alpha\right\} \subset \mathbb{C}$ be a sector with vertex in origin and angles δ and α.

(ii) The operator $T : H \to H$, $T \in \mathfrak{B}(H)$ in any Hilbert space H is said to be (γ,α)-sectorial if $W(T) \subseteq S(\gamma,\alpha)$.

(iii) The class of all (γ,α)-sectorial operators on Hilbert space H it will be denoted by $Sec_{(\gamma,\alpha)}(H)$.

Throughout this paper, in (ii) of Definition 3, δ will be taken as the largest δ satisfying the given condition, and α as the smallest α satisfying the given condition.

It is clear that if $T \in Sec_{(\gamma,\alpha)}(H)$, T is accretive. If $T \in Sec_{(\gamma,\alpha)}(H)$ and $\gamma = 0$, then T is accetive-dissipative. If $T \in Sec_{(\gamma,\alpha)}(H)$ and $\alpha = 0$, then T is accetive-accumulative.

In [11] and [12], some lower and upper bounds for 2×2 operator matrices on the direct sum of Hilbert spaces have been investigated.

In [19] and [26], the relationship between certain numerical characteristics – such as the numerical range, numerical radius, Crawford number, and sectoriality – of bounded linear operators on the direct sum of Hilbert spaces and those of their coordinate operators has been investigated.

In [20], the characterization of lower and upper bounds for the q-numerical radius of 2×2 operator matrices acting on the direct sum of Hilbert spaces has been investigated.

In [25], the relationship between q-numerical radii of infinitely direct sum of Hilbert space operators and their coordinate operators has been investigated.

In [14], the lower and upper bounds of the numerical radius for operators on finite direct sums of Hilbert spaces have been investigated.

In [3], some interesting new inequalities for the numerical radius of accretive matrices have been obtained.

The main goal of this work is to investigate the relationship between the δ-numerical radius and the δ-Crawford number functions of direct sum of Hilbert space operators and those of their coordinate operators. Furthermore, some evolutions formulas for lower and upper bounds of the δ-numerical radius and the δ-Crawford number functions of direct sum of Hilbert space operators have been obtained. Later on, similar problems have been investigated for sectorial cases of coordinate operators. The obtained results generalize many well-known conclusions concerning the classical numerical radius and the classical Crawford number functions for an operator and the infinite direct sum of Hilbert space operators found in the current mathematical literature (see, e.g., [3, 8, 16, 19, 23, 26]). In addition, the obtained results provide various useful techniques for further developments in this area.

This work is organized as follows. In Section 2, we first investigate some important properties of the δ-numerical radius and the δ-Crawford number functions of a bounded linear operator in a Hilbert space. In Theorems 8 and 9, we examine the relationships between the δ-numerical radius and the δ-Crawford number functions of the direct sum of Hilbert space operators and those of their coordinate operators. Subsequently, we obtain some generalization formulas for the lower and upper bounds of the δ-numerical radius and the δ-Crawford number functions of such direct sums. In Section 3, we obtain several formulas for the upper and lower bounds of the δ-numerical radius in the sectorial cases of coordinate operators.

Note that each operator $T \in \mathfrak{B}(H)$ can be expressed in the Cartesian decomposition form as $T = ReT + iImT$, where $ReT = \dfrac{T+T^*}{2}$ and $ImT = \dfrac{T-T^*}{2i}$. Here, T^* denotes the adjoint of T. Throughout this study we denote by $|T| = (T^*T)^{1/2}$ the absolute value of an operator $T \in \mathfrak{B}(H)$.

2. Bounds of the δ-numerical radius and the δ-Crawford number functions of direct sum of Hilbert space operators

In this section, firstly, we investigate some important properties of the δ-numerical radius and the δ-Crawford number functions of bounded linear operators on Hilbert spaces. Then, we explore the relationships between the δ-numerical radius and the δ-Crawford number functions of the direct sum of Hilbert space operators and those of their coordinate operators. Furthermore, we obtain some generalization formulas for lower and upper bounds of the δ-numerical radius and the δ-Crawford number functions of such direct sums.

Firstly, we give a well-known inequality that is essential for proving theorems.

LEMMA 1. ([4]) *If* $T \in \mathfrak{B}(H)$, $T \geqslant 0$, *then for any* $x \in H$

$$\|Tx\|^2 \leqslant \|T\|\langle Tx,x\rangle.$$

Now, we are in a position to present our results.

THEOREM 2. *Let* $T \in \mathfrak{B}(H)$.

(i) If $0 \leqslant \alpha \leqslant \delta \leqslant \|T\|$, *then* $0 \leqslant \omega_\alpha(T) - \omega_\delta(T) \leqslant \delta$.

In special case, if $\alpha = 0$, *then* $0 \leqslant \omega(T) - \omega_\delta(T) \leqslant \delta$.

(ii) If $\delta \in [0, \|T\|]$, *then* $\lim\limits_{\delta \to 0^+} \omega_\delta(T) = \omega(T)$.

(iii) If $\delta_1, \delta_2 \in [0, \|T\|]$, *then* $|\omega_{\delta_1}(T) - \omega_{\delta_2}(T)| \leqslant \delta_1 + \delta_2$.

(iv) If $\delta \in [0, \|T\|]$, *then* $0 \leqslant c_\delta(T) - c(T) \leqslant \delta$.

(v) If $\delta \in [0, \|T\|]$, *then* $\lim_{\delta \to 0^+} c_\delta(T) = c(T)$.

(vi) If $\delta_1, \delta_2 \in [0, \|T\|]$, *then* $|c_{\delta_1}(T) - c_{\delta_2}(T)| \leqslant \delta_1 + \delta_2$.

(vii) If $\delta \in [0, \|T\|]$, *then* $\omega_\delta^2(T) \leqslant \omega_{\delta^2}(T^*T)$.

(viii) If the operator T *is positive definite and* $\delta \in [0, \|T\|]$, *then* $c_{\delta^2}(T^*T) \leqslant \|T\| c_\delta(T)$.

(ix) If the operator T *is normal and* $\delta \in [0, \|T\|]$, *then*

$$\omega_\delta(T) = \omega_\delta(T^*) \quad \textit{and} \quad c_\delta(T) = c_\delta(T^*).$$

Proof.

(i) For $\alpha \leqslant \delta \leqslant \|T\|$, we have

$$\begin{aligned} \Delta_\alpha(T) &= \{x \in S_1(H) : \|Tx\| \geqslant \alpha\} \\ &= \{x \in S_1(H) : \alpha \leqslant \|Tx\| < \delta\} \cup \{x \in S_1(H) : \|Tx\| \geqslant \delta\}. \end{aligned}$$

Then, we get

$$\begin{aligned} \omega_\alpha(T) &= \sup_{x \in \Delta_\alpha(T)} |\langle Tx, x\rangle| \\ &\leqslant \max\{\sup\{|\langle Tx, x\rangle| : \alpha \leqslant \|Tx\| < \delta\}, \sup\{|\langle Tx, x\rangle| : \|Tx\| \geqslant \delta\}\} \\ &\leqslant \omega_\delta(T) + \delta. \end{aligned}$$

Hence, since the δ-numerical radius function is monotonically decreasing with respect to the variable δ, then we have

$$0 \leqslant \omega_\alpha(T) - \omega_\delta(T) \leqslant \delta.$$

(ii) The proof is clear from (i).

(iii) By (i) in this theorem we have

$$|\omega_{\delta_1}(T) - \omega_{\delta_2}(T)| \leqslant |\omega_{\delta_1}(T) - \omega(T)| + |\omega_{\delta_2} - \omega(T)| \leqslant \delta_1 + \delta_2.$$

(iv) It is known that for any $0 \leqslant \delta \leqslant \|T\|$ we have

$$0 \leqslant c(T) \leqslant c_\delta(T).$$

On the other hand, we get

$$\begin{aligned} \inf_{\Delta_\delta(T)} |\langle Tx, x\rangle| &= \inf_{S_1(H) \setminus \{x \in S_1(H) : 0 \leqslant \|Tx\| < \delta\}} |\langle Tx, x\rangle| \\ &\leqslant \inf_{S_1(H)} |\langle Tx, x\rangle| + \inf_{\{x \in S_1(H) : 0 \leqslant \|Tx\| < \delta\}} |\langle Tx, x\rangle|, \end{aligned}$$

i.e.,

$$c_\delta(T) \leqslant c(T) + \delta.$$

Therefore, it is obtained that for each $\delta \leqslant \|T\|$

$$0 \leqslant c_\delta(T) - c(T) \leqslant \delta.$$

(v)–(vi) Similarly to (ii) and (iii) the validity of (v) and (vi) it can be proved, respectively.

(vii) If the operator $T \in \mathfrak{B}(H)$, then for $x \in S_1(H)$ and $0 \leqslant \delta \leqslant \|T\|$ we have

$$\delta^2 \leqslant \|Tx\|^2 = \langle T^*Tx, x\rangle \leqslant \|T^*Tx\|.$$

The last relation shown that

$$\Delta_\delta(T) = \Delta_{\delta^2}(T^*T). \tag{3}$$

On the other hand,

$$|\langle Tx, x\rangle|^2 \leqslant \|Tx\|^2 = \langle T^*Tx, x\rangle. \tag{4}$$

Hence, from (3) and (4) it is obtained that

$$\omega_\delta^2(T) \leqslant \omega_{\delta^2}(T^*T).$$

(viii) If the operators $T \in \mathfrak{B}(H)$ is positive definite, then by Lemma 1 for $x \in S_1(H)$ and $0 \leqslant \delta \leqslant \|T\|$ we have

$$\langle T^*Tx, x\rangle = \|Tx\|^2 \leqslant \|T\|\langle Tx, x\rangle. \tag{5}$$

Hence, from (3) and (5) we have

$$c_{\delta^2}(T^*T) \leqslant \|T\| c_\delta(T).$$

(ix) The proof is clear from the definition of the δ-numerical radius and the δ-Crawford number functions. □

THEOREM 3. *Let $T, S \in \mathfrak{B}(H)$.*

(i) If $0 \leqslant \delta \leqslant \|\lambda T\|$ for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then

$$\omega_\delta(\lambda T) = |\lambda| \omega_{\frac{\delta}{|\lambda|}}(T) \quad \text{and} \quad c_\delta(\lambda T) = |\lambda| c_{\frac{\delta}{|\lambda|}}(T).$$

(ii) If $\delta \leqslant \|T+S\|$, then

$$\omega_\delta(T+S) \leqslant \omega_{\delta-\|S\|}(T) + \omega_{\delta-\|T\|}(S) \quad \text{and} \quad c_\delta(T+S) \geqslant c_{\delta-\|T\|}(S) - \omega_{\delta-\|S\|}(T).$$

Proof.

(i) If $\lambda \neq 0$, then

$$\omega_\delta(\lambda T) = \sup_{\Delta_\delta(\lambda T)} |\langle \lambda Tx, x\rangle| = |\lambda| \sup_{\Delta_\delta(\lambda T)} |\langle Tx, x\rangle|. \tag{6}$$

On the other hand, we get

$$\Delta_\delta(\lambda T) = \{x \in S_1(H) : \|\lambda Tx\| \geqslant \delta\} = \left\{x \in S_1(H) : \|Tx\| \geqslant \frac{\delta}{|\lambda|}\right\} \tag{7}$$

From (6) and (7) we have

$$\omega_\delta(\lambda T) = |\lambda| \sup_{\Delta_{\frac{\delta}{|\lambda|}}(T)} |\langle Tx, x\rangle| = |\lambda|\omega_{\frac{\delta}{|\lambda|}}(T).$$

Similarly, the second equality it can be proved.

(ii) Since for $x \in S_1(H)$,

$$\delta \leqslant \|(T+S)x\| \leqslant \|Tx\| + \|Sx\|,$$

then we have

$$\delta \leqslant \|Tx\| + \|S\| \quad \text{and} \quad \delta \leqslant \|T\| + \|Sx\|,$$

that is,

$$\delta - \|S\| \leqslant \|Tx\| \quad \text{and} \quad \delta - \|T\| \leqslant \|Sx\|.$$

Thus, we get

$$\Delta_\delta(T+S) \subset \Delta_{\delta-\|S\|}(T) \cap \Delta_{\delta-\|T\|}(S). \tag{8}$$

On the other hand, for $x \in S_1(H)$, we have

$$|\langle (T+S)x, x\rangle| \leqslant |\langle Tx, x\rangle| + |\langle Sx, x\rangle|. \tag{9}$$

Hence, from (8) and (9) we obtain

$$\sup_{\Delta_\delta(T+S)} |\langle (T+S)x, x\rangle| \leqslant \sup_{\Delta_{\delta-\|S\|}(T)} |\langle Tx, x\rangle| + \sup_{\Delta_{\delta-\|T\|}(S)} |\langle Sx, x\rangle|.$$

Thus, we get

$$\omega_\delta(T+S) \leqslant \omega_{\delta-\|S\|}(T) + \omega_{\delta-\|T\|}(S).$$

Also, for any $x \in H$ we have

$$|\langle Tx, x\rangle| \leqslant |\langle (T+S)x, x\rangle| + |\langle Sx, x\rangle|. \tag{10}$$

Hence, from (8) and (10) we obtain

$$\begin{aligned}
\inf_{\Delta_{\delta-\|S\|}(T)} |\langle Tx, x\rangle| &\leqslant \inf_{\Delta_\delta(T+S)} |\langle Tx, x\rangle| \\
&\leqslant \inf_{\Delta_\delta(T+S)} \left(|\langle (T+S)x, x\rangle| + |\langle Sx, x\rangle|\right) \\
&\leqslant \inf_{\Delta_\delta(T+S)} |\langle (T+S)x, x\rangle| + \sup_{\Delta_{\delta-\|T\|}(S)} |\langle Sx, x\rangle|
\end{aligned}$$

and

$$c_{\delta-\|S\|}(T) \leqslant c_\delta(T+S) + \omega_{\delta-\|T\|}(S),$$

i.e.

$$c_\delta(T+S) \geqslant c_{\delta-\|S\|}(T) - \omega_{\delta-\|T\|}(S). \quad \square$$

THEOREM 4. *Let* $T \in \mathfrak{B}(H)$ *and* $\delta \leqslant \|T\|$. *Then*

$$\omega_{\delta/2}(ReT) \leqslant \omega(T) \quad \text{and} \quad \omega_{\delta/2}(ImT) \leqslant \omega(T).$$

Proof. For any $x \in S_1(H)$ and $\delta \leqslant \|T\|$ we have

$$\delta \leqslant \|Tx\| = \|ReTx\| + \|ImTx\|.$$

Then, $\|ReTx\| \geqslant \frac{\delta}{2}$ or $\|ImTx\| \geqslant \frac{\delta}{2}$ for $x \in S_1(H)$. Hence

$$\Delta_\delta(T) \subset \Delta_{\delta/2}(ReT) \cup \Delta_{\delta/2}(ImT). \tag{11}$$

On the other hand, for $x \in H$

$$|\langle Tx,x\rangle| \geqslant |\langle ReTx,x\rangle| \quad \text{and} \quad |\langle Tx,x\rangle| \geqslant |\langle ImTx,x\rangle|. \tag{12}$$

Consequently, from (11) and (12) we obtained that

$$\sup_{x\in S_1(H)} |\langle Tx,x\rangle| \geqslant \sup_{\Delta_{\delta/2}(ReT)\cup\Delta_{\delta/2}(ImT)} |\langle ReTx,x\rangle| \geqslant \sup_{\Delta_{\delta/2}(ReT)} |\langle ReTx,x\rangle|.$$

This shows that

$$\omega_{\delta/2}(ReT) \leqslant \omega(T).$$

Similarly, it can be proved that

$$\omega_{\delta/2}(ImT) \leqslant \omega(T). \quad \square$$

THEOREM 5. *Let* $T \in \mathfrak{B}(H)$ *and* $0 \leqslant \delta \leqslant \|T\|$. *Then*

$$\omega_\delta^4(T) \leqslant \|\,|T|\,\|^3 \omega_\delta(|T|) \quad \text{and} \quad c_\delta^4(T) \leqslant \|\,|T|\,\|^3 c_\delta(|T|).$$

Proof. For any $x \in S_1(H)$ and $0 \leqslant \delta \leqslant \|T\|$ we have

$$\delta^2 \leqslant \|Tx\|^2 = \langle T^*Tx,x\rangle = \|(T^*Tx)^{1/2}\|^2 = \|\,|Tx|\,\|^2.$$

Then

$$\Delta_\delta(T) = \Delta_\delta(|T|). \tag{13}$$

On the other hand, for $x \in S_1(H)$

$$\begin{aligned} |\langle Tx,x\rangle|^4 &\leqslant \|Tx\|^4 \\ &= \langle T^*Tx,x\rangle^2 \\ &= \langle (T^*T)^{1/4}x,(T^*T)^{3/4}x\rangle^2 \\ &\leqslant \|(T^*T)^{1/4}x\|^2\|(T^*T)^{3/4}\|^2 \\ &= \langle (T^*T)^{1/2}x,x\rangle\|(T^*T)^{3/4}\|^2 \\ &= \|\,|T|\,\|^3\langle |T|x,x\rangle. \end{aligned} \tag{14}$$

Hence, from (13) and (14) the validities of theorem it is clear. $\square$

THEOREM 6. *Let* $T \in \mathfrak{B}(H)$ *and* $0 \leqslant \delta \leqslant \|T\|$. *Then*

$$\omega_\delta^2(T) \leqslant \omega_{\delta^2}(|T|^2).$$

Proof. For $x \in S_1(H)$ and $0 \leqslant \delta \leqslant \|T\|$, we have

$$\delta^2 \leqslant \|Tx\|^2 = \langle T^*Tx, x\rangle \leqslant \|T^*Tx\|.$$

Then,

$$\Delta_\delta(T) \subset \Delta_{\delta^2}(|T|^2). \tag{15}$$

On the other hand, for any $x \in S_1(H)$

$$|\langle Tx, x\rangle|^2 \leqslant \|Tx\|^2 = \langle T^*Tx, x\rangle = \langle |T|^2 x, x\rangle. \tag{16}$$

Thus, from (15) and (16) it is obtained that

$$\omega_\delta^2(T) \leqslant \omega_{\delta^2}(|T|^2). \quad \square$$

By Theorem 6, we get the following corollary.

COROLLARY 1. *If* $T \in \mathfrak{B}(H)$, $T = T^* \geqslant 0$, *and* $0 \leqslant \delta \leqslant \|T\|$, *then*

$$\omega_\delta^2(T) \leqslant \omega_{\delta^2}(T^2).$$

THEOREM 7. *Let* $T \in \mathfrak{B}(H)$, $T = T^* > 0$, *and* $\delta \leqslant \|T^2\|$. *Then*

$$\omega_\delta(T^2) \leqslant \|T\|\omega_{\frac{\delta}{\|T\|}}(T) \quad \text{and} \quad c^2_{\frac{\delta}{\|T\|}}(T) \leqslant c_\delta(T^2).$$

Proof. Since $\delta \leqslant \|T^2x\| \leqslant \|T\|\|Tx\|$, then

$$\Delta_\delta(T^2) \subset \Delta_{\frac{\delta}{\|T\|}}(T). \tag{17}$$

Moreover, by Lemma 1 we have

$$\langle T^2x, x\rangle = \|Tx\|^2 \leqslant \|T\|\langle Tx, x\rangle, \ \ x \in H. \tag{18}$$

Therefore, from the relations (17) and (18) it implies that

$$\omega_\delta(T^2) \leqslant \|T\|\omega_{\frac{\delta}{\|T\|}}(T).$$

It is known that if $T = T^* \geqslant 0$, then

$$\langle Tx, x\rangle^2 \leqslant \langle T^2x, x\rangle, \ \ x \in H. \tag{19}$$

Consequently, from (17) and (19) we have

$$c^2_{\frac{\delta}{\|T\|}}(T) \leqslant c_\delta(T^2). \quad \square$$

Now, we explore the relationships between the δ-numerical radius and the δ-Crawford number functions of the direct sum of Hilbert space operators and those of their coordinate operators. Furthermore, we obtain some generalization formulas for lower and upper bounds of the δ-numerical radius and the δ-Crawford number functions of such direct sums.

THEOREM 8. *Let* $A = \bigoplus_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$. *Then,*

(i) for any $0 \leqslant \delta_n \leqslant \|A_n\|$, $n \geqslant 1$, $\sup\limits_{n\geqslant 1} \omega_{\delta_n}(A_n) \leqslant \omega_{\delta^*}(A)$ *where* $\delta^* = \sup\limits_{n\geqslant 1} \delta_n$,

(ii) for any $0 \leqslant \delta \leqslant \|A\|$, $\omega_\delta(A) \leqslant \sup\limits_{n\geqslant 1} \omega_\delta(A_n)$,

(iii) for any $0 \leqslant \delta \leqslant \|A\|$, $\omega_\delta(A) = \sup\limits_{n\geqslant 1} \omega_\delta(A_n)$.

Proof.

(i) If $x^* = (0,0,\dots,0,x_n,0,\dots)$, $x_n \in \Delta_{\delta_n}(A_n)$, $n \geqslant 1$, then $x^* \in S_1(H)$ and

$$\delta_n \leqslant (0^2 + 0^2 + \ldots + 0^2 + \|A_n x_n\|^2 + 0^2 + \ldots)^{1/2} = \|Ax^*\|, \quad n \geqslant 1.$$

Hence, $\delta^* = \sup\limits_{n\geqslant 1} \delta_n \leqslant \|Ax^*\|$. So,

$$\left\{(0,\dots,0,x_n,0,\dots) : x_n \in \Delta_{\delta_n}(A_n)\right\} \subset \Delta_{\delta^*}(A). \tag{20}$$

On the other hand, for $x^* = (0,\dots,0,x_n,0,\dots)$, $x_n \in \Delta_{\delta_n}(A_n)$, $n \geqslant 1$, we have

$$|\langle A_n x_n, x_n\rangle| = |\langle Ax^*, x^*\rangle|. \tag{21}$$

Hence, from (20) and (21) we obtain

$$\sup_{\left\{(0,\dots,0,x_n,0,\dots):x_n\in\Delta_{\delta_n}(A_n)\right\}} |\langle A_n x_n, x_n\rangle| \leqslant \sup_{\Delta_{\delta^*}(A)} |\langle Ax, x\rangle|,$$

i.e.

$$\omega_{\delta_n}(A_n) \leqslant \omega_{\delta^*}(A), \quad n \geqslant 1.$$

Consequently,

$$\sup_{n\geqslant 1} \omega_{\delta_n}(A_n) \leqslant \omega_{\delta^*}(A).$$

(ii) If $x \in W_\delta(A)$, then

$$\delta^2 \leqslant \|Ax\|^2 = \sum_{n=1}^{\infty} \|A_n x_n\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 \left\| A_n \left(\frac{x_n}{\|x_n\|} \right) \right\|^2 \text{ if } x_n \neq 0. \tag{22}$$

Hence, for fixed $n \geqslant 1$ and $x_n \neq 0$

$$x_* = \left(0,\dots,0,\frac{x_n}{\|x_n\|},0,\dots\right) \in W_\delta(A).$$

Then, from (22) we obtain

$$\delta^2 \leqslant \|x_n\|^2 \left\|A_n\left(\frac{x_n}{\|x_n\|}\right)\right\|^2 \leqslant \left\|A_n\left(\frac{x_n}{\|x_n\|}\right)\right\|^2.$$

Consequently, $\frac{x_n}{\|x_n\|} \in W_\delta(A_n)$, $x_n \neq 0$. Then, we have

$$W_\delta(A) \subset \bigcup_{n=1}^{\infty} W_\delta(A_n).$$

From the last relation, we have

$$\omega_\delta(A) \leqslant \sup_{n\geqslant 1} \omega_\delta(A_n).$$

(iii) From (i) and (ii) in this theorem, the proof is clear. ☐

Similar to the proof of Theorem 8, the following theorem can also be proved.

THEOREM 9. *Let* $A = \bigoplus_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$. *Then,*

(i) for any $0 \leqslant \delta_n \leqslant \|A_n\|$, $n \geqslant 1$, $c_{\delta^*}(A) \leqslant \inf_{n\geqslant 1} c_{\delta_n}(A_n)$ *where* $\delta^* = \sup_{n\geqslant 1} \delta_n$,

(ii) for any $0 \leqslant \delta \leqslant \|A\|$, $\inf_{n\geqslant 1} c_\delta(A_n) \leqslant c_\delta(A)$,

(iii) for any $0 \leqslant \delta \leqslant \|A\|$, $c_\delta(A) = \inf_{n\geqslant 1} c_\delta(A_n)$.

By (i) of Theorem 8, we get the following corollary which is proved in [8, Thm. 9] in the special case.

COROLLARY 2. *Let* $A = A_1 \oplus \dots \oplus A_n \oplus 0 \oplus \dots \in \mathfrak{B}(H)$ *and* $\delta_m \leqslant \|A_m\|$, $1 \leqslant m \leqslant n$, *and* $\delta_m = 0$, $m \geqslant n+1$. *Then,*

$$\omega^2_{\delta_m}(A_m) \leqslant \omega^2_{\delta^*}(A), \ \ 1 \leqslant m \leqslant n,$$

where $\delta^* = \sup_{1\leqslant m\leqslant n} \delta_m$. *Hence,*

$$\omega_{\delta^*}(A) \geqslant \left(\frac{\omega^2_{\delta_1}(A_1) + \dots + \omega^2_{\delta_n}(A_n)}{n}\right)^{1/2}.$$

If $\delta_m = 0,\ 1 \leqslant m \leqslant n$, *then* $\delta = 0$ *and the last relation shows that*

$$\begin{aligned}\omega(A) &\geqslant \left(\frac{\sum\limits_{m=1}^{n} \omega^2(A_m)}{n}\right)^{1/2} \\ &\geqslant \frac{1}{\sqrt{n}}\left(\sum_{m=1}^{n} \frac{\|A_m\|^2}{4}\right)^{1/2} \\ &= \frac{1}{2\sqrt{n}}\left(\sum_{m=1}^{n} \|A_m\|^2\right)^{1/2} \\ &\geqslant \frac{\max\limits_{1\leqslant m\leqslant n} \|A_m\|}{2\sqrt{n}} \\ &= \frac{\omega(A)}{2\sqrt{n}}.\end{aligned}$$

If we take $\delta = 0$ in (iii) of Theorem 8 and Theorem 9, we get the following corollary which proved in [19, Thm. 4 and Thm. 7] and [26].

COROLLARY 3. *Let* $A = \bigoplus\limits_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$. *Then,*

$$\omega(A) = \sup_{n\geqslant 1} \omega(A_n) \quad \text{and} \quad c(A) = \inf_{n\geqslant 1} c(A_n).$$

THEOREM 10. *Let* $A = \bigoplus\limits_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$ *and* $0 \leqslant \delta \leqslant \|A\|$. *Then,*

$$\omega_\delta^2(A) \leqslant \omega_{\delta^2-\alpha}\left(Re^2 A + Im^2 A\right),$$

where $\alpha = \sum\limits_{n=1}^{\infty} \alpha_n,\ \alpha_n = \|ReA_nImA_n - ImA_nReA_n\|,\ n \geqslant 1$.

Proof. For $x \in S_1(H)$ and $n \geqslant 1$, we have

$$|\langle Ax, x\rangle|^2 \leqslant \langle \left(Re^2 A + Im^2 A\right) x, x\rangle. \tag{23}$$

On the other hand, for $x \in S_1(H)$ and $n \geqslant 1$ we have

$$\|A_n x_n\|^2 = \langle (ReA_n + iImA_n)x_n, (ReA_n + iImA_n)x_n\rangle \leqslant \langle \left(Re^2 A_n + Im^2 A_n\right) x_n, x_n\rangle + \alpha_n,$$

where $\alpha_n = \|ReA_nImA_n - ImA_nReA_n\|$, $n \geqslant 1$. Hence, it is obtained that

$$\begin{aligned}\delta^2 &\leqslant \|Ax\|^2\\ &= \sum_{n=1}^{\infty} \|A_nx_n\|^2\\ &\leqslant \langle \sum_{n=1}^{\infty} \left(Re^2A_n + Im^2A_n\right) x_n, x_n\rangle + \alpha\\ &\leqslant \left\| \sum_{n=1}^{\infty} \left(Re^2A_n + Im^2A_n\right) x_n \right\| + \alpha\\ &= \left\| \left(Re^2A + Im^2A\right) x\right\| + \alpha,\end{aligned}$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Consequently, we get

$$\delta^2 - \alpha \leqslant \left\| \left(Re^2A + Im^2A\right) x\right\|.$$

Thus,

$$\Delta_{\delta}(A) \subset \Delta_{\delta^2-\alpha}\left(Re^2A + Im^2A\right). \tag{24}$$

Finally, (23) and (24) show that

$$\omega_{\delta}^2(A) \leqslant \omega_{\delta^2-\alpha}\left(Re^2A + Im^2A\right). \quad \square$$

By Theorem 10, we get the following corollary.

COROLLARY 4. *If the operator* $A \in \mathfrak{B}(H)$ *is normal and* $0 \leqslant \delta \leqslant \|A\|$*, then*

$$\omega_{\delta}^2(A) \leqslant \omega_{\delta^2}\left(Re^2A + Im^2A\right).$$

Theorem 10 can be reformulated as form:

REMARK 1. Let $A = \bigoplus_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$ and $0 \leqslant \delta \leqslant \|A\|$. Then

$$\omega_{\delta}^2(A) \leqslant \omega_{\delta^2-\alpha}\left(Re^2A + Im^2A\right) \leqslant \|Re^2A + Im^2A\| = \sup_{n\geqslant 1} \|Re^2A_n + Im^2A_n\|.$$

If we take $n = 1$, i.e. $A = A_1 \oplus 0 \oplus \ldots$, $A_1 \in \mathfrak{B}(H_1)$ and $\delta = 0$ in Remark 1, then we get the following inequality which says that second inequality in (2)

$$\omega^2(A_1) \leqslant \|Re^2A_1 + Im^2A_1\| = \frac{1}{2}\|A_1^*A_1 + A_1A_1^*\|.$$

THEOREM 11. *Let* $A = \bigoplus_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$ *and* $0 \leqslant \delta \leqslant \|A\|$. *Then*

$$c_{\delta}^2(A) \geqslant \frac{1}{2} c_{\delta^2-\alpha}\left(Re^2A + Im^2A\right),$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$, $\alpha_n = \|ReA_nImA_n - ImA_nReA_n\|$, $n \geqslant 1$.

Proof. For $n \geqslant 1$ and $x \in H$ we have

$$|\langle Ax,x\rangle|^2 \geqslant \frac{1}{2}\left(|\langle ReAx,x\rangle|^2 + |\langle ImAx,x\rangle|^2\right). \tag{25}$$

On the other hand, from the proof of Theorem 10 it is known that

$$\Delta_\delta(A) \subset \Delta_{\delta^2-\alpha}\left(Re^2A + Im^2A\right). \tag{26}$$

Therefore, from (25) and (26) it is established that

$$c_\delta^2(A) \geqslant \frac{1}{2}c_{\delta^2-\alpha}\left(Re^2A + Im^2A\right). \quad \square$$

Theorem 11 can be reformulated as form:

REMARK 2. Let $A = \bigoplus_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$ and $0 \leqslant \delta \leqslant \|A\|$. Then,

$$c_\delta^2(A) \geqslant \frac{1}{4}c_{\delta^2-\alpha}\left(A^*A + AA^*\right),$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$, $\alpha_n = \|ReA_nImA_n - ImA_nReA_n\|$, $n \geqslant 1$.

By Theorem 11, we get the following corollary.

COROLLARY 5. *If the operator $A \in \mathfrak{B}(H)$ is normal and $0 \leqslant \delta \leqslant \|A\|$, then*

$$\begin{aligned} c_\delta^2(A) &\geqslant \frac{1}{4}c_{\delta^2}\left(Re^2A + Im^2A\right) \\ &= \frac{1}{4}c_{\delta^2}\left(A^*A + AA^*\right). \end{aligned}$$

Moreover, if $\delta = 0$, then

$$c^2(A) \geqslant \frac{1}{4}c^2\left(A^*A + AA^*\right).$$

Also, if $A = A_1 \oplus 0 \oplus \ldots$, $A_1 \in \mathfrak{B}(H_1)$ and $\delta = 0$, then

$$c^2(A_1) \geqslant \frac{1}{4}c(A_1^*A_1 + A_1A_1^*).$$

3. Bounds of the δ-numerical radius in sectorial cases of coordinate operators

In this section, we obtain some evolutions for upper and lower bounds of the δ-numerical radius function of direct sum of Hilbert space operators in the sectorial cases of coordinate operators.

THEOREM 12. *Let* $A = \bigoplus_{n=1}^{\infty} A_n \in \mathfrak{B}(H)$, A_n *be accretive-dissipative operator in* H_n *such that* $A_n \in Sec_{(0,\alpha_n)}(H_n)$, $0 \leqslant \alpha_n < \frac{\pi}{2}$, $n \geqslant 1$, $\alpha^* = \sup_{n\geqslant 1} \alpha_n < \frac{\pi}{2}$. *Then, for* $0 \leqslant \delta \leqslant \|A\|$

$$\omega_\delta(A) \leqslant \sec(\alpha^*)\omega_{\delta^*}(ReA),$$

where $\delta^* = \dfrac{\delta^2}{a(1+\tan(\alpha^*))}$ *and* $a = 2\max_{n\geqslant 1}\{\|ReA_n\|, \|ImA_n\|\} \neq 0$.

Proof. For any $x \in H$, we have

$$\begin{aligned}
|\langle Ax,x\rangle|^2 &= \sum_{n=1}^{\infty} |\langle A_n x_n, x_n\rangle|^2 \\
&= \sum_{n=1}^{\infty} \left(|\langle ReA_n x_n, x_n\rangle|^2 + |\langle ImA_n x_n, x_n\rangle|^2\right) \\
&\leqslant \sum_{n=1}^{\infty} \left(|\langle ReA_n x_n, x_n\rangle|^2 + \tan^2(\alpha_n)|\langle ReA_n x_n, x_n\rangle|^2\right) \\
&\leqslant \left(1+\tan^2(\alpha^*)\right) \sum_{n=1}^{\infty} |\langle ReA_n x_n, x_n\rangle|^2 \\
&= \frac{1}{\cos^2(\alpha^*)} \sum_{n=1}^{\infty} |\langle ReA_n x_n, x_n\rangle|^2 \\
&= \sec^2(\alpha^*)|\langle ReAx,x\rangle|^2,
\end{aligned}$$

where $\alpha^* = \sup_{n\geqslant 1}\alpha_n < \frac{\pi}{2}$. That is, for each $x \in H$

$$|\langle Ax,x\rangle| \leqslant \sec(\alpha^*)|\langle ReAx,x\rangle|. \tag{27}$$

On the other hand, since the ReA_n and ImA_n are accretive-dissipative operators, then from Lemma 1 for each $x \in S_1(H)$ and $0 \leqslant \delta \leqslant \|A\|$ we have

$$\begin{aligned}
\delta^2 &\leqslant \|Ax\|^2 \\
&= \sum_{n=1}^{\infty} \|A_n x_n\|^2 \\
&\leqslant \sum_{n=1}^{\infty} (\|ReA_n x_n\| + \|ImA_n x_n\|)^2
\end{aligned}$$

$$\begin{aligned}
&\leqslant 2\left(\sum_{n=1}^{\infty}\|ReA_nx_n\|^2+\sum_{n=1}^{\infty}\|ImA_nx_n\|^2\right)\\
&\leqslant 2\left(\sum_{n=1}^{\infty}\|ReA_nx\|\langle ReA_nx_n,x_n\rangle+\sum_{n=1}^{\infty}\|ImA_nx\|\langle ImA_nx_n,x_n\rangle\right)\\
&\leqslant a\left(\sum_{n=1}^{\infty}\langle ReA_nx_n,x_n\rangle+\sum_{n=1}^{\infty}\tan(\alpha_n)\langle ReA_nx_n,x_n\rangle\right)\\
&\leqslant a\,(1+\tan(\alpha^*))\sum_{n=1}^{\infty}\langle ReA_nx_n,x_n\rangle\\
&= a\,(1+\tan(\alpha^*))\,\langle ReAx,x\rangle\\
&\leqslant a\,(1+\tan(\alpha^*))\,\|ReAx\|
\end{aligned}$$

where $a=2\max\{\|ReA_n\|,\|ImA_n\|\}$. Therefore

$$\delta^*=\frac{\delta^2}{a\,(1+\tan(\alpha^*))}\leqslant\|ReAx\|.$$

Hence, we have

$$\Delta_\delta(A)\subset\Delta_{\delta^*}(ReA). \tag{28}$$

Consequently, from (27) and (28) it implies that

$$\omega_\delta(A)\leqslant\sec(\alpha^*)\omega_{\delta^*}(ReA). \quad \square$$

By similar technique the following result can be proved.

THEOREM 13. *Let* $A=\bigoplus_{n=1}^{\infty}A_n\in\mathfrak{B}(H)$, A_n *be accretive-dissipative operator in* H_n *and* $A_n\in Sec_{(0,\alpha_n)}(H_n)$, $0<\alpha_n<\dfrac{\pi}{2}$, $n\geqslant 1$, $\alpha_*=\inf_{n\geqslant 1}\alpha_n>0$. *Then, for* $0\leqslant\delta\leqslant\|A\|$

$$\omega_\delta(A)\leqslant\csc(\alpha_*)\omega_{\delta_*}(ImA),$$

where $\delta_*=\dfrac{\delta^2}{a\,(1+\cot(\alpha_*))}$ *and* $a=2\max\{\|ReA_n\|,\|ImA_n\|\}\neq 0$.

If we take $A=A_1\oplus 0\oplus\ldots$, $A_1\in Sec_{(0,\alpha_1)}(H_1)$ and $\delta=0$ in Theorem 12 and Theorem 13, we get the following corollary, whose item (i) was previously proved in [3, Corollary 3.2] in case of $n\times n$ matrix.

COROLLARY 6.

(i) Under the assumptions of Theorem 12

$$\omega(A_1)\leqslant\sec(\alpha_1)\|ReA_1\|,$$

(ii) Under the assumptions of Theorem 13

$$\omega(A_1) \leqslant \csc(\alpha_1)\|ImA_1\|.$$

If we take $\delta = 0$ in Theorem 12 and Theorem 13, we get the following corollaries.

COROLLARY 7. *(i) Under the assumptions of Theorem 12,*

$$\begin{aligned}\omega^2(A) &\leqslant \frac{1}{\cos^2(\alpha^*)}\omega^2(ReA)\\ &= \frac{1}{\cos^2(\alpha^*)}\|ReA\|^2\\ &= \frac{1}{\cos^2(\alpha^*)}\sup_{n\geqslant 1}\|ReA_n\|^2\\ &\leqslant \frac{1}{\cos^2(\alpha^*)}\sup_{n\geqslant 1}\left(\|ReA_n\|^2+\|ImA_n\|^2\right)\\ &= \frac{1}{2\cos^2(\alpha^*)}\sup_{n\geqslant 1}\|A_n^*A_n+A_nA_n^*\|\\ &= \frac{1}{2\cos^2(\alpha^*)}\|A^*A+AA^*\|,\ 0\leqslant \alpha^* < \frac{\pi}{2}.\end{aligned}$$

(ii) Under the assumptions of Theorem 13,

$$\begin{aligned}\omega^2(A) &\leqslant \frac{1}{\sin^2(\alpha_*)}\omega^2(ImA)\\ &= \frac{1}{\sin^2(\alpha_*)}\|ImA\|^2\\ &= \frac{1}{\sin^2(\alpha_*)}\sup_{n\geqslant 1}\|ImA_n\|^2\\ &\leqslant \frac{1}{\sin^2(\alpha_*)}\sup_{n\geqslant 1}\left(\|ReA_n\|^2+\|ImA_n\|^2\right)\\ &= \frac{1}{2\sin^2(\alpha_*)}\sup_{n\geqslant 1}\|A_n^*A_n+A_nA_n^*\|\\ &= \frac{1}{2\sin^2(\alpha_*)}\|A^*A+AA^*\|,\ \ 0< \alpha_* < \frac{\pi}{2}.\end{aligned}$$

Note that in case of $A = A_1 \oplus 0 \oplus \ldots,\ A_1 \in Sec_{(0,\alpha_1)}(H_1)$ and $\alpha^* = \alpha_1 = 0$, item (i) of Corollary 7 was proved by Kittaneh in [16].

COROLLARY 8. *Under the assumptions of Theorem 12 and Theorem 13,*

$$\begin{aligned}\omega^2(A) &\leqslant \frac{1}{2}\left(\sec^2(\alpha^*)\|ReA\|^2+\csc^2(\alpha_*)\|ImA\|^2\right)\\ &\leqslant \frac{1}{2}\sup\left\{\sec^2(\alpha^*),\csc^2(\alpha_*)\right\}\left(\|ReA\|^2+\|ImA\|^2\right)\\ &= \frac{1}{4}\sup\left\{\sec^2(\alpha^*),\csc^2(\alpha_*)\right\}\|A^*A+AA^*\|,\ \ 0<\alpha_*,\alpha^*<\frac{\pi}{2}.\end{aligned}$$

In additionally, if $A=A_1\oplus 0\oplus\dots,\ A_1\in Sec_{\left(0,\frac{\pi}{4}\right)}(H_1)$, *then from the last inequality*

$$\omega^2(A_1)\leqslant \frac{1}{2}\|A_1^*A_1+A_1A_1^*\|.$$

THEOREM 14. *Let* $A=\bigoplus_{n=1}^{\infty}A_n\in\mathfrak{B}(H)$ *and for some* $0<\gamma_n\leqslant\alpha_n<\dfrac{\pi}{2},\ A_n\in Sec_{(\gamma_n,\alpha_n)}(H_n),\ n\geqslant 1,\ \gamma_*=\inf\limits_{n\geqslant 1}\gamma_n>0$, *and* $\alpha^*=\sup\limits_{n\geqslant 1}\alpha_n$.

(i)
- *If* $\delta\leqslant\|ReA\|$, *then* $\omega_\delta(ReA)\leqslant\cos(\gamma_*)\omega(A)$,
- *If* $\delta\leqslant\|ImA\|$, *then* $\omega_\delta(ImA)\leqslant\sin(\alpha^*)\omega(A)$,

(ii)
- *If* $\delta\leqslant\|ReA\|$, *then* $c_\delta(ReA)\geqslant\cos(\alpha^*)c(A)$,
- *If* $\delta\leqslant\|ImA\|$, *then* $c_\delta(ImA)\geqslant\sin(\gamma_*)c(A)$.

Proof.

(i) For any $x\in H$, we have

$$|\langle Ax,x\rangle|^2=\sum_{n=1}^{\infty}|\langle A_nx_n,x_n\rangle|^2=\sum_{n=1}^{\infty}\left(\langle ReA_nx_n,x_n\rangle^2+\langle ImA_nx_n,x_n\rangle^2\right).$$

Also, since

$$|\langle ImA_nx_n,x_n\rangle|\geqslant\tan(\gamma_n)|\langle ReA_nx_n,x_n\rangle|,\ \ x_n\in H_n,\ \ n\geqslant 1,$$

then

$$\begin{aligned}|\langle Ax,x\rangle|^2 &\geqslant \sum_{n=1}^{\infty}(1+\tan^2(\gamma_n))|\langle ReA_nx_n,x_n\rangle|^2\\ &\geqslant (1+\tan^2(\gamma_*))\sum_{n=1}^{\infty}|\langle ReA_nx_n,x_n\rangle|^2\\ &= \sec^2(\gamma_*)|\langle ReAx,x\rangle|^2, \qquad (29)\end{aligned}$$

where $\gamma_*=\inf\limits_{n\geqslant 1}\gamma_n$. Consequently, from (29) and the inclusion $\Delta_\delta(ReA)\subset S_1(H)$ we obtain

$$\sup_{S_1(H)}|\langle Ax,x\rangle|\geqslant\sec(\gamma_*)\sup_{\Delta_\delta(ReA)}|\langle ReAx,x\rangle|.$$

Hence, it is obtained that

$$\omega(A) \geqslant \sec(\gamma_*)\omega_\delta(ReA),$$

that is

$$\omega_\delta(ReA) \leqslant \cos(\gamma_*)\omega(A).$$

Similarly, since

$$|\langle Ax,x\rangle|^2 = \sum_{n=1}^{\infty}\left(\langle ReA_nx_n,x_n\rangle^2 + \langle ImA_nx_n,x_n\rangle^2\right)$$

and

$$|\langle ImA_nx_n,x_n\rangle| \leqslant \tan(\alpha_n)|\langle ReA_nx_n,x_n\rangle|, x_n \in H_n,$$

then, we obtain

$$\begin{aligned}|\langle Ax,x\rangle|^2 &\geqslant \sum_{n=1}^{\infty}(1+\cot^2(\alpha_n))|\langle ImA_nx_n,x_n\rangle|^2\\ &\geqslant (1+\cot^2(\alpha^*))\sum_{n=1}^{\infty}|\langle ImA_nx_n,x_n\rangle|^2\\ &= \csc^2(\alpha^*)|\langle ImAx,x\rangle|^2, \end{aligned} \tag{30}$$

where $\alpha^* = \sup\limits_{n\geqslant 1}\alpha_n$. From (30) and the inclusion $\Delta_\delta(ImA) \subset S_1(H)$, it is established that

$$\omega(A) \geqslant \csc(\alpha^*)\omega_\delta(ImA),$$

that is

$$\omega_\delta(ImA) \leqslant \sin(\alpha^*)\omega(A).$$

(ii) Since $A_n \in Sec_{(\gamma_n,\alpha_n)}(H_n)$, $n \geqslant 1$ we have for $x \in H$

$$|\langle Ax,x\rangle|^2 \leqslant (1+\tan^2(\alpha^*))\sum_{n=1}^{\infty}|\langle ReA_nx_n,x_n\rangle|^2 = \sec^2(\alpha^*)|\langle ReAx,x\rangle|^2. \tag{31}$$

Thus, from (31) it implies that

$$\inf_{S_1(H)}|\langle Ax,x\rangle| \leqslant \sec(\alpha^*)\inf_{\Delta_\delta(ReA)}|\langle ReAx,x\rangle|.$$

Hence

$$c(A) \leqslant \sec(\alpha^*)c_\delta(ReA),$$

that is

$$c_\delta(ReA) \geqslant \cos(\alpha^*)c(A),$$

where $\alpha^* = \sup\limits_{n\geqslant 1}\alpha_n$.

On the other hand, since for $n \geqslant 1$

$$|\langle ReA_n x_n, x_n\rangle| \leqslant \cot(\gamma_*)|\langle ImA_n x_n, x_n\rangle|,\ x_n \in H_n,$$

then

$$\begin{aligned}|\langle Ax,x\rangle|^2 &= \sum_{n=1}^{\infty} |\langle A_n x_n, x_n\rangle|^2 \\ &\leqslant (1+\cot^2(\gamma_*))|\langle ImAx,x\rangle|^2 \\ &= \csc^2(\gamma_*)|\langle ImAx,x\rangle|^2, \end{aligned} \tag{32}$$

where $\gamma_* = \inf_{n\geqslant 1} \gamma_n$. Thus, from (32) we have

$$\inf_{S_1(H)} |\langle Ax,x\rangle| \leqslant \csc(\gamma_*) \inf_{\Delta_\delta(ImA)} |\langle ImAx,x\rangle|.$$

Finally, we have

$$c(A) \leqslant \csc(\gamma_*) c_\delta(ImA),$$

that is

$$c_\delta(ImA) \geqslant \sin(\gamma_*)c(A). \quad \square$$

If we take $\delta = 0$ in (i) of Theorem 14, we get the following corollary.

COROLLARY 9. *If* $A = A_1 \oplus 0 \oplus \ldots$, $A_1 \in Sec_{(\frac{\pi}{2}-\alpha_1,\alpha_1)}(H_1)$, $0 < \alpha_1 < \dfrac{\pi}{2}$, *then*

$$\omega(ReA_1) \leqslant \sin(\alpha_1)\omega(A_1) \quad \text{and} \quad \omega(ImA_1) \leqslant \sin(\alpha_1)\omega(A_1).$$

Note that since ReA_1 and ImA_1 are self-adjoint operators, then $\omega(ReA_1) = \|ReA_1\|$ and $\omega(ImA_1) = \|ImA_1\|$. Hence by Corollary 9 we get the following corollary which proved in [23, Lemma 3.1] in the case of matrix.

COROLLARY 10. *If* $A = A_1 \oplus 0 \oplus \ldots$, $A_1 \in Sec_{(\frac{\pi}{2}-\alpha_1,\alpha_1)}(H_1)$, $0 < \alpha_1 < \dfrac{\pi}{2}$, *then*

$$\|ReA_1\| \leqslant \sin(\alpha_1)\omega(A_1) \quad \text{and} \quad \|ImA_1\| \leqslant \sin(\alpha_1)\omega(A_1).$$

On the other hand, since $\|A_1\| \leqslant \|ReA_1\| + \|ImA_1\|$, then

$$\begin{aligned} &\frac{\csc(\alpha_1)}{2}\|A_1\| + \frac{\csc(\alpha_1)}{2}(\|ImA_1\| - \|ReA_1\|) \\ &= \frac{\csc(\alpha_1)}{2}\|ImA_1\| + \frac{\csc(\alpha_1)}{2}(\|A_1\| - \|ReA_1\|) \\ &\leqslant \frac{\csc(\alpha_1)}{2}\|ImA_1\| + \frac{\csc(\alpha_1)}{2}\|ImA_1\| \\ &= \csc(\alpha_1)\|ImA_1\|. \end{aligned}$$

Hence, by Corollary 10 we get the following corollary which proved in [23, Theorem 3.5].

COROLLARY 11. *If* $A = A_1 \oplus 0 \oplus \ldots$, $A_1 \in Sec_{(\frac{\pi}{2}-\alpha_1,\alpha_1)}(H_1)$, $0 < \alpha_1 < \frac{\pi}{2}$, *then*

$$\omega(A_1) \geqslant \csc(\alpha_1)\|ImA_1\| \geqslant \frac{\csc(\alpha_1)}{2}\|A_1\| + \frac{\csc(\alpha_1)}{2}\left(\|ImA_1\| - \|ReA_1\|\right).$$

Also, we obtain the following corollary from Theorem 4 and Theorem 14.

COROLLARY 12. *Under the assumptions of Theorem 14*

$$\omega^2(A) \geqslant \frac{\omega^2_{\delta/2}(ReA) + \omega^2_{\delta/2}(ImA)}{\sin^2(\alpha^*) + \cos^2(\gamma_*)},$$

$$\omega^2(A) \geqslant \frac{\omega^2_{\delta/2}(ReA) + \omega^2_{\delta/2}(ImA)}{1 + \cos^2(\gamma_*)},$$

and

$$\omega^2(A) \geqslant \frac{\omega^2_{\delta/2}(ReA) + \omega^2_{\delta/2}(ImA)}{1 + \sin^2(\alpha^*)}.$$

If we take $\delta = 0$ in Corollary 12, then the following result is obtained.

COROLLARY 13. *If* $A = A_1 \oplus 0 \oplus \ldots$, $A_1 \in Sec_{(\gamma_1,\alpha_1)}(H_1)$, $0 < \gamma_1 \leqslant \alpha_1 < \frac{\pi}{2}$, *then*

$$\omega^2(A_1) \geqslant \frac{\|ReA_1\|^2 + \|ImA_1\|^2}{\sin^2(\alpha_1) + \cos^2(\gamma_1)},$$

$$\omega^2(A_1) \geqslant \frac{\|ReA_1\|^2}{1 + \cos^2(\gamma_1)},$$

and

$$\omega^2(A_1) \geqslant \frac{\|ImA_1\|^2}{1 + \sin^2(\alpha_1)}.$$

If we take $\delta = 0$ in Theorem 4 and Theorem 14, we get the following corollary, whose item (i) was previously proved in the matrix case in [23, Theorem 3.4].

COROLLARY 14. *If* $A = A_1 \oplus 0 \oplus \ldots$, $A_1 \in Sec_{(\gamma_1,\alpha_1)}(H_1)$, $0 < \gamma_1 \leqslant \alpha_1 < \frac{\pi}{2}$, *then*

$$\omega^2(A_1) \geqslant \frac{\omega^2(ReA_1) + \omega^2(ImA_1)}{1 + \sin^2(\alpha_1)} = \frac{\|ReA_1\|^2 + \|ImA_1\|^2}{1 + \sin^2(\alpha_1)} = \frac{\|A_1^*A_1 + A_1A_1^*\|}{2(1 + \sin^2(\alpha_1))}$$

and

$$\omega^2(A_1) \geqslant \frac{\omega^2(ReA_1) + \omega^2(ImA_1)}{1 + \cos^2(\gamma_1)} = \frac{\|ReA_1\|^2 + \|ImA_1\|^2}{1 + \cos^2(\gamma_1)} = \frac{\|A_1^*A_1 + A_1A_1^*\|}{2(1 + \cos^2(\gamma_1))}.$$

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