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UPPER BOUNDS FOR ERDÉLYI'S MULTIVARIATE LAGUERRE POLYNOMIALS

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Abstract. We establish in this paper two inequalities for the multivariate Laguerre polynomials introduced and studied by Arthur Erdélyi [Sitzungsber. Akad. Wiss. Wien, Math.-Naturw. Kl., Abt. IIa 146 (1937), 431–467]. These inequalities generalize the well-known Szegő's inequality for the Laguerre polynomials $L_n^{(\alpha)}(x)$. We also mention briefly few insightful remarks giving a comparative analysis concerning the upper bounds of the derived inequalities in the concluding section.

1. Introduction

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are usually defined by the generating function [14, p. 449, Eq. (18.12.13)]

$$(1-z)^{-\alpha-1}\exp\left(-\frac{xz}{1-z}\right)=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)z^n,\ |z|<1.$$

Explicitly, we have

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix};x\right],$$

where ${}_1F_1$ denotes the confluent hypergeometric function (see [14, p. 443, Eq. (18.5.12)]).

For the Laguerre polynomial $L_n^{(\alpha)}(x)$, the following inequality is well-known:

$$|L_n^{(\alpha)}(x)|\leqslant\frac{(\alpha+1)_n}{n!}\mathrm{e}^{x/2}, \tag{1}$$

where $\alpha\geqslant 0$, $x\geqslant 0$ and $n\in\mathbb{Z}_{\geqslant 0}:=\{0,1,2,\cdots\}$. Inequality (1) is usually called *Szegő's inequality*. There are several different proofs of this famous result (see [7], [17], [24] and [25]). Several improvements have been suggested regarding the inequality (1). Rooney [19] extended the range of α to negative numbers, namely,

$$|L_n^{(\alpha)}(x)|\leqslant 2^{-\alpha}\mathrm{e}^{x/2}, \tag{2}$$

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where $\alpha \leqslant 0$, $x \geqslant 0$ and $n \in \mathbb{Z}_{\geqslant 0}$. Later, Rooney in [20] obtained the inequality that

$$|L_n^{(\alpha)}(x)| \leqslant q_n 2^{-\alpha} \mathrm{e}^{x/2}, \tag{3}$$

where $\alpha \leqslant -1/2$, $x \geqslant 0$, $n \in \mathbb{Z}_{\geqslant 0}$ and

$$q_n := \frac{((2n)!)^{1/2}}{2^{n+1/2} n!} \sim \frac{1}{\sqrt[4]{4\pi n}} \; (n \to +\infty). \tag{4}$$

As Rooney [20] has pointed out that the inequality (3) is stronger than (2) if $\alpha \leqslant -1/2$. For other inequalities for $L_n^{(\alpha)}(x)$, the interested readers may refer to [8], [9] and [13].

In addition, there are some useful inequalities for the Laguerre function $L_\nu^{(\alpha)}(z)$ defined by

$$L_\nu^{(\alpha)}(z) = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)} {}_1F_1\left[\begin{matrix} -\nu \\ \alpha+1 \end{matrix}; z\right],$$

where α, ν and z are allowed complex values ($\alpha \notin \mathbb{Z}_{\leqslant -1}$). One may refer to one such inequality due to Love [11, p. 295, Theorem 2] as follows:

$$|L_\nu^{(\alpha)}(x)| \leqslant \frac{\Gamma(\Re(\alpha+\nu)+1)\Gamma(\Re(\alpha)+1/2)}{|\Gamma(\nu+1)|\Gamma(\Re(\alpha)+1)|\Gamma(\alpha+1/2)|} \mathrm{e}^x \tag{5}$$

$$(x > 0, \; \Re(\alpha) > -1/2, \; \Re(\alpha+\nu) > -1),$$

see also [18]. These inequalities for $L_\nu^{(\alpha)}(z)$ can be used to derive some bounds for the Jacobi function of the first kind [21]. Note that the upper bound in (5) is $\mathscr{O}(\mathrm{e}^x)$ as $x \to +\infty$, which is obviously larger than the bound in (1) even when $\nu = n \in \mathbb{Z}_{\geqslant 0}$ and $\alpha > -1$. Such a marked contrast has been well-explained by Love [11, p. 297, Theorem 3].

There are many multivariate generalizations of the classical Laguerre polynomials in the literature (see, for example, [1, p. 532, Eq. (1.4)], [4, p. 14, Eq. (1.1)], [5, p. 324, Definition 1] and [10, p. 387, Eq. (30)]). But in the present investigation, we focus ourselves on Erdélyi's multivariate Laguerre polynomials [6] which defines and introduces the Laguerre polynomials in k variables by means of the multivariate generating function

$$(1-z_1-\cdots-z_k)^{-\alpha-1} \exp\left(-\frac{x_1 z_1+\cdots+x_k z_k}{1-z_1-\cdots-z_k}\right) = \sum_{n_1,\cdots,n_k=0}^{\infty} L_{n_1,\cdots,n_k}^{(\alpha)}(x_1,\cdots,x_k) z_1^{n_1}\cdots z_k^{n_k},$$

where $|z_1|+\cdots+|z_k| < 1$. For convenience, we let $\mathbf{x} = (x_1,\cdots,x_k)$ so that $\lambda\mathbf{x} = (\lambda x_1,\cdots,\lambda x_k)$ $(\lambda \in \mathbb{R})$. Erdélyi [6, p. 458, Eq. (11.3)] has shown that

$$L_{n_1,\cdots,n_k}^{(\alpha)}(\mathbf{x}) = \frac{(\alpha+1)_{n_1+\cdots+n_k}}{n_1!\cdots n_k!} \Phi_2^{(k)}[-n_1,\cdots,-n_k;\alpha+1;\mathbf{x}],$$

where $\Phi_2^{(k)}$ is defined by (see [6, p. 446] and [22, p. 34])

$$\Phi_2^{(k)}[b_1,\cdots,b_k;c;\mathbf{x}] := \sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty} \frac{(b_1)_{j_1}\cdots(b_k)_{j_k}}{(c)_{j_1+\cdots+j_k}} \frac{x_1^{j_1}}{j_1!}\cdots\frac{x_k^{j_k}}{j_k!}. \tag{6}$$

It may be pointed out here that Erdélyi also obtained an elegant generalization of *Hardy-Hille formula* in [6] for $L^{(\alpha)}_{n_1,\cdots,n_k}(\mathbf{x})$ which in our opinion makes this set of polynomials particularly useful and of importance. Carlitz [3] further provided an elementary proof of this formula and considered other new generalizations.

Our main result is contained in the following theorem.

THEOREM 1.1. *Let* $\alpha>0,\ \mathbf{x}=(x_1,\cdots,x_k)\ \ (x_j\geqslant 0,\ j=1,\cdots,k)$ *and* $\|\mathbf{x}\|:=\max\limits_{1\leqslant j\leqslant k}\{|x_j|\}$. *Then*

$$\left|L^{(\alpha)}_{n_1,\cdots,n_k}(\mathbf{x})\right|\leqslant 2^{k-1}\frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{k})_{n_1}\cdots(\frac{1}{k})_{n_k}}\mathrm{e}^{\|\mathbf{x}\|/2}. \tag{7}$$

When $k=1$, (7) reduces to Szegő's inequality (1). Another variation of (7) can be considered which holds for certain extended range of parameter α. This result is given by the following theorem.

THEOREM 1.2. *Let* $\alpha>-1,\ \mathbf{x}=(x_1,\cdots,x_k)\ \ (x_j\geqslant 0,\ j=1,\cdots,k)$ *and* $\|\mathbf{x}\|:=\max\limits_{1\leqslant j\leqslant k}\{|x_j|\}$. *Then*

$$\left|L^{(\alpha)}_{n_1,\cdots,n_k}(\mathbf{x})\right|\leqslant q_{n_1}\cdots q_{n_k}2^{k-\frac{1}{2}}\frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{2k})_{n_1}\cdots(\frac{1}{2k})_{n_k}}\mathrm{e}^{\|\mathbf{x}\|/2}, \tag{8}$$

where q_n *is given by* (4).

The proofs of Theorem 1.1 and Theorem 1.2 will be presented in Section 2 and Section 3, respectively. A comparative analysis of the upper bounds for the multivariate Laguerre polynomials defined above along with some insightful remarks are mentioned in Section 4.

2. Proof of Theorem 1.1

Our starting point is an integral representation due to Srivastava and Niukkanen [23, p. 249, Eq. (22)]. To state their result in a compact form, we introduce the concept of Dirichlet measure [2, p. 64, Definition 4.4-1]. Let E_k be the standard simplex in $\mathbb{R}^k$. The Dirichlet measure $\mu_{(b_1,\cdots,b_k,\beta)}(\mathbf{u})$ is defined on E by

$$\mathrm{d}\mu_{(b_1,\cdots,b_k,\beta)}(\mathbf{u})=\frac{\Gamma(b_1+\cdots+b_k+\beta)}{\Gamma(b_1)\cdots\Gamma(b_k)\Gamma(\beta)}u_1^{b_1-1}\cdots u_k^{b_k-1}(1-u_1-\cdots-u_k)^{\beta-1}\mathrm{d}u_1\cdots\mathrm{d}u_k, \tag{9}$$

where $b_j>0\ \ (j=1,\cdots,k)$ and $\beta>0$. We have

$$\int\cdots\int_{E_k}\mathrm{d}\mu_{(b_1,\cdots,b_k,\beta)}(\mathbf{u})=1.$$

The integral representation given by Srivastava and Niukkanen [23] can now be used for the multivariable Laguerre polynomials, and in view of (9), we can express

$$\begin{aligned} L_{n_1,\cdots,n_k}^{(\alpha_1+\cdots+\alpha_k+\beta+k)}(\mathbf{x}) &= \frac{(\alpha_1+\cdots+\alpha_k+\beta+k+1)_{n_1+\cdots+n_k}}{(\alpha_1+1)_{n_1}\cdots(\alpha_k+1)_{n_k}} \\ &\qquad \cdot\int\cdots\int_{E_k}\prod_{j=1}^{k} L_{n_j}^{(\alpha_j)}(x_ju_j)\,\mathrm{d}\mu_{(\alpha_1+1,\cdots,\alpha_k+1,\beta+1)}(\mathbf{u}), \end{aligned} \tag{10}$$

where $\alpha_j > -1$ $(j = 1,\cdots,k)$ and $\beta > -1$.

If we set $\alpha_1 = \cdots = \alpha_k = (1-k)/k \in (-1,0]$ and $\beta = \alpha - 1$ $(\alpha > 0)$ in (10), then we get

$$L_{n_1,\cdots,n_k}^{(\alpha)}(\mathbf{x}) = \frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{k})_{n_1}\cdots(\frac{1}{k})_{n_k}}\int\cdots\int_{E_k}\prod_{j=1}^{k} L_{n_j}^{(\frac{1-k}{k})}(x_ju_j)\,\mathrm{d}\mu_{(\frac{1}{k},\cdots,\frac{1}{k},\alpha)}(\mathbf{u}).$$

Making use of Rooney's inequality (2), we have

$$\begin{aligned} \left|L_{n_1,\cdots,n_k}^{(\alpha)}(\mathbf{x})\right| &\leqslant \frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{k})_{n_1}\cdots(\frac{1}{k})_{n_k}}\int\cdots\int_{E_k}\prod_{j=1}^{k}\left|L_{n_j}^{(\frac{1-k}{k})}(x_ju_j)\right|\mathrm{d}\mu_{(\frac{1}{k},\cdots,\frac{1}{k},\alpha)}(\mathbf{u}) \\ &\leqslant \frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{k})_{n_1}\cdots(\frac{1}{k})_{n_k}}\cdot 2^{k-1}\int\cdots\int_{E_k}\mathrm{e}^{(x_1u_1+\cdots+x_ku_k)/2}\mathrm{d}\mu_{(\frac{1}{k},\cdots,\frac{1}{k},\alpha)}(\mathbf{u}), \end{aligned} \tag{11}$$

where $\alpha > 0$.

Note that

$$\begin{aligned} &\int\cdots\int_{E_k}\mathrm{e}^{(x_1u_1+\cdots+x_ku_k)/2}\mathrm{d}\mu_{(\frac{1}{k},\cdots,\frac{1}{k},\alpha)}(\mathbf{u}) \\ &\quad = \sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}\frac{(\frac{1}{2}x_1)^{j_1}}{j_1!}\cdots\frac{(\frac{1}{2}x_k)^{j_k}}{j_k!}\int\cdots\int_{E_k}u_1^{j_1}\cdots u_k^{j_k}\mathrm{d}\mu_{(\frac{1}{k},\cdots,\frac{1}{k},\alpha)}(\mathbf{u}) \\ &\quad = \sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}\frac{(\frac{1}{k})_{j_1}\cdots(\frac{1}{k})_{j_k}}{(\alpha+1)_{j_1+\cdots+j_k}}\frac{(\frac{1}{2}x_1)^{j_1}}{j_1!}\cdots\frac{(\frac{1}{2}x_k)^{j_k}}{j_k!} \\ &\quad = \Phi_2^{(k)}\left[\frac{1}{k},\cdots,\frac{1}{k};\alpha+1;\frac{1}{2}\mathbf{x}\right], \end{aligned} \tag{12}$$

where $\Phi_2^{(k)}$ is defined by (6). Thus, it follows from (11) and (12) that

$$\left|L_{n_1,\cdots,n_k}^{(\alpha)}(\mathbf{x})\right| \leqslant 2^{k-1}\frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{k})_{n_1}\cdots(\frac{1}{k})_{n_k}}\Phi_2^{(k)}\left[\frac{1}{k},\cdots,\frac{1}{k};\alpha+1;\frac{1}{2}\mathbf{x}\right]. \tag{13}$$

In order to simplify the upper bound in (13), we note that

$$\Phi_2^{(k)}\left[\frac{1}{k},\cdots,\frac{1}{k};\alpha+1;\frac{1}{2}\mathbf{x}\right] \leqslant \sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}\frac{(\frac{1}{k})_{j_1}\cdots(\frac{1}{k})_{j_k}}{(\alpha+1)_{j_1+\cdots+j_k}}\frac{(\frac{1}{2}\|\mathbf{x}\|)^{j_1+\cdots+j_k}}{j_1!\cdots j_k!}.$$

Now to reduce the multiple series into a single series, we use the following identity due to Panda [15, p. 166, Theorem 2]:

$$\sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty} C(j_1+\cdots+j_k)(\alpha_1)_{j_1}\cdots(\alpha_k)_{j_k}\frac{x^{j_1+\cdots+j_k}}{j_1!\cdots j_k!}=\sum_{j=0}^{\infty} C(j)(\alpha_1+\cdots+\alpha_k)_j\frac{x^j}{j!},$$

and therefore

$$\Phi_2^{(k)}\left[\frac{1}{k},\cdots,\frac{1}{k};\alpha+1;\frac{1}{2}\mathbf{x}\right]\leqslant\sum_{j=0}^{\infty}\frac{(1)_j}{(\alpha+1)_j}\frac{(\frac{1}{2}\|\mathbf{x}\|)^j}{j!}={}_1F_1\left[\frac{1}{\alpha+1};\frac{1}{2}\|\mathbf{x}\|\right]\leqslant \mathrm{e}^{\|\mathbf{x}\|/2}. \tag{14}$$

Combining now (13) and (14), we obtain the desired inequality (7). This completes the proof. □

3. Proof of Theorem 1.2

Let us set $\alpha_1=\cdots=\alpha_k=(1-2k)/(2k)\in(-1,-1/2]$ and $\beta=\alpha-(1/2)$ $(\alpha>-1/2)$ in (10) so that

$$L_{n_1,\cdots,n_k}^{(\alpha)}(\mathbf{x})=\frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{2k})_{n_1}\cdots(\frac{1}{2k})_{n_k}}\int\cdots\int_{E_k}\prod_{j=1}^{k}L_{n_j}^{(\frac{1-2k}{2k})}(x_ju_j)\mathrm{d}\mu_{(\frac{1}{2k},\cdots,\frac{1}{2k},\alpha+\frac{1}{2})}(\mathbf{u}).$$

Then, making use of Rooney's inequality (3), we have

$$\begin{aligned}\left|L_{n_1,\cdots,n_k}^{(\alpha)}(\mathbf{x})\right|&\leqslant\frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{2k})_{n_1}\cdots(\frac{1}{2k})_{n_k}}\int\cdots\int_{E_k}\prod_{j=1}^{k}\left|L_{n_j}^{(\frac{1-2k}{2k})}(x_ju_j)\right|\mathrm{d}\mu_{(\frac{1}{2k},\cdots,\frac{1}{2k},\alpha+\frac{1}{2})}(\mathbf{u})\\&\leqslant\frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{2k})_{n_1}\cdots(\frac{1}{2k})_{n_k}}\cdot q_{n_1}\cdots q_{n_k}2^{\frac{2k-1}{2}}\\&\quad\cdot\int\cdots\int_{E_k}\mathrm{e}^{(x_1u_1+\cdots+x_ku_k)/2}\mathrm{d}\mu_{(\frac{1}{2k},\cdots,\frac{1}{2k},\alpha+\frac{1}{2})}(\mathbf{u}),\end{aligned}$$

where $\alpha>-1/2$.

Note that

$$\begin{aligned}&\int\cdots\int_{E_k}\mathrm{e}^{(x_1u_1+\cdots+x_ku_k)/2}\mathrm{d}\mu_{(\frac{1}{2k},\cdots,\frac{1}{2k},\alpha+\frac{1}{2})}(\mathbf{u})\\&\quad=\sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}\frac{(\frac{1}{2}x_1)^{j_1}}{j_1!}\cdots\frac{(\frac{1}{2}x_k)^{j_k}}{j_k!}\int\cdots\int_{E_k}u_1^{j_1}\cdots u_k^{j_k}\mathrm{d}\mu_{(\frac{1}{2k},\cdots,\frac{1}{2k},\alpha+\frac{1}{2})}(\mathbf{u})\\&\quad=\sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}\frac{(\frac{1}{2k})_{j_1}\cdots(\frac{1}{2k})_{j_k}}{(\alpha+1)_{j_1+\cdots+j_k}}\frac{(\frac{1}{2}x_1)^{j_1}}{j_1!}\cdots\frac{(\frac{1}{2}x_k)^{j_k}}{j_k!}\\&\quad=\Phi_2^{(k)}\left[\frac{1}{2k},\cdots,\frac{1}{2k};\alpha+1;\frac{1}{2}\mathbf{x}\right],\end{aligned}$$

where $\Phi_2^{(k)}$ is defined by (6). Thus

$$\left|L^{(\alpha)}_{n_1,\cdots,n_k}(\mathbf{x})\right| \leqslant \frac{(\alpha+1)_{n_1+\cdots+n_k}}{(\frac{1}{2k})_{n_1}\cdots(\frac{1}{2k})_{n_k}} \cdot q_{n_1}\cdots q_{n_k}2^{\frac{2k-1}{2}}\Phi_2^{(k)}\left[\frac{1}{2k},\cdots,\frac{1}{2k};\alpha+1;\frac{1}{2}\mathbf{x}\right]. \tag{15}$$

Following similar approach as in the proof of Theorem 1.1 in Section 2 above, the function $\Phi_2^{(k)}\left[\frac{1}{2k},\cdots,\frac{1}{2k};\alpha+1;\frac{1}{2}\mathbf{x}\right]$ occurring in the upper bound of (15) simplifies to

$$\begin{aligned}\Phi_2^{(k)}\left[\frac{1}{2k},\cdots,\frac{1}{2k};\alpha+1;\frac{1}{2}\mathbf{x}\right] &\leqslant \sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}\frac{(\frac{1}{2k})_{j_1}\cdots(\frac{1}{2k})_{j_k}}{(\alpha+1)_{j_1+\cdots+j_k}}\frac{(\frac{1}{2}\|\mathbf{x}\|)^{j_1+\cdots+j_k}}{j_1!\cdots j_k!}\\ &= \sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(\alpha+1)_j}\frac{(\frac{1}{2}\|\mathbf{x}\|)^j}{j!} = {}_1F_1\left[\begin{matrix}\frac{1}{2}\\ \alpha+1\end{matrix};\frac{1}{2}\|\mathbf{x}\|\right]\\ &\leqslant \mathrm{e}^{\|\mathbf{x}\|/2}.\end{aligned} \tag{16}$$

Using (15) and (16), we obtain the desired inequality (8). This completes the proof. $\square$

4. Concluding remarks

(i) We have mentioned in Section 1 that Szegő's inequaltiy (1) is reproduced when taking $k=1$ in (7). However, letting $k=1$ in inequality (8) of Theorem 1.2 gives

$$\left|L_n^{(\alpha)}(x)\right| \leqslant \left(\frac{(1)_n}{(\frac{1}{2})_n}\right)^{1/2}\frac{(\alpha+1)_n}{n!}\mathrm{e}^{x/2}.$$

The additional factor $((1)_n/(\frac{1}{2})_n)^{1/2}$ makes the upper bound a little worse than (1) when $\alpha>0$. But it is still better than the bound provided by Love's inequality in the real case for large x. In addition, we can show that, for $k\geqslant 2$, inequality (8) could be better than (7). For this purpose, we let $n_1=\cdots=n_k=n$ in Theorem 1.1 and Theorem 1.2. The resulting inequalities are

$$\left|L^{(\alpha)}_{n,\cdots,n}(\mathbf{x})\right| \leqslant \mathsf{A}_n(\alpha,k)\mathrm{e}^{\|\mathbf{x}\|/2} \quad\text{and}\quad \left|L^{(\alpha)}_{n,\cdots,n}(\mathbf{x})\right| \leqslant \mathsf{B}_n(\alpha,k)\mathrm{e}^{\|\mathbf{x}\|/2},$$

where

$$\mathsf{A}_n(\alpha,k) := 2^{k-1}\frac{(\alpha+1)_{kn}}{((\frac{1}{k})_n)^k} \quad\text{and}\quad \mathsf{B}_n(\alpha,k) := (q_n)^k 2^{k-\frac{1}{2}}\frac{(\alpha+1)_{kn}}{((\frac{1}{2k})_n)^k}.$$

We have after a little computation that

$$\frac{\mathsf{A}_n(\alpha,k)}{\mathsf{B}_n(\alpha,k)} \sim 2^{\frac{k-1}{2}}\pi^{\frac{k}{4}}\left(\frac{\Gamma(\frac{1}{2k})}{\Gamma(\frac{1}{k})}\right)^k n^{\frac{k}{4}-\frac{1}{2}} \ (n\to+\infty).$$

It is therefore clear that when $k\geqslant 2$, the upper bound provided by (8) would be better than the one given by (7). In addition, if we want to find an accurate estimate of $L^{(\alpha)}_{n,\cdots,n}(\mathbf{x})$, it would be possible to apply the so-called *diagonal method* (see [16, Section 13.1]). We may have to first find a closed form for the *diagonal generating function*

$$\sum_{n=0}^{\infty} L_{n,\cdots,n}^{(\alpha)}(\mathbf{x})z^n. \tag{17}$$

It seems at this point very difficult to find such a closed-form result for (17) for $k \geqslant 3$, and therefore we leave it for future research.

(ii) Our main technique employed in the present investigation was to make the parameters $\alpha_1,\cdots,\alpha_k$ in the integral representation (10) sufficiently close to -1. Based on this observation, it seems necessary and worthwhile to mention here the following inequality discovered by Lewandowski and Szynal [9, p. 532]:

$$|L_n^{(\alpha)}(x)| \leqslant \frac{(\alpha+1)_n}{n!}\sigma_n^{(\alpha)}(\mathrm{e}^x), \tag{18}$$

where $\alpha \geqslant -1/2$, $x \geqslant 0$, $n \in \mathbb{Z}_{\geqslant 0}$ and

$$\sigma_n^{(\alpha)}(\mathrm{e}^x) = \frac{n!}{(\alpha+1)_n}\sum_{k=0}^{n}\frac{(\alpha+1)_{n-k}}{(n-k)!}\frac{x^k}{k!}.$$

This inequality (18) is better than Szegő's inequality (1) for large x, because $\sigma_n^{(\alpha)}(\mathrm{e}^x)$ is a polynomial. Inequality (18) has a remarkable application. By using (18), Luo and Raina [12] established an inequality for the associated Pollaczek polynomials. Since the inequality (18) holds for all $\alpha \geqslant -1/2$, it can therefore also be used to obtain other inequalities for the multivariate Laguerre polynomials. We choose to omit such results here because derivations of these results would not involve any new ideas in their proofs.

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