Journal of Mathematical Inequalities
Volume 20, Number 1 (2026), 155–180 doi:10.7153/jmi-2026-20-11

FORMABLE INTEGRAL TRANSFORM AND ULAM STABILITY OF nTH ORDER LINEAR DIFFERENTIAL EQUATIONS

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(*Communicated by N. Elezović*)

Abstract. In this current manuscript, we prove the Hyers-Ulam stability of homogeneous and nonhomogeneous linear differential equations by using a new transform called formable integral transform. Further, we present counter-examples of formable integral transforms for the Hyers-Ulam stability approach.

1. Introduction

In 1940, Ulam [23] posed the stability problem of of functional equations: Given a group G_1 and a metric group G_2 with metric $\rho(.,.)$. Given $\varepsilon > 0$; does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(x),h(x)) < \delta$ for all $x,y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x),h(x)) < \varepsilon$, for all $x,y \in G_1$? The problem for approximately additive mappings, on Banach spaces, was solved by Hyers [7]. The result obtained by Hyers was generalized by Rassias [18]. During the last two decades many mathematicians have extensively investigated the stability problems of functional equations (see [5, 8, 9, 10]).

Alsina and Ger [2] were the first mathematicians who investigated the Hyers-Ulam stability of the differential equation $g' = g$. They proved that if a differentiable function $y : I \to \mathbb{R}$ satisfies $|y' - y| \leqslant \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|g - y| \leqslant 3\varepsilon$ for all $t \in I$. This result of Alsina and Ger was generalized by Takahasi *et al.* [22] to the case of the complex Banach space valued differential equation $y' = \lambda y$. Furthermore, the results of Hyers-Ulam stability of differential equations of first order were also generalized by Miura *et al.* [14], Jung [11] and Wang *et al.* [24]. Li [12] established the stability of linear differential equation of second order in the sense of the Hyers and Ulam $y'' = \lambda y$. Li and Shen [13] proved the stability of nonhomogeneous linear differential equation of second order $y'' + p(x)y' + q(x)y + r(x) = 0$ in the sense of the Hyers and Ulam, while Gavruta *et al.* [6] proved the Hyers-Ulam stability of the equation $y'' + \beta(x)y = 0$ with boundary and initial conditions.

Mathematics subject classification (2020): 26D10, 44A10, 39B82, 34A40, 39A30, 34K20.
Keywords and phrases: Hyers-Ulam stability, nth order linear differential equation, formable (integral) transform.
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Qarawani [16] established the Hyers-Ulam stability of the equations of the second order

$$z'' = F(x,z)$$

with the initial conditions $z(x_0) = 0 = z'(x_0)$. For more information on the Hyers-Ulam stability for several type of differential equations and integral equations, see [4].

We use basic symbols and notations in Table 1.

Table 1. Basic symbols and notations

constant	$t_0, t_1, c, \varepsilon, \alpha, \beta, k, M, T, a_1, \cdots, a_n$
variable	t, x, s, u, w
function	$y, z, f, g, \mu, g_1, g_2, m, l, \sigma$
differential	$[y] = (y, y', y'', \cdots, y^{(n-1)})$

In this paper, we investigate the Hyers-Ulam stability of the following nonlinear differential equation of nth order

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}) \tag{1}$$

with the initial conditions

$$y(t_0) = y_0, \;\; y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}, \tag{2}$$

where $y \in C^{(n)}(I)$, $I = [t_0, t_1]$, $(t, [y]) \equiv (t, y, y', y'', \cdots, y^{(n-1)}) \in D$, $t \in I$, $-\infty < t_0 < t_1 < \infty$, and $f(t, y, y', y'', \ldots, y^{(n-1)})$ is defined on a closed bounded set $D \subset \mathbb{R}^{n+1}$ that satisfies the condition

$$|f(t,[y]) - f(t,[z])| \leqslant g(t) \frac{|y(t) - z(t)|}{(t_1 - t_0)^{n-1}}, \tag{3}$$

where $g(t) : I \to (0,1)$ is an integrable function.

Moreover, we establish the Hyers-Ulam stability of the problem (1), (2) for f satisfying the Lipschitz condition

$$|f(t,[y]) - f(t,[z])| \leqslant A_0 \sum_{i=0}^{n-1} \left| y^{(i)}(t) - z^{(i)(t)} \right|, \tag{4}$$

where $A_0 > 0$.

DEFINITION 1. We say that the differential equation (1) has the Hyers-Ulam stability if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0$, $y \in C^{(n)}(I)$, if

$$|y^{(n)} - f(t;[y])| \leqslant \varepsilon$$

with the initial condition (2), then there exists a solution $z(t) \in C^{(n)}(I)$ of the differential equation (1) such that $|y(t) - z(t)| \leqslant K\varepsilon$, where K is a constant that does not depend on ε nor on $y(t)$.

2. Preliminaries

By using various integral transforms such as Sumudu transform [3], Aboodh transform [15], Mohand transform [20] and Laplace transform [21], the Hyers-Ulam stability problems of several differential equations have been solved.

In Table 2, integral transforms to solve the stability problems of differential equations are given.

Table 2. Integral transforms

Sumudu transform	$S(f(t)) = F(\xi) = \int_0^\infty f(t\xi)e^{-t}dt$
Aboodh transform	$A(f(t)) = F(u) = \frac{1}{u}\int_0^\infty f(t)e^{-ut}dt$
Mohand transform	$M(f(t)) = F(v) = v^2\int_0^\infty f(t)e^{-vt}dt$
Laplace transform	$L(f(t)) = F(s) = \int_0^{-\infty} e^{-st}f(t)dt$

In this section, we introduce the definition of a new transform, called the formable (integral) transform, and study some theorems and properties of the new transform.

A function $g(t)$ is said to be of exponential order c if there exist constants M and T such that $|g(t)| \leqslant Me^{ct}$ for all $t \geqslant T$.

DEFINITION 2. The formable (integral) transform of a function $g(t)$ of exponential order is defined over the set of functions

$$W = \{g(t) : \exists N \in (0,\infty),\ \tau_i > 0,\ i = 1,2,\ |g(t)| < Ne^{t\tau_i^{-1}},\ \text{if}\ t \in [0,\infty)\}$$

in the following form:

$$R[g(t)] = B(s,u) = s\int_0^\infty e^{-st}g(ut)dt. \tag{5}$$

This is equivalent to

$$R[g(t)] = \frac{s}{u}\int_0^\infty e^{-stu^{-1}}g(t)dt,\ \ s > 0,\ \ u > 0,$$

where s and u are the formable transform's variables. The term *formable* emphasizes the transform's flexibility in handling diverse classes of differential equations. In addition, it has a duality with other well known transforms that will be considered later. To show the applicability of the formable transform, we compute the transform for several functions and we compare the results with other values from some well-known transforms and illustrate them in Section 3.

The inverse formable transform of a function $g(t)$ is given by

$$R^{-1}[B(s,u)] = g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s}e^{stu^{-1}}B(s,u)ds.$$

From the definition of the Fourier transform, we know

$$F[g(t)] = F(W) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwt} g(t)dt,$$

$$g(t) = F^{-1}[F(W)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwt} F(w)dw, \tag{6}$$

where $g(t)$ is a function defined on the domain $(-\infty, \infty)$, so that for $t \in (-\infty, 0)$, we assume that $g(t) = 0$. Hence for $t > 0$, let $g(t) = g(t)\mu(t)e^{-ct}$, where $\mu(t)$ is the unit step function and c is any constant, so that Eq. (6) becomes

$$g(t)\mu(t)e^{-ct} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} \left[\int_{0}^{\infty} e^{-(c+iw)t} g(t)dt \right] dw. \tag{7}$$

Multiplying both sides of Eq. (7) by e^{ct}, we obtain

$$g(t)\mu(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(c+iw)t} \left[\int_{0}^{\infty} e^{-(c+iw)t} g(t)dt \right] dw. \tag{8}$$

Letting $\frac{s}{u} = c + iw$, $\frac{ds}{u} = idw$ and $dw = \frac{1}{iu}ds$ in (8), we obtain

$$\begin{aligned} g(t)u(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} \frac{1}{u} \left[\int_{0}^{\infty} e^{\frac{-st}{u}} g(t)dt \right] ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{\frac{st}{u}} \left[\frac{s}{u} \int_{0}^{\infty} e^{\frac{-st}{u}} g(t)dt \right] ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{\frac{st}{u}} B(s,u)ds. \end{aligned}$$

Defining $g(t)$ on $(0, \infty)$, we obtain

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{stu^{-1}} B(s,u)ds.$$

Hence

$$R^{-1}[B(s,u)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{stu^{-1}} B(s,u)ds$$

and

$$R^{-1}[R(g(t))] = g(t).$$

THEOREM 1. (Sufficient conditions for the existence of the formable transform) *If a function $g(t)$ is a piecewise continuous function in every finite interval $[0, \alpha]$ and is of exponential order β, then the formable (integral) transform $B(s,u)$ of $g(t)$ exists.*

Proof. Let α be any positive number. Then we have

$$\begin{aligned} B(s,u) &= \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}g(t)dt \\ &= \frac{s}{u}\int_0^\alpha e^{\frac{-st}{u}}g(t)dt + \frac{s}{u}\int_\alpha^\infty e^{\frac{-st}{u}}g(t)dt. \end{aligned}$$

Since the function $g(t)$ is a piecewise continuous function in every finite interval $[0,\alpha]$, the integral $\frac{s}{u}\int_0^\alpha e^{\frac{-st}{u}}g(t)dt$ exists, and since $g(t)$ is of exponential order β, we have

$$\begin{aligned} \left|\frac{s}{u}\int_0^\alpha e^{\frac{-st}{u}}g(t)dt\right| &\leqslant \left|\frac{s}{u}\right|\int_\alpha^\infty \left|e^{\frac{-st}{u}}g(t)\right|dt \\ &= \left|\frac{s}{u}\right|\int_\alpha^\infty e^{\frac{-st}{u}}\,|g(t)|\,dt \\ &\leqslant \left|\frac{s}{u}\right|\int_\alpha^\infty e^{\frac{-st}{u}}Ne^{\beta t}dt \\ &= \left|\frac{s}{u}\right|N\int_0^\infty e^{-t\left(\frac{s}{u}-\beta\right)}dt \\ &\leqslant \frac{s}{u}N\int_0^\infty e^{-t\left(\frac{s}{u}-\beta\right)}dt \\ &= \frac{s}{u}N\left(\frac{1}{\frac{s}{u}-\beta}\right) \\ &= sN/(s-\beta u). \end{aligned}$$

This completes the proof. □

Now, we introduce some basic properties.

PROPOSITION 1. (Linearity property) *Let $\alpha g_1(t)$ and $\beta g_2(t)$ be two functions in a set W. Then $(\alpha g_1(t)+\beta g_2(t)) \in W$, where α and β are nonzero arbitrary constants, and*

$$R[\alpha g_1(t)+\beta g_2(t)] = \alpha R[g_1(t)] + \beta R[g_2(t)]. \tag{9}$$

Proof. Using the definition of the formable (integral) transform, we have

$$\begin{aligned} R[\alpha g_1(t)+\beta g_2(t)] &= \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}(\alpha g_1(t)+\beta g_2(t))dt \\ &= \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}\alpha g_1(t)dt + \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}\beta g_2(t))dt \\ &= \alpha\frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}g_1(t)dt + \beta\frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}g_2(t))dt \\ &= \alpha R[g_1(t)] + \beta R[g_2(t)]. \end{aligned}$$

This completes the proof. □

PROPOSITION 2. (Change of scale) *Let $g(\alpha t)$ be a function in the set W, where α is an arbitrary constant. Then*

$$R[g(\alpha t)] = B\left(\frac{s}{\alpha}, u\right) = B(s, \alpha u).$$

Proof. Let

$$R[g(\alpha t)] = \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}} g(\alpha t)dt. \tag{10}$$

Letting $\delta = at$ in Eq. (10), we have

$$\begin{aligned} R[g(\alpha t)] &= \frac{s}{u}\int_0^\infty e^{\frac{-s\delta}{u\alpha}} g(\delta)\frac{ds}{\alpha} \\ &= \frac{s}{\alpha u}\int_0^\infty e^{\frac{-s\delta}{u\alpha}} g(\delta)ds \\ &= B\left(\frac{s}{\alpha}, u\right) \\ &= B(s, \alpha u). \end{aligned}$$

This completes the proof. □

PROPOSITION 3. (Formable transform of the derivative) *If $g^{(n)}(t)$ is the nth derivative of a function $g(t)$, where $g^{(n)}(t) \in W$, for $n = 0, 1, 2, \ldots$ with respect to t, then*

$$R\left[g^{(n)}(t)\right] = \frac{s^n}{u^n}B(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k} g^{(k)}(0).$$

Proof. For $n = 1$, we have

$$\begin{aligned} R[g'(t)] &= \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}} g'(t)dt \\ &= \frac{s}{u}\left[-g(0) + B(s,u)\right]. \end{aligned}$$

Thus

$$R[g'(t)] = \frac{s}{u}B(s,u) - \frac{s}{u}g(0). \tag{11}$$

Similarly, for $n = 2$, we have

$$\begin{aligned} R[g''(t)] &= \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}} g''(t)dt \\ &= \frac{s}{u}\left[-g'(0) + \frac{s}{u}B(s,u) - \frac{s}{u}g(0)\right]. \end{aligned}$$

Thus

$$R[g'(t)] = \frac{s^2}{u^2}B(s,u) - \frac{s^2}{u^2}g(0) - \frac{s}{u}g'(0). \tag{12}$$

Assume that Eq. (11) is true for $n = k$. We show that it is true for $n = k+1$. By Eq. (12), we have

$$\begin{aligned}
R\left[g^{(k+1)}(t)\right] &= R\left[\left(g^{(k)}(t)\right)'\right] \\
&= \frac{s}{u}R\left[g^{(k)}(t)\right] - \frac{s}{u}g^{(k)}(0) \\
&= \frac{s}{u}\left[\frac{s^k}{u^k}B(s,u) - \sum_{i=0}^{k-1}\left(\frac{s}{u}\right)^{k-i} g^{(i)}(0)\right] - \frac{s}{u}g^{(k)}(0) \\
&= \frac{s^{k+1}}{u^{k+1}}B(s,u) - \sum_{i=0}^{k}\left(\frac{s}{u}\right)^{K+1-i} g^{(i)}(0).
\end{aligned}$$

This implies that Eq. (10) holds for $n = k+1$, which completes the proof. □

PROPOSITION 4. (Formable transform of the convolution) *If* $F(s,u)$ *and* $G(s,u)$ *are the formable transforms of functions* $f(t)$ *and* $g(t)$*, respectively, then*

$$R[f(t) * g(t)] = \frac{u}{s}F(s,u)G(s,u),$$

where $f(t) * g(t)$ *is the convolution of the functions* $f(t)$ *and* $g(t)$ *defined by*

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Proof. Using the definition of the formable transform in Eq. (5), we obtain

$$\begin{aligned}
R[f(t) * g(t)] &= s\int_0^\infty e^{-st}(f * g)(ut)dt \\
&= s\int_0^\infty e^{-st}\int_0^{ut} f(\tau)g(ut-\tau)d\tau dt.
\end{aligned} \tag{13}$$

Letting $\tau = ux$ and $d\tau = udx$ in Eq. (13), we obtain

$$\begin{aligned}
R[f(t) * g(t)] &= s\int_0^\infty e^{-st}\int_0^t f(ux)g(ut-ux)d(ux)dt \\
&= s\int_0^\infty e^{-st}\int_0^t f(ux)g(u(t-x))udxdt.
\end{aligned} \tag{14}$$

Letting $y = t - x$ and $dy = dt$ in Eq. (14), we obtain

$$\begin{aligned} R[f(t)*g(t)] &= s\int_0^\infty\int_0^\infty e^{(-s(x+y)}f(ux)g(uy)udxdy \\ &= su\int_0^\infty\int_0^\infty e^{-s(x+y)}f(ux)g(uy)dxdy \\ &= su\int_0^\infty e^{-sx}f(ux)dx\int_0^\infty e^{-sy}g(uy)dy \\ &= \frac{u}{s}\int_0^\infty e^{-sx}f(ux)dxs\int_0^\infty e^{-sy}g(uy)dy \\ &= \frac{u}{s}F(s,u)G(s,u). \end{aligned}$$

This completes the proof. □

COROLLARY 1. *The formable transform of* $(f*g)'$ *is given by*

$$B\left[(f*g)'\right] = F(s,u)G(s,u). \tag{15}$$

Proof. Applying (3) and (4), we obtain

$$B\left[(f*g)'\right] = \frac{s}{u}R(f(t)*g(t)) - \frac{s}{u}(f*g)(0).$$

But $(f*g)(0) = 0$, and hence

$$\begin{aligned} B\left[(f*g)'\right] &= \frac{s}{u}\frac{u}{s}F(s,u)G(s,u) \\ &= F(s,u)G(s,u). \end{aligned}$$

This completes the proof. □

Here, if we put $g(t) = f(t)$ in Eq. (15), then we have

$$B\left[(f*f)'\right] = F^2(s,u).$$

DEFINITION 3. (Inverse formable transform) A function $g(t)$ is called the inverse formable transform of a function $B(s,u)$ if it has the following property $R[g(t)] = B(s,u)$. In symbolic form, we can write as $R^{-1}[B(s,u)] = g(t)$.

We summarize basic properties of the formable transform in Table 3.

Table 3. General properties of formable transform

Definition	$B(s,u) = \frac{s}{u}\int_0^\infty e^{(\frac{-st}{u}}g(t)dt$
Inverse	$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}e^{\frac{st}{u}}B(s,u)ds$
Derivative	$R[g^n(t)] = s^nB(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0)$
Convolution	$B[f*g] = \frac{u}{s}F(s,u)G(s,u)$

By simple computations, we obtain the formable (integral) transforms for some functions as follows:

$$\begin{aligned}
R\left[\frac{t^n}{n!}\right] &= \frac{u^n}{s^n}, \quad n=0,1,2,\cdots,\\
R\left[e^{\alpha t}\right] &= \frac{s}{s-\alpha u},\\
R\left[t^n e^{\alpha t}\right] &= \frac{su^n n!}{(s-\alpha u)^{n+1}}, \quad n=0,1,2,\cdots,\\
R[\sin(\alpha t)] &= \frac{\alpha su}{s^2+\alpha^2u^2},\\
R[\cos(\alpha t)] &= \frac{s^2}{s^2+\alpha^2u^2},\\
R[sinh(\alpha t)] &= \frac{\alpha su}{s^2-\alpha^2u^2},\\
R[cosh(\alpha t)] &= \frac{s^2}{s^2-\alpha^2u^2},\\
R\left[e^{\beta t}\sin(\alpha t)\right] &= \frac{\alpha su}{(s-\beta u)^2+\alpha^2u^2},\\
R\left[e^{\beta t}\cos(\alpha t)\right] &= \frac{\alpha s(s-\beta u)}{(s-\beta u)^2+\alpha^2u^2},\\
R\left[e^{\beta t}sinh(\alpha t)\right] &= \frac{\alpha su}{(s-\beta u)^2-\alpha^2u^2},\\
R\left[e^{\beta t}cosh(\alpha t)\right] &= \frac{\alpha s(s-\beta u)}{(s-\beta u)^2-\alpha^2u^2}.
\end{aligned}$$

Based on the above results, our main goal is to prove more efficiently the Hyers-Ulam stability of the following nth order linear differential equations

$$g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)=0 \tag{16}$$

and

$$g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)=m(t). \tag{17}$$

In this section, we introduce some standard notations and definitions which will be useful to prove our main results. Assume that $F:=\mathbb{R}$ or $\mathbb{C}$ and $R(\alpha)$ denotes the real part of α.

DEFINITION 4. [1, 17, 19] The Mittag-Leffler function of one parameter is denoted by $E_\beta(t)$ and defined by, $E_\beta(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\beta k+1)}$ for $t,\beta\in\mathbb{C}$ with $R(\beta)>0$.

If we put $\beta=1$, then the above equation becomes $E_1(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{t^k}{k!}$ $=e^t$.

Now, we give the different definitions of Hyers-Ulam stability and Hyers-Ulam σ-stability of the differential equations (16) and (17).

Throughout this section, consider

$$B = \{g : [0,\infty) \to F \mid g \quad \text{is a continuously differential function of exponential order}\}.$$

DEFINITION 5.

(i) The linear differential equation (16) is said to have the Hyers-Ulam stability (for class B) if there exists a constant $k > 0$ such that the following statement is true: For every $\varepsilon > 0$, if a function $g \in B$ satisfies the inequality.

$$\left|g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1} g'(t) + a_n g(t)\right| \leqslant \varepsilon \tag{18}$$

for all $t \geqslant 0$, then there exists a solution $f : [0,\infty) \to F$ of the differential equation (16) such that $f \in B$ and $|g(t) - f(t)| \leqslant k\varepsilon$ for all $t \geqslant 0$, where k is called as Hyers-Ulam constant.

(ii) We say that the non-homogeneous linear differential equation (17) has the Hyers-Ulam stability if there exists a constant $k > 0$ such that the following statement is true: For every $\varepsilon > 0$, if a function $g \in B$ satisfies the inequality

$$\left|g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1} g'(t) + a_n g(t) - m(t)\right| \leqslant \varepsilon \tag{19}$$

for all $t \geqslant 0$, then there exists a solution $f : [0,\infty) \to F$ of the differential equation (17) such that $f \in B$ and $|g(t) - f(t)| \leqslant k\varepsilon$ for all $t \geqslant 0$, where k is called as Hyers-Ulam constant.

DEFINITION 6. Let $\sigma : [0,\infty) \to (0,\infty)$ be a function.

(i) We say that the homogeneous linear differential equation (16) has the Hyers-Ulam σ-stability (for the class B) if there exists a constant $k > 0$ such that the following statement is true: For every $\varepsilon > 0$, if a function $g \in B$ satisfies the inequality.

$$\left|g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1} g'(t) + a_n g(t)\right| \leqslant \sigma(t)\varepsilon \tag{20}$$

for all $t \geqslant 0$, then there exists a solution $f : [0,\infty) \to F$ of the differential equation (16) such that $f \in B$ and $|g(t) - f(t)| \leqslant k\sigma(t)\varepsilon$ for all $t \geqslant 0$, where k is called as Hyers-Ulam σ-constant.

(ii) We say that the non-homogeneous linear differential equation (17) has the Hyers-Ulam σ-stability if there exists a constant $k > 0$ such that the following statement is true: For every $\varepsilon > 0$, if a function $g \in B$ satisfies the inequality

$$\left|g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1} g'(t) + a_n g(t) - m(t)\right| \leqslant \sigma(t)\varepsilon \tag{21}$$

for all $t \geqslant 0$, then there exists a solution $f : [0,\infty) \to F$ of the differential equation (17) such that $f \in B$ and $|g(t) - f(t)| \leqslant \sigma(t)\varepsilon$ for all $t \geqslant 0$, where k is called as Hyers-Ulam σ-constant.

Now we introduce the definitions of Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam σ-stability of the differential equations (16) and (17).

DEFINITION 7. Let $E_\beta(t)$ be the Mittag-Leffler function.

(i) We say that the differential equation (16) has the Mittag-Leffler-Hyers-Ulam stability if there exists a constant $k>0$ such that the following statement holds true: For every $\varepsilon>0$, if a function $g\in B$ satisfies the inequality.

$$\left|g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)\right|\leqslant E_\beta(t)\varepsilon \tag{22}$$

for all $t\geqslant 0$, then there exists a solution $f:[0,\infty)\to F$ of the differential equation (16) such that $f\in B$ and $|g(t)-f(t)|\leqslant kE_\beta(t)\varepsilon$ for all $t\geqslant 0$, where the constant k is called as Mittag -Leffler-Hyers-Ulam constant.

(ii) We say that the non-homogeneous linear differential equation (17) has the Mittag-Leffler-Hyers-Ulam stability if there exists a constant $k>0$ such that the following statement is true:. For every $\varepsilon>0$, if a function $g\in B$ satisfies the inequality

$$\left|g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)-m(t)\right|\leqslant E_\beta(t)\varepsilon \tag{23}$$

for all $t\geqslant 0$, then there exists a solution $f:[0,\infty)\to F$ of the differential equation (17) such that $|g(t)-f(t)|\leqslant kE_\beta(t)\varepsilon$ for all $t\geqslant 0$, where the constant k is called as Mittag-Leffler-Hyers-Ulam constant.

DEFINITION 8. Let $E_\beta(t)$ be the Mittag-Leffler function and $\sigma:[0,\infty)\to(0,\infty)$ be a function.

(i) We say that the differential equation (16) has the Mittag-Leffler-Hyers-Ulam σ-stability if there exists a constant $k>0$ such that the following statement holds true: For every $\varepsilon>0$, if a function $g\in B$ satisfies the inequality.

$$\left|g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)\right|\leqslant \sigma(t)E_\beta(t)\varepsilon \tag{24}$$

for all $t\geqslant 0$, then there exists a solution $f:[0,\infty)\to F$ of the differential equation (16) such that $f\in B$ and $|g(t)-f(t)|\leqslant k\sigma(t)E_\beta(t)\varepsilon$ for all $t\geqslant 0$, where the constant k is called as Mittag-Leffler-Hyers-Ulam σ-constant.

(ii) We say that the non-homogeneous linear differential equation (17) has the Mittag-Leffler-Hyers-Ulam σ-stability if there exists a constant $k>0$ such that the following statement is true: For every $\varepsilon>0$, if a function $g\in B$ satisfies the inequality

$$\left|g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)-m(t)\right|\leqslant \sigma(t)E_\beta(t)\varepsilon \tag{25}$$

for all $t\geqslant 0$, then there exists a solution $f:[0,\infty)\to F$ of the differential equation (17) such that $f\in B$ and $|g(t)-f(t)|\leqslant k\sigma(t)E_\beta(t)\varepsilon$ for all $t\geqslant 0$, where the constant k is called as Mittag-Leffler-Hyers-Ulam σ-constant.

3. Hyers-Ulam stability of (16)

In this section, we prove several types of Hyers-Ulam stability of the homogeneous nth order linear differential equation (16) by using the formable (integral) transform.

We note that in this section and the next section we investigate various types of Hyers-Ulam stability for the class B, where B is the class of all continuously differentiable functions. Assume that $f:[0,\infty)\to F$ is of exponential order.

THEOREM 2. *Assume that $a_1+\ldots+a_{n-1}+a_n$ is a constant with $R(a_1+\ldots+a_{n-1}+a_n)>0$. Then the homogeneous linear differential equation (16) is Hyers-Ulam stable in the class B.*

Proof. Assume that $g\in B$ satisfies the inequality (18) for all $t\geqslant 0$. Let us define a function $e(t):[0,\infty)\to F$ by

$$e(t)=g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)$$

for all $t\geqslant 0$. In view of (18), the inequality $|e(t)|\leqslant\varepsilon$ holds for all $t\geqslant 0$. The formable transform of $e(t)$ gives

$$R\{e(t)\}=D(s,u)=R\{g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)\},$$

$$D(s,u)=R\{g^{(n)}(t)\}+a_1R\{g^{(n-1)}(t)\}+\ldots+a_{n-1}R\{g'(t)\}+a_nR\{g(t)\},$$

where

$$\begin{aligned}
R\{g(t)\}&=B_1(s,u),\\
R\{g'(t)\}&=\frac{s}{u}B_1(s,u)-\frac{s}{u}g(0),\\
R\{g''(t)\}&=\frac{s^2}{u^2}B_1(s,u)-\frac{s^2}{u^2}g(0)-\frac{s'}{u}g(0),\\
&\vdots\\
R\{g^{(n)}(t)\}&=\frac{s^n}{u^n}B_1(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0).
\end{aligned}$$

Thus we have

$$D(s,u)=\frac{s^n}{u^n}B_1(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0)+a_{n-1}\left(\frac{s}{u}B_1(s,u)-\frac{s}{u}g(0)\right)+a_nB_1(s,u),$$

$$\begin{aligned}
&\frac{s^n}{u^n}B_1(s,u)+\ldots+a_{n-1}\frac{s}{u}B_1(s,u)+a_nB_1(s,u)\\
&\qquad=D(s,u)+\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0)+\ldots a_{n-1}\frac{s}{u}g(0),
\end{aligned}$$

$$B_1(s,u)\left[\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right]$$
$$=D(s,u)+\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0)+\ldots+a_{n-1}\frac{s}{u}g(0),$$

$$B_1(s,u)=\frac{D(s,u)+\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0)+\ldots a_{n-1}\frac{s}{u}g(0)}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}. \tag{26}$$

If we put $t=0$ in $f(t)=e^{-(a_1+\ldots+a_{n-1}+a_n)t}g(t)$, then $f(t)=g(t)$ and $f\in B$. The formable transform of $f(t)$ gives

$$R(f(t))=B_2(s,u)=\frac{\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+\ldots a_{n-1}\frac{s}{u}f(0)}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}. \tag{27}$$

Thus

$$R\{f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)\}$$
$$=\frac{s^n}{u^n}B_2(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+a_{n-1}\left(\frac{s}{u}B_2(s,u)-\frac{s}{u}f(0)\right)+a_nB_2(s,u).$$

So we have

$$\frac{s^n}{u^n}B_2(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+a_{n-1}\left(\frac{s}{u}B_2(s,u)-\frac{s}{u}f(0)\right)+a_nB_2(s,u)=0,$$

$$\frac{s^n}{u^n}B_2(s,u)+\ldots+a_{n-1}\frac{s}{u}B_2(s,u)+a_nB_2(s,u)$$
$$=\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+\ldots+a_{n-1}\frac{s}{u}f(0),$$

$$B_2(s,u)\left[\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right]=\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+\ldots+a_{n-1}\frac{s}{u}f(0),$$
$$B_2(s,u)=\frac{\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+\ldots+a_{n-1}\frac{s}{u}f(0)}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}.$$

By (27), we have

$$R\{f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)\}=0.$$

Since R is a one to one operator,

$$f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)=0.$$

Here $f(t)$ is a solution of the differential equation (16).

By (26) and (27), we obtain

$$\begin{aligned}
R\{g(t)\} - R\{f(t)\} &= B_1(s,u) - B_2(s,u) \\
&= \frac{D(s,u)}{\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)} \\
&= \frac{\frac{u}{s}D(s,u)}{\frac{u}{s}\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)} \\
&= \frac{u}{s}D(s,u)C(s,u) \\
&= R\{D*C\}(s,u) \\
&= R[e(t)*l(t)],
\end{aligned}$$

where

$$\begin{aligned}
R\{l(t)\} = C(s,u) &= \frac{s}{u\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)}, \\
l(t) &= R^{-1}\left\{\frac{s}{u\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)}\right\} \\
&= R^{-1}\left\{\frac{s}{\left(\frac{s^n}{u^{n-1}} + \ldots + a_{n-1}\frac{s}{u} + a_n u\right)}\right\} \\
&= e^{-(a_1+\ldots+a_{n-1}+a_n)t}.
\end{aligned}$$

Consequently,

$$R\{g(t) - f(t)\} = R\{e(t)*l(t)\}$$

and thus

$$g(t) - f(t) = e(t)*l(t).$$

Taking modulus on both sides, we have

$$\begin{aligned}
|g(t) - f(t)| &= |e(t)*l(t)| \\
&= \left|\int_0^t e(s)l(t-s)ds\right| \\
&\leqslant \int_0^t |e(s)||l(t-s)|ds \\
&\leqslant \varepsilon \int_0^t l(t-s)ds,
\end{aligned}$$

since

$$\begin{aligned}
l(t) &\leqslant e^{-(a_1+\ldots+a_{n-1}+a_n)t} \\
&\leqslant \varepsilon \int_0^t e^{-R(a_1+\ldots+a_{n-1}+a_n)(t-s)}ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \varepsilon \int_0^t e^{-R(a_1+\ldots+a_{n-1}+a_n)(t)} e^{-R(a_1+\ldots+a_{n-1}+a_n)(s)} ds \\
&\leqslant \varepsilon e^{-R(a_1+\ldots+a_{n-1}+a_n)(t)} \int_0^t e^{-R(a_1+\ldots+a_{n-1}+a_n)(s)} ds \\
&\leqslant \varepsilon \left[\frac{e^{R(a_1+\ldots+a_{n-1}+a_n)(t)} - 1}{R(a_1+\ldots+a_{n-1}+a_n)} \right] \\
&\leqslant k\varepsilon
\end{aligned}
$$

for all $t \geqslant 0$, where $k = \frac{1}{R(a_1+\ldots+a_{n-1}+a_n)}$.

This implies that the homogeneous linear differential equation (16) has the Hyers-Ulam stability for the class B. □

We note that if $-R(a_1+\ldots+a_{n-1}+a_n) < 0$, then

$$
\frac{\varepsilon}{R(a_1+\ldots+a_{n-1}+a_n)} \left(1 - e^{-R(a_1+\ldots+a_{n-1}+a_n)(t)}\right)
$$

diverges to infinity as t tends to infinity. Hence in the case of $-R(a_1+\ldots+a_{n-1}+a_n) < 0$, we cannot prove the Hyers-Ulam stability by applying the formable transform method.

Similar to Theorem 2, we prove the Hyers-Ulam σ-stability for the differential equation (16). For the sake of the completeness of this paper, the proof is introduced here in detail.

THEOREM 3. *Assume that $\sigma : [0,\infty) \to (0,\infty)$ is an increasing function and $a_1 + \ldots + a_{n-1} + a_n$ is a constant with $R(a_1+\ldots+a_{n-1}+a_n) > 0$. Then the differential equation* (16) *has the Hyers-Ulam σ-stability for the class B.*

Proof. Assume that $g \in B$ and $\sigma : [0,\infty) \to (0,\infty)$ is an increasing function satisfying the inequality (20) for all $t \geqslant 0$. If we define a function $e(t) : [0,\infty) \to F$ by

$$
e(t) = g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1} g'(t) + a_n g(t)
$$

for all $t \geqslant 0$, then we have $|e(t)| \leqslant \sigma(t)\varepsilon$ for all $t \geqslant 0$.

As we did in the first part of Theorem 2, we can prove that

$$
f(t) = e^{-(a_1+\ldots+a_{n-1}+a_n)t} g(t)
$$

is a solution of the differential equation (16).

By a similar method to the proof of Theorem 2, we can show that

$$
\begin{aligned}
|g(t) - f(t)| &= |e(t) * e^{-(a_1+\ldots+a_{n-1}+a_n)t}| \\
&= \left| \int_0^t e(s) * e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)} ds \right| \\
&\leqslant \int_0^t |e(s)||e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)}| ds
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \sigma(t)\varepsilon e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\int_0^t e^{R(a_1+\ldots+a_{n-1}+a_n)s}ds\\
&\leqslant \frac{\sigma(t)\varepsilon}{R(a_1+\ldots+a_{n-1}+a_n)}\left(1-e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\right)\\
&\leqslant k\sigma(t)\varepsilon
\end{aligned}$$

for all $t \geqslant 0$, where $k = \frac{1}{R(a_1+\ldots+a_{n-1}+a_n)}$. This completes the proof. □

Now, we are going to establish the Mittag-Leffler-Hyers-Ulam stability of the differential equation (16) by using the formable (integral) transform.

THEOREM 4. *Let $a_1+\ldots+a_{n-1}+a_n$ and β be constants with $R(a_1+\ldots+a_{n-1}+a_n) > 0$ and $\beta > 0$. Then the homogeneous linear differential equation (16) has the Mittag-Leffler-Hyers-Ulam stability for the class B.*

Proof. Assume that $g \in B$ satisfies the inequality (22) for all $t \geqslant 0$. Let $e(t) : [0,\infty) \to F$ be a function defined by

$$e(t) = g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1}g'(t) + a_n g(t)$$

for all $t \geqslant 0$. In view of (22), we have $|e(t)| \leqslant \varepsilon$ holds for all $t \geqslant 0$. The formable transform of $e(t)$ gives

$$R\{e(t)\} = D(s,u) = R\{g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1}g'(t) + a_n g(t)\}.$$

Then

$$D(s,u) = \frac{s^n}{u^n}B_1(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k} g^{(k)}(0) + a_{n-1}\left(\frac{s}{u}B_1(s,u) - \frac{s}{u}g(0)\right) + a_n B_1(s,u),$$

and

$$B_1(s,u) = \frac{D(s,u) + \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k} g^{(k)}(0) + \ldots a_{n-1}\frac{s}{u}g(0)}{\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)}. \tag{28}$$

If we put $f(t) = e^{-(a_1+\ldots+a_{n-1}+a_n)t}g(t)$, then $f(0) = g(0)$ and $f \in B$. The formable transform of $f(t)$ gives

$$R(f(t)) = B_2(s,u) = \frac{\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k} f^{(k)}(0) + \ldots a_{n-1}\frac{s}{u}f(0)}{\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)}. \tag{29}$$

It follows from (29) that

$$\begin{aligned}
&R\{f^{(n)}(t) + a_1 f^{(n-1)}(t) + \ldots + a_{n-1}f'(t) + a_n f(t)\}\\
&= \frac{s^n}{u^n}B_2(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k} f^{(k)}(0) + a_{n-1}\left(\frac{s}{u}B_2(s,u) - \frac{s}{u}f(0)\right) + a_n B_2(s,u).
\end{aligned}$$

Since R is a one to one operator, we have

$$f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)=0.$$

Then

$$C(s,u)=\frac{s}{u\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)},$$

$$\begin{aligned}
R\{l(t)\}&=\frac{s}{u\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}\\
&=R^{-1}\left\{\frac{s}{\left(\frac{s^n}{u^{n-1}}+\ldots+a_{n-1}\frac{s}{u}+a_nu\right)}\right\},
\end{aligned}$$

$$l(t)=e^{-(a_1+\ldots+a_{n-1}+a_n)t}.$$

Moreover, it follows from (28) and (29) that

$$\begin{aligned}
R\{g(t)\}-R\{f(t)\}&=B_1(s,u)-B_2(s,u)\\
&=\frac{D(s,u)}{\left(\frac{s^n}{u^{n-1}}+\ldots+a_{n-1}\frac{s}{u}+a_nu\right)}\\
&=\frac{u}{s}D(s,u)\left(\frac{s}{u\left(\frac{s^n}{u^{n-1}}+\ldots+a_{n-1}\frac{s}{u}+a_nu\right)}\right)\\
&=\frac{u}{s}D(s,u)C(s,u).
\end{aligned}$$

Put

$$R\{g(t)\}-R\{f(t)\}=R\{e(t)*l(t)\}.$$

Then we have

$$\begin{aligned}
g(t)-f(t)&=e(t)*l(t)\\
&=e(t)*e^{-(a_1+\ldots+a_{n-1}+a_n)t}.
\end{aligned}$$

Taking modulus on both sides and using the fact that $|e(t)|\leqslant\varepsilon E_\beta(t)$ for $t\geqslant 0$ and $E_\beta(t)$ is increasing for $t\geqslant 0$, we have

$$\begin{aligned}
|g(t)-f(t)|&=|e(t)*e^{-(a_1+\ldots+a_{n-1}+a_n)t}|\\
&=\left|\int_0^t e(s)*e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)}ds\right|\\
&\leqslant\int_0^t |e(s)||e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)}|ds
\end{aligned}$$

$$\begin{aligned}
&\leqslant E_\beta(t)\varepsilon e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\int_0^t e^{R(a_1+\ldots+a_{n-1}+a_n)s}ds\\
&\leqslant \frac{E_\beta(t)\varepsilon}{R(a_1+\ldots+a_{n-1}+a_n)}\left(1-e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\right)\\
&\leqslant kE_\beta(t)\varepsilon
\end{aligned}$$

for all $t \geqslant 0$, where $k = \frac{1}{R(a_1+\ldots+a_{n-1}+a_n)}$. Then by (19), we can confirm that the homogeneous linear differential equation (16) has the Mittag-Leffler-Hyers-Ulam stability for class B. $\square$

Similar to the case of Theorem 4, the Mittag-Leffler-Hyers-Ulam σ-stability for the linear differential equation (16) can be proved.

THEOREM 5. *Assume that $\sigma : [0,\infty) \to (0,\infty)$ is an increasing function and $a_1 + \ldots + a_{n-1} + a_n$ and β are constants with $R(a_1+\ldots+a_{n-1}+a_n) > 0$. Then the differential equation (16) has the Mittag-Leffler-Hyers-Ulam σ-stability for the class B.*

Proof. Assume that $g \in B$ and $\sigma : [0,\infty) \to (0,\infty)$ is a function and $g(t)$ and $f(t)$ satisfy the inequality (24) for all $t \geqslant 0$. We will prove that there exist a positive integer $k > 0$ (independent of ε) and a solution $f : [0,\infty) \to F$ of the differential equation (16) such that $f \in B$ and

$$|g(t) - f(t)| \leqslant k\sigma(t)\varepsilon E_\beta(t)$$

for all $t \geqslant 0$. If we define a function $e : [0,\infty) \to k$ by

$$e(t) = g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1}g'(t) + a_n g(t)$$

for all $t \geqslant 0$, then we have $|e(t)| \leqslant \sigma(t)\varepsilon E_\beta(t)$ for all $t \geqslant 0$.

Then by applying the same method as in the proof of Theorem 4, we can easily get

$$\begin{aligned}
|g(t)-f(t)| &= |e(t) * e^{-(a_1+\ldots+a_{n-1}+a_n)t}|\\
&= \left|\int_0^t e(s) * e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)}ds\right|\\
&\leqslant \int_0^t |e(s)||e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)}|ds\\
&\leqslant \sigma(t)E_\beta(t)\varepsilon e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\int_0^t e^{R(a_1+\ldots+a_{n-1}+a_n)s}ds\\
&\leqslant \frac{\sigma(t)E_\beta(t)\varepsilon}{R(a_1+\ldots+a_{n-1}+a_n)}\left(1-e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\right)\\
&\leqslant k\sigma(t)E_\beta(t)\varepsilon
\end{aligned}$$

for all $t \geqslant 0$, where $k = \frac{1}{R(a_1+\ldots+a_{n-1}+a_n)}$. Then by (20), we can confirm that the homogeneous linear differential equation (16) has the Mittag-Leffler-Hyers-Ulam σ-stability for the class B. $\square$

4. Hyers-Ulam stability of (17)

In this section, we prove several types of Hyers-Ulam stability of the non-homogeneous nth order linear differential equation (17) by using the formable (integral) transform.

THEOREM 6. *Assume that $m:[0,\infty)\to(0,\infty)$ is a continuous function of exponential order and $a_1+\ldots+a_{n-1}+a_n$ is a constant with $R(a_1+\ldots+a_{n-1}+a_n)>0$. The non-homogeneous linear differential equation (17) has the Hyers-Ulam stability for the class B.*

Proof. Suppose that $g\in B$ satisfies the inequality (19) for all $t\geqslant 0$. Consider the function $e:[0,\infty)\to F$ by

$$e(t)=g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)-m(t)$$

for all $t\geqslant 0$. Then $|e(t)|\leqslant\varepsilon$ holds for all $t\geqslant 0$. The formable transform of $e(t)$ gives

$$R\{e(t)\}=D(s,u)=R\{g^n(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)-m(t)\}.$$

This implies that

$$R\{g(t)\}=B_1(s,u)=\frac{D(s,u)+\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0)+\ldots a_{n-1}\frac{s}{u}g(0)}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}. \tag{30}$$

If we set $f(t)=e^{-(a_1+\ldots+a_{n-1}+a_n)t}g(t)+(m(t)*e^{-(a_1+\ldots+a_{n-1}+a_n)t})$, then $f(0)=g(0)$ and $f\in B$. Formable transform of $f(t)$ gives

$$R(f(t))=B_2(s,u)=\frac{\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+\ldots a_{n-1}\frac{s}{u}f(0)}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}. \tag{31}$$

On the other hand, we have

$$\begin{aligned}&R\{f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)\}\\&=\frac{s^n}{u^n}B_2(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+a_{n-1}\left(\frac{s}{u}B_2(s,u)-\frac{s}{u}f(0)\right)+a_nB_2(s,u).\end{aligned}$$

By (31), we have

$$R\{f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)\}=R[m(t)]=M(s,u)$$

and thus

$$f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)=m(t).$$

Hence $f(t)$ is a solution of the differential equation (17).

In addition, by applying (30) and (31), we can obtain

$$\begin{aligned}
R\{g(t)\} - R\{f(t)\} &= B_1(s,u) - B_2(s,u) \\
&= \frac{D(s,u)}{\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)} \\
&= \frac{\frac{u}{s}D(s,u)}{\frac{u}{s}\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)} \\
&= \frac{u}{s}D(s,u)C(s,u) \\
&= R\{D * C\}(s,u) \\
&= R[e(t) * l(t)],
\end{aligned}$$

where

$$R\{l(t)\} = C(s,u) = \frac{s}{u\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)},$$

$$\begin{aligned}
l(t) &= R^{-1}\left\{\frac{s}{\left(\frac{s^n}{u^{n-1}} + \ldots + a_{n-1}\frac{s}{u} + a_n u\right)}\right\} \\
&= e^{-(a_1 + \ldots + a_{n-1} + a_n)t}.
\end{aligned}$$

Therefore, we have

$$R\{g(t) - f(t)\} = R\{e(t) * l(t)\},$$

which yields

$$g(t) - f(t) = e(t) * e^{-(a_1 + \ldots + a_{n-1} + a_n)t}.$$

Furthermore,

$$\begin{aligned}
|g(t) - f(t)| &= |e(t) * e^{-(a_1 + \ldots + a_{n-1} + a_n)t}| \\
&= \left|\int_0^t e(s) e^{-(a_1 + \ldots + a_{n-1} + a_n)(t-s)} ds\right| \\
&\leqslant \int_0^t |e(s)||e^{-(a_1 + \ldots + a_{n-1} + a_n)(t-s)}| ds \\
&\leqslant \varepsilon e^{-R(a_1 + \ldots + a_{n-1} + a_n)t} \int_0^t e^{-R(a_1 + \ldots + a_{n-1} + a_n)s} ds \\
&\leqslant \frac{\varepsilon}{R(a_1 + \ldots + a_{n-1} + a_n)}\left(1 - e^{-R(a_1 + \ldots + a_{n-1} + a_n)t}\right) \\
&\leqslant k\varepsilon
\end{aligned}$$

for all $t \geqslant 0$, where $k = \frac{1}{R(a_1 + \ldots + a_{n-1} + a_n)}$. This completes the proof. □

For the Hyers-Ulam σ-stability of the non-homogeneous linear differential equation (17), we obtain the following theorem.

THEOREM 7. *Assume that* $m : [0,\infty) \to k$ *is a continuous function of exponential order,* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_1 + \ldots + a_{n-1} + a_n$ *is a constant with* $R(a_1 + \ldots + a_{n-1} + a_n) > 0$*. Then the differential equation* (17) *has the Hyers-Ulam stability for the class* B*.*

Proof. We consider an arbitrary function $g \in B$ that satisfies the inequality (21) for all $t \geqslant 0$. Now we define a function $e : [0,\infty) \to F$ by

$$e(t) = g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1} g'(t) + a_n g(t) - m(t)$$

for all $t \geqslant 0$. Then $|e(t)| \leqslant \sigma(t)\varepsilon$ holds for all $t \geqslant 0$. It is not difficult to check that

$$R\{g(t)\} = B_1(s,u) = \frac{D(s,u) + \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-k} g^{(k)}(0) + \ldots a_{n-1} \frac{s}{u} g(0)}{\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)}. \tag{32}$$

If we set $f(t) = e^{-(a_1+\ldots+a_{n-1}+a_n)t} g(t) + (m(t) * e^{-(a_1+\ldots+a_{n-1}+a_n)t})$, then $f \in B$. Further, we apply the formable transform on both sides to get

$$R(f(t)) = B_2(s,u) = \frac{\sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-k} f^{(k)}(0) + \ldots a_{n-1} \frac{s}{u} f(0)}{\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)}. \tag{33}$$

On the other hand, we have

$$\begin{aligned} &R\{f^{(n)}(t) + a_1 f^{(n-1)}(t) + \ldots + a_{n-1} f'(t) + a_n f(t)\} \\ &= \frac{s^n}{u^n} B_2(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-k} f^{(k)}(0) + a_{n-1}\left(\frac{s}{u} B_2(s,u) - \frac{s}{u} f(0)\right) + a_n B_2(s,u). \end{aligned}$$

The relation (33) implies that

$$R\{f^{(n)}(t) + a_1 f^{(n-1)}(t) + \ldots + a_{n-1} f'(t) + a_n f(t)\} = R[m(t)] = N(s,u)$$

and thus

$$f^{(n)}(t) + a_1 f^{(n-1)}(t) + \ldots + a_{n-1} f'(t) + a_n f(t) = m(t).$$

That is, $f(t)$ is a solution of the differential equation (17). By (32) and (33), we obtain

$$\begin{aligned} R\{g(t)\} - R\{f(t)\} &= B_1(s,u) - B_2(s,u) \\ &= R[e(t) * l(t)], \end{aligned}$$

where

$$R\{l(t)\} = C(s,u) = \frac{s}{u\left(\frac{s^n}{u^n} + \ldots + a_{n-1}\frac{s}{u} + a_n\right)}.$$

This gives

$$\begin{aligned} l(t) &= R^{-1}\left\{ \frac{s}{\left(\frac{s^n}{u^{n-1}} + \ldots + a_{n-1}\frac{s}{u} + a_n u\right)} \right\} \\ &= e^{-(a_1+\ldots+a_{n-1}+a_n)t}. \end{aligned}$$

Therefore, we have

$$R\{g(t)-f(t)\}=R\{e(t)*l(t)\},$$

which gives

$$g(t)-f(t)=e(t)*l(t).$$

Similar to the proof of Theorem 3, we have

$$\begin{aligned}|g(t)-f(t)|&=|e(t)*l(t)|\\&\leqslant k\sigma(t)\varepsilon\end{aligned}$$

for all $t\geqslant 0$, where $k=\frac{1}{R(a_1+\ldots+a_{n-1}+a_n)}$. This completes the proof. □

Now, we prove the Mittag-Leffler-Hypers-Ulam stability of the non-homogeneous linear differential equation (17) by using the formable transform method.

THEOREM 8. *Assume that* $m:[0,\infty)\to(0,\infty)$ *is a continuous function of exponential order and* $a_1+\ldots+a_{n-1}+a_n$ *and* β *are constants with* $R(a_1+\ldots+a_{n-1}+a_n)>0$. *Then the non-homogeneous linear differential equation* (17) *has the Mittag-Leffler-Hyers-Ulam stability for the class* B.

Proof. Suppose that $g\in B$ satisfies the inequality (23) for all $t\geqslant 0$. Consider a function $e:[0,\infty)\to F$ defined by

$$e(t)=g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)-m(t)$$

for all $t\geqslant 0$. It follows from (23) that $|e(t)|\leqslant E_\beta(t)\varepsilon$ holds for all $t\geqslant 0$. The formable transform of $e(t)$ gives

$$R\{e(t)\}=D(s,u)=R\{g^{(n)}(t)+a_1g^{(n-1)}(t)+\ldots+a_{n-1}g'(t)+a_ng(t)-m(t)\}.$$

That is,

$$R\{g(t)\}=B_1(s,u)=\frac{D(s,u)+\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0)+\ldots a_{n-1}\frac{s}{u}g(0)}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}. \tag{34}$$

If we set $f(t)=e^{-(a_1+\ldots+a_{n-1}+a_n)t}g(t)+(m(t)*e^{-(a_1+\ldots+a_{n-1}+a_n)t})$, then $f(0)=g(0)$ and $f\in B$. Applying the formable transform on both sides of the last equality, we get

$$R(f(t))=B_2(s,u)=\frac{\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+\ldots a_{n-1}\frac{s}{u}f(0)}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}. \tag{35}$$

On the other hand, we have

$$\begin{aligned}&R\{f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)\}\\&=\frac{s^n}{u^n}B_2(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}f^{(k)}(0)+a_{n-1}\left(\frac{s}{u}B_2(s,u)-\frac{s}{u}f(0)\right)+a_nB_2(s,u).\end{aligned}$$

By (35), we have

$$R\{f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)\}=R[m(t)]=N(s,u)$$

and thus

$$f^{(n)}(t)+a_1f^{(n-1)}(t)+\ldots+a_{n-1}f'(t)+a_nf(t)=m(t).$$

Hence $f(t)$ is a solution of the differential equation (17).

In addition, by applying (34) and (35), we can obtain

$$\begin{aligned}R\{g(t)\}-R\{f(t)\}&=B_1(s,u)-B_2(s,u)\\&=R[e(t)*l(t)],\end{aligned}$$

where

$$R\{l(t)\}=C(s,u)=\frac{s}{\left(\frac{s^n}{u^n}+\ldots+a_{n-1}\frac{s}{u}+a_n\right)}.$$

This gives

$$\begin{aligned}l(t)&=R^{-1}\left\{\frac{s}{\left(\frac{s^n}{u^{n-1}}+\ldots+a_{n-1}\frac{s}{u}+a_nu\right)}\right\}\\&=e^{-(a_1+\ldots+a_{n-1}+a_n)t}.\end{aligned}$$

Therefore, we have

$$R\{g(t)-f(t)\}=R\{e(t)*l(t)\},$$

which yields $g(t)-f(t)=e(t)*l(t)$ for all $t\geqslant 0$. Furthermore,

$$\begin{aligned}|g(t)-f(t)|&=|e(t)*e^{-(a_1+\ldots+a_{n-1}+a_n)t}|\\&=\left|\int_0^t e(s)e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)}ds\right|\\&\leqslant\int_0^t|e(s)||e^{-(a_1+\ldots+a_{n-1}+a_n)(t-s)}|ds\\&\leqslant E_\beta(t)\varepsilon e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\int_0^t e^{-R(a_1+\ldots+a_{n-1}+a_n)s}ds\\&\leqslant\frac{E_\beta(t)\varepsilon}{R(a_1+\ldots+a_{n-1}+a_n)}\left(1-e^{-R(a_1+\ldots+a_{n-1}+a_n)t}\right)\\&\leqslant kE_\beta(t)\varepsilon\end{aligned}$$

for all $t\geqslant 0$, where $k=\frac{1}{R(a_1+\ldots+a_{n-1}+a_n)}$. This completes the proof. □

Similar to the case of Theorem 8, the Mittag-Leffler-Hyers-Ulam σ-stability for the linear differential equation (17) can be proved.

THEOREM 9. *Assume that* $m : [0,\infty) \to (0,\infty)$ *is a continuous function of exponential order and* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_1 + \ldots + a_{n-1} + a_n$ *and* β *are constants with* $R(a_1 + \ldots + a_{n-1} + a_n) > 0$. *Then the non-homogeneous linear differential equation* (17) *has the Mittag-Leffler-Hyers-Ulam* σ*-stability for the class* B.

Proof. Assume that $g \in B$ satisfies the inequality (25) for all $t \geqslant 0$. It is easy to prove that there exist a constant $k > 0$ (independent of ε) and a solution $f : [0,\infty) \to F$ of the differential equation (17) such that $f \in B$ and

$$|g(t) - f(t)| \leqslant k\sigma(t) \in E_\beta(t)$$

for all $t \geqslant 0$. If we define a function $e : [0,\infty) \to F$ by

$$e(t) = g^{(n)}(t) + a_1 g^{(n-1)}(t) + \ldots + a_{n-1} g'(t) + a_n g(t) - m(t)$$

for all $t \geqslant 0$, then we have $|e(t)| \leqslant \sigma(t) E_\beta(t)\varepsilon$ for all $t \geqslant 0$. By applying a similar method as in the proof of Theorem 8, we can easily prove that there exists a solution $f : [0,\infty) \to F$ of (17) satisfying $f \in B$ and

$$\begin{aligned} |g(t) - f(t)| &= |e(t) * e^{-(a_1 + \ldots + a_{n-1} + a_n)t}| \\ &\leqslant k\sigma(t) E_\beta(t)\varepsilon \end{aligned}$$

for all $t \geqslant 0$, where $k = \frac{1}{R(a_1 + \ldots + a_{n-1} + a_n)}$. This completes the proof. □

Conclusion

In this paper, we proved the Hyers-Ulam stability, Hyers-Ulam σ-stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam σ-stability of the linear differential equation of nth order with constant coefficients by using the formable (integral) transform method. In other words, we established sufficient criteria for the Hyers-Ulam stability of nth order linear differential equations with constant coefficients by using the formable (integral) transform method. Moreover, this paper provided a new method to investigate the Hyers-Ulam stability of differential equations.

Declarations

Availability of data and materials. Not applicable.

Human and animal rights. We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest. The authors declare that they have no competing interests.

Fundings. S. Donganont was supported by the University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2257/2568).

Authors' contributions. The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Acknowledgements. We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that will help to improve the quality of the manuscript.

REFERENCES

[1] S. R. ADERYANI, L. P. CASTRO, R. SAADATI, C. PARK, *Aggregated multi-stability for a class of J-Hilfer fractional differential equations via Mittag-Leffler and hypergeometric type functions*, Afr. Mat. **36** (2025), no. 1, Paper No. 19.

[2] C. ALSINA, R. GER, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373–380.

[3] S. BASKARAN, R. MURALI, C. PARK, A. P. SELVAN, *Sumudu transform and the stability of second order linear differential equations*, J. Math. Inequal. **18** (2024), no. 3, 847–864.

[4] F. DEVELI, *Existence and Ulam-Hyers stability results for nonlinear fractional Langevin equation with modified argument*, Math. Methods Appl. Sci. **45** (2022), no. 7, 3417–3425.

[5] P. GAVRUTA, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), no. 3, 431–436.

[6] P. GĂVRUTA, S. M. JUNG, Y. LI, *Hyers-Ulam stability for second-order linear differential equations with boundary conditions*, Electron. J. Differ. Equ. **2011** (2011), Paper No. 80.

[7] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[8] K. JUN, Y. LEE, *A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratice quations*, J. Math. Anal. Appl. **297** (2004), no. 1, 70–86.

[9] S. M. JUNG, *On the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **204** (1996), 221–226.

[10] S. M. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fl. 2001.

[11] S. M. JUNG, *Hyers-Ulam stability of linear differential equations of first order*, J. Math. Anal. Appl. **311** (2005), 139–146.

[12] Y. LI, *Hyers-Ulam stability of linear differential equations*, Thai J. Math. **8** (2010), no. 2, 215–219.

[13] Y. LI, Y. SHEN, *Hyers-Ulam stability of nonhomogeneous linear differential equations of second order*, Int. J. Math. Math. Sci. **2009** (2009), Article ID 576852.

[14] T. MIURA, S. MIYAJIMA, S.-E. TAKAHASI, *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286** (2007), 136–146.

[15] R. MURALI, A. P. SELVAN, S. BASKARAN, C. PARK, J. R. LEE, *Hyers-Ulam stability of first-order linear differential equations using Aboodh transform*, J. Inequal. Appl. **2021** (2021), Paper No. 133.

[16] M. N. QARAWANI, *Hyers-Ulam stability of a generalized second- order non-linear differential equation*, Appl. Math. **3** (2012), no. 12, 1857–1861, https://doi.org/10.4236/am.2012.312252.

[17] S. RASHID, T. KHADIJA, K. M. ABUALNAJA, *Fractional view of heat-like equations via the Elzaki transform in the settings of the Mittag-Leffler function*, Math. Methods Appl. Sci. **46** (2023), no. 10, 11420–11441.

[18] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.

[19] R. RIZWAN, F. LIU, C. PARK, *Exploring the dynamics of implusive fractional Langevin equations via Mittag-Leffler functions*, Math. Methods Appl. Sci. **48** (2025), no. 16, 14828–14845.

[20] S. SABARINATHAN, A. SELVAM, S. PINELAS, *Stability analyis of differential equations using Mohand integral transform*, Analysis and Applied Mathematics, pp. 171–181, Trands Math., Res. Perspect Ghent Anal. PDE Cent., **6**, Birkhäuser/Springer, Cham, 2024.

[21] Y. SAYYARI, M. DEHGHANIAN, C. PARK, *Some stabilities of system of differential equations using Laplace transform*, J. Appl. Math. Comput. **69** (2023), no. 4, 3113–3129.

[22] S.-E. TAKAHASI, T. MIURA, S. MIYAJIMA, *On the Hyers-Ulam stability of the Banach space-valued differential equation* $y' = \lambda y$, Bull. Korean Math. Soc. **39** (2002), no. 2, 309–315.

[23] S. M. ULAM, *Problems in Modern Mathematics*, John Wiley & Sons, New York, 1964.

[24] G. WANG, M. ZHOU, L. SUN, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **21** (2008), 1024–1028.

(Received July 27, 2025)

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