

INTEGRAL OPERATORS AND THEIR COMMUTATORS IN VARIABLE EXPONENT GENERALIZED WEIGHTED MORREY–GULIYEV SPACES

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Abstract. In this paper we consider the variable exponent generalized weighted Morrey-Guliyev spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$ with variable exponents $p(x)$, $\theta(r)$ and a general function $\omega(x,r)$ defining the Morrey-type norm. In case of unbounded sets $\Omega \subseteq \mathbb{R}^n$ we prove the boundedness of Hardy-Littlewood maximal operator, generalized fractional integral operators and Calderón-Zygmund singular integral operators in weighted Morrey-Guliyev spaces. Furthermore, we investigate the boundedness of the commutators of abovementioned integral operators in weighted Morrey-Guliyev spaces.

1. Introduction

Morrey spaces $\mathcal{M}^{p,\lambda}(\Omega)$ are basic instruments in the analysis of local properties of solutions of partial differential equations. Morrey [40] introduced given these spaces in 1938 by the norm

$$\|f\|_{\mathcal{M}^{p,\lambda}(\Omega)} := \sup_{x,r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(\tilde{B}(x,r))}, \quad \tilde{B}(x,r) = B(x,r) \cap \Omega,$$

where $0 \leq \lambda < n$, $1 \leq p < \infty$, $\Omega \subseteq \mathbb{R}^n$ is an open set and $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Morrey spaces have been expanded over the years for studying some solutions of PDEs and so-called Morrey-type spaces have been formed. Since studying of variable exponent spaces and operators have become very popular in last two decades, variable exponent Morrey-type spaces have come to the literature, interested readers can refer to the sources [3, 22, 28, 32, 39].

Various Morrey spaces are given in this study. Guliyev, Mizuhara and Nakai ([19, 38, 41]) established generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [20, 21, 43, 45]). Recently, Komori and Shirai [33] defined the weighted Morrey spaces $L_w^{p,K}(\mathbb{R}^n)$ and investigated the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Moreover, Guliyev

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in [23] first presented the generalized weighted Morrey spaces $M_w^{p,\varphi}(\mathbb{R}^n)$ and investigated the boundedness of the some integral operators and their commutators in these variable exponent spaces.

Later some researchers investigated Morrey spaces from a different viewpoint. These examinations lead to the concept of local and global counterpart.

The local Morrey-type space $LM^{p,q,\omega(\cdot)}(\mathbb{R}^n)$ is defined by Guliyev [[19], pp. 75–76], (see also [[20], pp. 123]) as the norm

$$\|f\|_{LM^{p,q,\omega(\cdot)}} = \|\omega(r)\|\chi_{B(0,r)}f\|_{L^p}\|_{L^q(\mathbb{R}_+)} < \infty,$$

where ω is a positive measurable function defined on $(0, \infty)$. Also, Guliyev in [19] (see also [20]) obtained the sufficient conditions for the boundedness of the singular and potential operators in $LM^{p,q,\omega(\cdot)}(\mathbb{R}^n)$. Laterly, Burenkov and H.V. Guliyev [[4], pp. 157] defined the global Morrey-type space $GM^{p,\theta,\omega(\cdot)}(\mathbb{R}^n)$ as the finite quasi-norm

$$\|f\|_{GM^{p,\theta,\omega(\cdot)}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{LM^{p,\theta,\omega(\cdot)}} = \sup_{x \in \mathbb{R}^n} \|\omega(r)\|f\chi_{B(x,r)}\|_{L^p}\|_{L^\theta(0,\infty)}$$

for all the Lebesgue measurable functions f on \mathbb{R}^n .

The spaces $GM^{p,q,w(\cdot)}(\mathbb{R}^n)$ and $LM^{p,q,w(\cdot)}(\mathbb{R}^n)$ may be also referred to (for the notation see [5, 6, 24]) *Morrey-Guliyev* and *local Morrey-Guliyev spaces*, respectively, by Rafeiro and Samko [42] depending on the progress of these spaces.

The local Morrey type space $LM^{p,q,\lambda}(\mathbb{R}^n) = LM^{p,q,t^{-\lambda}}(\mathbb{R}^n)$ is defined by Adams in [[1], p. 44] and also called *local Morrey-Adams spaces*, see, for example, [42, 44].

As is known that the commutator is an basic integral operator and has an important role in harmonic analysis. Maximal commutator is a main tool in the study of commutators of singular integral operators with *BMO* symbols (see, for example [18], [36]). The commutator of Calderón-Zygmund operators is very important in investigating the regularity of solutions of elliptic partial differential equations of second order (see, for example, [7, 8, 10, 11, 12, 17, 27, 37]). Therefore, one of the proposes of this paper is to establish the boundedness of the commutators of maximal operator, fractional integral operators and Calderón-Zygmund operators in the spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$.

This paper is devoted into 5 sections. In Section 3 we acquire the boundedness of the maximal operator and its commutator in $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$ spaces and obtain an embedding between these spaces. In Section 4 it is acquired boundedness of generalized fractional integral operators and their commutators in the spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$. Finally, in Section 5 we establish the boundedness of singular integral operators and their commutators in the spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$. Our main results are introduced in Theorems 4, 5, 9, 10, 11, 12, 13, 14, 16, and 18. During the paper, the various positive constants are expressed by c, c_1, c_2, C etc. We mean by an open set Ω is unbounded, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $p(\cdot)$ is a measurable function on Ω with values in $(1, \infty)$ and $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, is the conjugate exponent.

2. Background and basic notations

Let

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{1}$$

where $p_- := \operatorname{ess\,inf}_{x \in \Omega} p(x)$, $p_+ := \operatorname{ess\,sup}_{x \in \Omega} p(x)$. The space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty$$

will be denoted by $L^{p(\cdot)}(\Omega)$, which is a Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \eta > 0 : I_{p(\cdot)} \left(\frac{f}{\eta} \right) \leq 1 \right\}.$$

We refer to the sources [13, 15, 29, 35, 46] for the basic properties of variable exponent Lebesgue spaces and the boundedness of operators in these spaces:

- the set of bounded measurable functions $p : \Omega \rightarrow [1, \infty)$ is denoted by $\mathcal{P}(\Omega)$;
- the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \Omega, \tag{2}$$

where $A = A(p) > 0$ does not depend on x, y is denoted by $\mathcal{P}^{\log}(\Omega)$;

- the set of bounded exponents $p : \Omega \rightarrow \mathbb{R}^n$ satisfying the condition (2) is denoted by $\mathcal{A}^{\log}(\Omega)$;
- the set of exponents $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$ is denoted by $\mathbb{P}^{\log}(\Omega)$;
- the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_{\infty}}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n, \tag{3}$$

where $p_{\infty} = \lim_{x \rightarrow \infty} p(x) > 1$ are denoted by $\mathcal{P}_{\infty}(\Omega)$, $\mathcal{P}_{\infty}^{\log}(\Omega)$, $\mathbb{P}_{\infty}^{\log}(\Omega)$, $\mathcal{A}_{\infty}^{\log}(\Omega)$.

We will also use the calculation provided by the following lemma (see [15], Corollary 4.5.9).

$$\|\chi_{\tilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq Cr^{\eta_p(x,r)}, \quad x \in \Omega, \quad p \in \mathbb{P}_{\infty}^{\log}(\Omega), \tag{4}$$

where $\eta_p(x, r) = \begin{cases} \frac{n}{p(x)}, & r \leq 1, \\ \frac{n}{p(\infty)}, & r \geq 1 \end{cases}$.

A positive, locally integrable function with domain Ω will be denoted by φ . The weighted Lebesgue space $L_{\varphi}^{p(\cdot)}(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L_{\varphi}^{p(\cdot)}(\Omega)} = \|f\varphi\|_{L^{p(\cdot)}(\Omega)}.$$

Now we define the class $A_{p(\cdot)}(\mathbb{R}^n)$ (see [14], [34]) consisting those weights φ with

$$[\varphi]_{A_{p(\cdot)}} \equiv \sup_B |B|^{-1} \|\varphi\|_{L^{p(\cdot)}(B(x,r))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(x,r))} < \infty.$$

A weight function φ belongs to the class $A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if

$$[\varphi]_{A_{p(\cdot),q(\cdot)}} \equiv \sup_B r^{\eta p(x,r) - \eta q(x,r) - n} \|\varphi\|_{L^{q(\cdot)}(B(x,r))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(x,r))} < \infty.$$

The following two theorems state the boundedness of maximal operator M and singular integral operator T in the spaces $L_\varphi^{p(\cdot)}(\Omega)$.

THEOREM 1. ([29, Theorem 1.1]) *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$. Then $M : L_\varphi^{p(\cdot)}(\Omega) \rightarrow L_\varphi^{p(\cdot)}(\Omega)$ if and only if $\varphi \in A_{p(\cdot)}(\Omega)$.*

THEOREM 2. ([13, Theorem 4.8]) *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}^{log}(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

DEFINITION 1. Let $1 \leq p(x) < \infty$, $x \in \Omega$. The variable exponent generalized Morrey space $M^{p(\cdot),\omega(\cdot)}(\Omega)$ and variable exponent generalized weighted Morrey space $M_\varphi^{p(\cdot),\omega(\cdot)}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{M^{p(\cdot),\omega(\cdot)}} = \sup_{x \in \Omega, r > 0} \frac{1}{\omega(x,r)r^{\eta p(x,r)}} \|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))},$$

$$\|f\|_{M_\varphi^{p(\cdot),\omega(\cdot)}} = \sup_{x \in \Omega, r > 0} \frac{1}{\omega(x,r)\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))}} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))},$$

respectively.

DEFINITION 2. Let $\omega(x,r) : \Omega \times (0,\infty) \rightarrow [0,\infty)$ and $\theta(r) : (0,\infty) \rightarrow [1,\infty]$ be measurable functions. We define the variable exponent generalized weighted Morrey-Guliyev spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$ as the set of functions which have the finite norm

$$\|f\|_{GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)} = \sup_{x \in \Omega} \left\| \frac{\omega(x,r)}{\|\varphi\|_{L^{p(\cdot)}(B(x,r))}} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta(\cdot)}(0,\infty)}.$$

Note that if for all $r \in (0,\infty)$, $\theta(r) = \infty$, then $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega) = M_\varphi^{p(\cdot),\frac{1}{\omega(\cdot)}}(\Omega)$.

According to this definition, we recover the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\theta(r) = \infty$, $\omega(x,r) = r^{\eta p(x,r) - \frac{\lambda(x)}{p(x)}}$, $\varphi \equiv 1$:

$$L^{p(\cdot),\lambda(\cdot)}(\Omega) = GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega) \Big|_{\theta(r)=\infty, \omega(x,r)=r^{\eta p(x,r) - \frac{\lambda(x)}{p(x)}}, \varphi(x)=1}.$$

Everywhere in this paper we assume that $\omega(x, r)$ satisfies the condition

$$\sup_{x \in \Omega} \|\omega(x, \cdot)\|_{L^{\theta(\cdot)}(0, \infty)} < \infty \tag{5}$$

under which the space $GM^{p(\cdot), \theta(\cdot), \omega(\cdot)}(\Omega)$ contains bounded functions.

The sharp maximal function M^\sharp is defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x, r)|^{-1} \int_{\tilde{B}(x, r)} |f(y) - f_{\tilde{B}(x, r)}| dy,$$

where $f_{\tilde{B}(x, t)}(x) = |\tilde{B}(x, t)|^{-1} \int_{\tilde{B}(x, t)} f(y) dy$.

DEFINITION 3. $BMO(\Omega)$ space is defined as the set of all locally integrable functions f with finite norm

$$\|f\|_* = \sup_{x \in \Omega} M^\sharp f(x) = \sup_{x \in \Omega, r>0} |B(x, r)|^{-1} \int_{\tilde{B}(x, r)} |f(y) - f_{\tilde{B}(x, r)}| dy.$$

DEFINITION 4. $BMO_{p(\cdot), \varphi}(\Omega)$ space is defined as the set of all locally integrable functions f with finite norm

$$\|f\|_{*_{p(\cdot), \varphi}} = \sup_{x \in \Omega, r>0} \frac{\|(f(\cdot) - f_{\tilde{B}(x, r)}) \chi_{\tilde{B}(x, r)}\|_{L_\varphi^{p(\cdot)}(\Omega)}}{\|\chi_{\tilde{B}(x, r)}\|_{L_\varphi^{p(\cdot)}(\Omega)}}.$$

THEOREM 3. [30, Theorem 4.4] *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$ and φ be a Lebesgue measurable function. If $\varphi \in A_{p(\cdot)}(\Omega)$, then the norms $\|\cdot\|_{*_{p(\cdot), \varphi}}$ and $\|\cdot\|_*$ are mutually equivalent.*

The weighted Hardy operator is defined by

$$\bar{H}_{v, u} g(t) = v(t) \int_t^\infty g(\xi) u(\xi) d\xi, \quad t > 0.$$

Now we give the following lemma which was proved in [16].

LEMMA 1. *Let $\theta(t)$ and $r(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists δ positive number a such that $\theta(t) = \theta = \text{const}$, $r(t) = r = \text{const}$ for $x > \delta$, and $1 < \theta_- \leq \tilde{\theta}(t) \leq r(t) \leq r_+ < \infty$ for a.a. $x \in (0, \infty)$. If*

$$\sup_{0 < t < \infty} \int_0^t v(\xi) r(\xi) \left(\int_t^\infty u^{[\tilde{\theta}(\xi)]'}(r) dr \right)^{\frac{r(\xi)}{|\tilde{\theta}(\xi)'|}} d\xi < \infty,$$

where $\tilde{\theta}(\xi) = \inf_{s \in (\xi, \infty)} \theta(s)$, then the operator $\bar{H}_{v, u}$ is bounded from $L^{\theta(\cdot)}(0, \infty)$ to $L^{r(\cdot)}(0, \infty)$.

Notice that Lemma 1 contains no assumption on the continuity of the exponents $\theta(\cdot)$ and $r(\cdot)$.

3. Maximal commutators in $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$

In this section, we consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |f(y)| dy$$

and its commutator. First we give the boundedness of M in the spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$.

The following theorem was proved in [25].

THEOREM 4. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ and $\varphi \in A_{p(\cdot)}(\Omega)$. Then*

$$\|Mf\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^{\infty} \|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}$$

for every $f \in L_{\varphi}^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t .

In the following theorem we extend the result obtained in [3] to the spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$.

THEOREM 5. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption (1), $\varphi \in A_{p(\cdot)}(\Omega)$ and $\theta_1(t)$ and $\theta_2(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists δ positive number a such that $\theta_1(t) = \theta_1 = const$, $\theta_2(t) = \theta_2 = const$ for $x > \delta$, and $1 < \theta_1^- \leq \tilde{\theta}_1(t) \leq \theta_2(t) \leq \theta_2^+ < \infty$ for a.a. $x \in (0, \infty)$. If the condition*

$$\sup_{x \in \Omega, t > 0} \int_0^t \omega_2(x, \xi)^{\theta_2(\xi)} \left(\int_t^{\infty} \frac{dr}{[r\omega_1(x, r)]^{|\tilde{\theta}_1(\xi)|'}} \right)^{\frac{\theta_2(\xi)}{|\tilde{\theta}_1(\xi)|'}} d\xi < \infty \tag{6}$$

holds, where $\tilde{\theta}_1(\xi) = \inf_{s \in (\xi, \infty)} \theta_1(s)$, then the operator M is bounded from

$GM^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot),\varphi}(\Omega)$ to $GM^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot),\varphi}(\Omega)$.

Proof. Let $f \in GM^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot),\varphi}(\Omega)$. Hence, by choosing $v(r) = \omega_2(x, r)$, $g(r) = \omega_1(x, r) \|\varphi\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,r))}^{-1} \|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,r))}$, $u(r) = \omega_1(x, r)^{-1} r^{-1}$ in Lemma 1, and using Theorem 4 we obtain

$$\begin{aligned} \|Mf\|_{GM^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot),\varphi}} &= \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|Mf\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^{\infty} \|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \sup_{x \in \Omega} \left\| \frac{\omega_1(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta_1(\cdot)}(0, \infty)} = C \|f\|_{GM^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot),\varphi}}. \quad \square \end{aligned}$$

COROLLARY 1. *Under the assumptions of Theorem 5, the embedding*

$$GM^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot),\varphi}(\Omega) \hookrightarrow GM^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot),\varphi}(\Omega)$$

holds.

Proof. It is an immediate consequence of the inequality $f(x) \leq Mf(x)$. That is

$$\|f\|_{GM^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot),\varphi}} \leq \|Mf\|_{GM^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot),\varphi}} \leq C\|f\|_{GM^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot),\varphi}} \quad \square$$

The commutator generated by M and a suitable function b is formally defined by

$$[M, b]f = M(bf) - bM(f).$$

Given a measurable function b the maximal commutator is defined by

$$M_b(f)(x) := \sup_{r>0} |B(x, r)|^{-1} \int_{B(x, r)} |b(x) - b(y)| |f(y)| dy$$

for all $x \in \mathbb{R}^n$.

The following statement was proved by Pu Zhang and Jianglong Wu in [48].

THEOREM 6. [48, Theorem 3.1] *Let $b \in L_1^{loc}(\mathbb{R}^n)$ and $p \in \mathbb{P}_\infty^{log}(\mathbb{R}^n)$, then the operator M_b is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself if and only if $b \in BMO(\mathbb{R}^n)$.*

THEOREM 7. *Let $b \in BMO(\mathbb{R}^n)$. Then, there exists a positive constant C such that*

$$M(M_b f)(x) \leq C \|b\|_{BMO} \left[(M(Mf)^s)^{\frac{1}{s}}(x) + M(M|f|^s)^{\frac{1}{s}}(x) \right]$$

for almost every $x \in \mathbb{R}^n$ and for all $f \in L_1^{loc}(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$, $B_t = B(0, t)$. The maximal commutator is defined by

$$\begin{aligned} M_b(f)(z) &:= \sup_{t>0} |B(z, t)|^{-1} \int_{B(z, t)} |b(z) - b(y)| |f(y)| dy \\ &\approx \sup_{t>0} |B(0, t)|^{-1} \int_{B(0, t)} |b(z) - b(z-y)| |f(z-y)| dy \end{aligned}$$

we get

$$\begin{aligned} &|B(0, t)|^{-1} \int_{B(0, t)} |b(z) - b(z-y)| |f(z-y)| dy \\ &\leq |B(0, t)|^{-1} \int_{B(0, t)} |b(z-y) - b_{B_t}| |f(z-y)| dy \\ &\quad + |B(0, t)|^{-1} \int_{B(0, t)} |b(z) - b_{B_t}| |f(z-y)| dy \end{aligned}$$

$$\begin{aligned} &\leq C|B(0,t)|^{-1} \left(\int_{B(0,t)} |b(z-y) - b_{B_t}|^{s'} dy \right)^{\frac{1}{s'}} \left(\int_{B(0,t)} |f(z-y)|^s dy \right)^{\frac{1}{s}} \\ &\quad + |B(0,t)|^{-1} |b(z) - b_{B_t}| \int_{B(0,t)} |f(z-y)| dy \leq \\ &\leq C \|b\|_{BMO} (M|f|^s)^{\frac{1}{s}}(z) + |b(z) - b_{B_t}| Mf(z). \end{aligned}$$

By the Hölder inequality we have

$$\begin{aligned} &|B(0,t)|^{-1} \int_{B(0,t)} \left(C \|b\|_{BMO} (M|f|^s)^{\frac{1}{s}}(x-z) + |b(x-z) - b_{B_t}| Mf(x-z) \right) dz \\ &\leq C \|b\|_{BMO} |B(0,t)|^{-1} \int_{B(0,t)} (M|f|^s)^{\frac{1}{s}}(x-z) dz \\ &\quad + |B(0,t)|^{-1} \int_{B(0,t)} |b(x-z) - b_{B_t}| Mf(x-z) dz \\ &\leq C \|b\|_{BMO} M (M|f|^s)^{\frac{1}{s}}(x) + |B(0,t)|^{-1} \int_{B(0,t)} |b(x-z) - b_{B_t}| Mf(x-z) dz \\ &\leq C \|b\|_{BMO} M (M|f|^s)^{\frac{1}{s}}(x) \\ &\quad + |B(0,t)|^{-1} \left(\int_{B(0,t)} |b(x-z) - b_{B_t}|^{s'} dz \right)^{\frac{1}{s'}} \left(\int_{B(0,t)} (Mf(x-z))^s dz \right)^{\frac{1}{s}} \\ &\leq C \|b\|_{BMO} \left[(M(Mf)^s)^{\frac{1}{s}}(x) + M (M|f|^s)^{\frac{1}{s}}(x) \right]. \quad \square \end{aligned}$$

The following theorem is valid.

THEOREM 8. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $b \in BMO(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$, then the operator M_b is bounded in $L_{p(\cdot),\omega}(\Omega)$.*

Proof. It is an immediate consequence of the inequality $f(x) \leq Mf(x)$. Using the Theorem 7 and Theorem 1, we get

$$\begin{aligned} \|M_b f\|_{L_{p,\omega}(\Omega)} &\leq \|M(M_b f)\|_{L_{p,\omega}(\Omega)} \\ &\leq C \|b\|_{BMO} \left\| (M(Mf)^s)^{\frac{1}{s}} + M(M|f|^s)^{\frac{1}{s}} \right\|_{L_{p,\omega}(\Omega)} \\ &\leq C \|b\|_{BMO} \left(\left\| (M(Mf)^s)^{\frac{1}{s}} \right\|_{L_{p,\omega}(\Omega)} + \left\| M(M|f|^s)^{\frac{1}{s}} \right\|_{L_{p,\omega}(\Omega)} \right) \\ &\leq C \|b\|_{BMO} \|f\|_{L_{p,\omega}(\Omega)}. \quad \square \end{aligned}$$

Before proving the main theorems, we need the following lemma.

LEMMA 2. [31] *Let $b \in BMO(\Omega)$. Then there is a constant $C > 0$ such that*

$$\left| b_{\tilde{B}(x,r)} - b_{\tilde{B}(x,t)} \right| \leq C \|b\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t,$$

where C is independent of b , x , r , and t .

The following weighted local estimates are valid.

THEOREM 9. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $\varphi \in A_{p(\cdot)}(\Omega)$ and $b \in BMO(\Omega)$. Then*

$$\|M_b f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|b\|_* \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^{\infty} \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,s))} ds}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))} s} \tag{7}$$

for every $f \in L_{\varphi}^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t .

Proof. The proof of inequality (7) can be done by representing f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\tilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega \setminus \tilde{B}(x,2t)}(y), \quad t > 0. \tag{8}$$

and have

$$\|M_b f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,t))} \leq \|M_b f_1\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,t))} + \|M_b f_2\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,t))}.$$

By Theorem 8 we obtain

$$\|M_b f_1\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,t))} \leq \|M_b f_1\|_{L_{\varphi}^{p(\cdot)}(\Omega)} \leq C \|b\|_* \|f_1\|_{L_{\varphi}^{p(\cdot)}(\Omega)} = C \|b\|_* \|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,2t))}, \tag{9}$$

where C does not depend on f . From (9) we obtain

$$\|M_b f_1\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|b\|_* \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^{\infty} \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,s))} ds}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))} s}, \tag{10}$$

easily obtained from the fact that $\|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,2t))}$ is non-decreasing in t , so that $\|f\|_{L_{\varphi}^{p(\cdot)}(\tilde{B}(x,2t))}$ on the right-hand side of (9) is dominated by the right-hand side of (10). For $z \in \tilde{B}(x,t)$, choosing $\delta > 0$ it is easily seen that

$$\begin{aligned} M_b f_2(z) &= \sup_{r>0} |B(z,r)|^{-1} \int_{\tilde{B}(z,r)} |b(z) - b(y)| |f_2(y)| dy \\ &\leq C \sup_{r \geq 2t} \int_{\mathbb{C}_{\tilde{B}(x,2t)} \cap \tilde{B}(z,r)} |y - z|^{-n} |b(z) - b(y)| |f(y)| dy \\ &\leq C \sup_{r \geq 2t} \int_{\tilde{B}(x,2t) \cap \tilde{B}(z,r)} |x - y|^{-n} |b(z) - b(y)| |f(y)| dy \\ &\leq \delta \int_{\mathbb{C}_{\tilde{B}(x,2t)}} |b(z) - b(y)| |f(y)| \left(\int_{|x-y|}^{\infty} s^{-\delta-1} ds \right) dy \\ &\leq C \int_t^{\infty} s^{-n-1} \left(\int_{\{y \in \Omega: 2t \leq |x-y| \leq s\}} |b(z) - b(y)| |f(y)| dy \right) ds \\ &\leq C \int_t^{\infty} s^{-n-1} \left(\int_{\{y \in \Omega: 2t \leq |x-y| \leq s\}} |b(y) - b_{\tilde{B}(x,s)}| |f(y)| dy \right) ds \\ &\quad + C \int_t^{\infty} s^{-n-1} \left(\int_{\{y \in \Omega: 2t \leq |x-y| \leq s\}} |b(z) - b_{\tilde{B}(x,s)}| |f(y)| dy \right) ds = I_1 + I_2. \end{aligned}$$

From Hölder’s inequality and Theorem 3 we obtain

$$\begin{aligned}
 I_1 &= \int_t^\infty s^{-n-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |b(y) - b_{\tilde{B}(x,s)}| |f(y)| dy \right) ds \\
 &\leq C \|b\|_* \int_t^\infty s^{-n-1} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,s))} ds.
 \end{aligned}$$

To estimate I_2 , by Lemma 2 we get

$$\begin{aligned}
 I_2 &= \int_t^\infty s^{-n-1} |b(z) - b_{\tilde{B}(x,s)}| \left(\int_{\{y \in \Omega : 2t \leq |x-y| \leq s\}} |f(y)| dy \right) ds \\
 &\leq CM_b \chi_{\tilde{B}(x,t)}(z) \int_t^\infty s^{-n-1} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,s))} ds \\
 &\quad + C \|b\|_* \int_t^\infty s^{-n-1} \ln \frac{s}{t} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,s))} ds.
 \end{aligned}$$

Then from Theorem 8 we obtain

$$\begin{aligned}
 \|M_b f_2\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} &\leq \|I_1\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} + \|I_2\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} \\
 &\leq C \|b\|_* \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t} \right) \frac{\|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} ds}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))} s}. \tag{11}
 \end{aligned}$$

Therefore from (10) and (11) we get (7). \square

THEOREM 10. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption (1), $\varphi \in A_{p(\cdot)}(\Omega)$ and $\theta_1(t)$ and $\theta_2(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists a positive number δ such that $\theta_1(t) = \theta_1 = const$, $\theta_2(t) = \theta_2 = const$ for $x > \delta$, and $1 < \theta_1^- \leq \tilde{\theta}_1(t) \leq \theta_2(t) \leq \theta_2^+ < \infty$ for a.a. $x \in (0, \infty)$. If the condition*

$$\sup_{x \in \Omega, t > 0} \int_0^t \omega_2(x, \xi)^{\theta_2(\xi)} \left(\int_t^\infty \left(1 + \ln \frac{r}{t} \right)^{[\tilde{\theta}_1(\xi)]'} \frac{dr}{[r\omega_1(x, r)]^{[\tilde{\theta}_1(\xi)]'}} \right)^{\frac{\theta_2(\xi)}{[\tilde{\theta}_1(\xi)]'}} d\xi < \infty \tag{12}$$

holds, where $\tilde{\theta}_1(\xi) = \inf_{s \in (\xi, \infty)} \theta_1(s)$, then the operator M_b is bounded from

$GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$ to $GM^{p(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}(\Omega)$.

Proof. Let $f \in GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$. By using Theorem 9 we obtain

$$\begin{aligned}
 \|M_b f\|_{GM^{p(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}} &= \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))}} \|M_b f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\
 &\leq C \|b\|_* \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))}} \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))} \int_t^\infty \left(1 + \ln \frac{s}{t} \right) \frac{\|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} ds}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))} s} \right\|_{L^{\theta_2(\cdot)}(0, \infty)}.
 \end{aligned}$$

By choosing $v(r) = \omega_2(x, r)$, $g(r) = \omega_1(x, r) \|\varphi\|_{L^p(\cdot)(\tilde{B}(x,r))}^{-1} \|f\|_{L^p(\cdot)(\tilde{B}(x,r))}$ and $u(r) = (1 + \ln \frac{r}{t}) \omega_1(x, r)^{-1} r^{-1}$ in Lemma 1, we get

$$\begin{aligned} \|M_b f\|_{GM^{p(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}} &\leq C \|b\|_* \sup_{x \in \Omega} \left\| \frac{\omega_1(x, r)}{\|\varphi\|_{L^p(\cdot)(\tilde{B}(x,t))}} \|f\|_{L^p(\cdot)(\tilde{B}(x,r))} \right\|_{L^{\theta_1(\cdot)}(0, \infty)} \\ &= C \|b\|_* \|f\|_{GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}}. \quad \square \end{aligned}$$

4. Commutators of generalized fractional integral operators in $GM^{p(\cdot), \theta(\cdot), \omega(\cdot), \varphi}(\Omega)$

In this section we consider the operators M_ρ , I_ρ and their commutators.

For a measurable function $\rho : (0, \infty) \rightarrow (0, \infty)$ the generalized fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ are defined by

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{\tilde{B}(x,t)} |f(y)| dy, \quad I_\rho f(x) = \int_\Omega \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy$$

for any suitable function f on \mathbb{R}^n . If $\rho(t) = t^\alpha$, then $M^\alpha = M_{t^\alpha}$ is the fractional maximal operator and $I^\alpha = I_{t^\alpha}$ is the Riesz potential.

We assume that

$$\int_1^\infty \frac{\rho(t)}{t^n} \frac{dt}{t} < \infty, \tag{13}$$

therefore generalized fractional integral operator I_ρ is well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n \setminus B(0,1)}(x)}{|x|^{2n}}.$$

Furthermore, we also assume that ρ satisfies the growth condition: there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r/2 < t \leq 3r/2} \frac{\rho(t)}{t^n} \leq C_1 \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t^n} \frac{dt}{t}, \quad r > 0. \tag{14}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C > 0$ such that

$$\frac{1}{C_1} \frac{\rho(t)}{t^n} \leq \frac{\rho(r)}{r^n} \leq C_1 \frac{\rho(t)}{t^n}, \quad r > 0. \tag{15}$$

whenever r and t satisfy $r, t > 0$ and $\frac{1}{2}t \leq r \leq 2t$. In the sequel for the generalized fractional integral operator I_ρ we always assume that ρ satisfies the condition (14).

Following theorem was proved in [26].

THEOREM 11. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{\text{log}}(\Omega)$, $1 < p(x) < q(x) < \infty$, $\varphi \in A_{p(\cdot), q(\cdot)}(\Omega)$, the function ρ satisfy the conditions (13)–(15). If the condition*

$$\rho(r) \leq Cr^{\eta_p(x,r) - \eta_q(x,r)} \tag{16}$$

is satisfied, then

$$\|I_\rho f\|_{L_\varphi^{q(\cdot)}(\tilde{B}(x,t))} \leq C \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}, \tag{17}$$

where C does not depend on f , x and t .

Proof. We represent f as in (8) and have

$$I_\rho f(x) = I_\rho f_1(x) + I_\rho f_2(x).$$

First we estimate $|I_\rho f_1(x)|$. By using Hölder’s inequality we have

$$\begin{aligned} |I_\rho f_1(x)| &= \int_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy \\ &\leq \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x, 2^{j+1}t) \setminus \tilde{B}(x, 2^j t))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x, 2^{j+1}t) \setminus \tilde{B}(x, 2^j t))} \\ &\leq C \sum_{j=-\infty}^{-1} \frac{\rho(2^j t) \eta_p(x,r) - \eta_q(x,r)}{(2^j t)^n} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x, 2^{j+1}t) \setminus \tilde{B}(x, 2^j t))} \\ &\quad \times \|\varphi^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x, 2^{j+1}t) \setminus \tilde{B}(x, 2^j t))} \\ &\leq C \sum_{j=-\infty}^{-1} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x, 2^{j+1}t) \setminus \tilde{B}(x, 2^j t))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, 2^{j+1}t) \setminus \tilde{B}(x, 2^j t))}^{-1} \\ &\leq C \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))}^{-1}. \end{aligned} \tag{18}$$

Then

$$\|I_\rho f_1\|_{L_\varphi^{q(\cdot)}(\tilde{B}(x,t))} \leq C \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))},$$

where the constant C is independent of f . Taking into account that, we get

$$\|I_\rho f_1\|_{L_\varphi^{q(\cdot)}(\tilde{B}(x,t))} \leq C \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \tag{19}$$

When $|x-z| \leq t$, $|z-y| \geq 2t$, we have $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$, and therefore we get

$$|I_\rho f_2(z)| \leq \int_{\Omega \setminus \tilde{B}(x, 2t)} \frac{\rho(|z-y|)}{|z-y|^n} |f(y)| dy \leq C \int_{\Omega \setminus \tilde{B}(x, 2t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy.$$

By estimating the integral $\int_{\Omega \setminus \tilde{B}(x, 2t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy$ we have

$$\begin{aligned} \int_{\Omega \setminus \tilde{B}(x, 2t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy &\leq C \int_{\Omega \setminus \tilde{B}(x, 2t)} |f(y)| \left(\int_{|x-y|}^{\infty} \frac{\rho(s)}{s^{n+1}} ds \right) dy \\ &\leq C \int_{2t}^{\infty} \frac{\rho(s)}{s^n} \left(\int_{\{y \in \mathbb{R}^n: 2t \leq |x-y| \leq s\}} |f(y)| dy \right) \frac{ds}{s} \\ &\leq C \int_2^{\infty} \frac{\rho(s)}{s^n} \|\chi_{\tilde{B}(x, s)} \varphi^{-1}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s} \\ &\leq C \int_t^{\infty} \frac{\rho(s)}{s^{\eta_p(x, r) - \eta_q(x, r)}} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s}. \end{aligned}$$

Hence we get

$$\|I_{\rho} f_2\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \leq C \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \int_t^{\infty} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s}.$$

Finally we obtain

$$\|I_{\rho} f\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \leq C \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \int_t^{\infty} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s}, \tag{20}$$

which together with (19) yields (17). \square

THEOREM 12. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}^{log}(\Omega)$, $1 < p(x) < q(x) < \infty$, $\varphi \in A_{p(\cdot), q(\cdot)}(\Omega)$, the function ρ satisfy the conditions (13)–(15). Let the condition (16) be fulfilled and $\theta_1(t)$ and $\theta_2(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists δ positive number a such that $\theta_1(t) = \theta_1 = const$, $\theta_2(t) = \theta_2 = const$ for $x > \delta$, and $1 < \theta_1^- \leq \tilde{\theta}_1(t) \leq \theta_2(t) \leq \theta_2^+ < \infty$ for a.a. $x \in (0, \infty)$. If the condition*

$$\sup_{x \in \Omega, t > 0} \int_0^t \omega_2(x, \xi)^{\theta_2(\xi)} \left(\int_t^{\infty} \left(\frac{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, r))}}{r \omega_1(x, r) \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, r))}} \right)^{[\tilde{\theta}_1(\xi)]'} dr \right)^{\frac{\theta_2(\xi)}{[\tilde{\theta}_1(\xi)]'}} d\xi < \infty \tag{21}$$

holds, where $\tilde{\theta}_1(\xi) = \inf_{s \in (\xi, \infty)} \theta_1(s)$, then the operator I_{ρ} is bounded from

$$GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega) \text{ to } GM^{q(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}(\Omega).$$

Proof. Let $f \in GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$. Hence, by choosing $v(r) = \omega_2(x, r)$, $g(r) = \frac{\omega_1(x, r) \|f\|_{L^{p(\cdot)}(\tilde{B}(x, r))}}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, r))}}$, $u(r) = \frac{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, r))}}{\omega_1(x, r) r \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, r))}}$ in Lemma 1, and using Theorem

11 we obtain

$$\begin{aligned} \|I_\rho f\|_{GM^{q(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}} &= \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, r))}} \|I_\rho f\|_{L_\varphi^{q(\cdot)}(\tilde{B}(x, r))} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, r))}} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \int_t^\infty \frac{\|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x, s))} ds}{\|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))} s} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \sup_{x \in \Omega} \left\| \frac{\omega_1(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, r))}} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x, r))} \right\|_{L^{\theta_1(\cdot)}(0, \infty)} = C \|f\|_{GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}}. \quad \square \end{aligned}$$

COROLLARY 2. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $1 < p(x) < q(x) < \infty$, $\varphi \in A_{p(\cdot), q(\cdot)}(\Omega)$, the function ρ satisfy the conditions (13)–(15). Let the condition (16) be fulfilled and $\theta_1(t)$ and $\theta_2(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists δ positive number a such that $\theta_1(t) = \theta_1 = const$, $\theta_2(t) = \theta_2 = const$ for $x > \delta$, and $1 < \theta_1^- \leq \tilde{\theta}_1(t) \leq \theta_2(t) \leq \theta_2^+ < \infty$ for a.a. $x \in (0, \infty)$. If the condition (21) holds, then the operator M_ρ is bounded from $GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$ to $GM^{q(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}(\Omega)$.*

From the inequality $M_\rho f(x) \leq C(I_\rho)|f|(x)$, we get the following corollary.

COROLLARY 3. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $1 < p(x) < q(x) < \infty$, $\varphi \in A_{p(\cdot), q(\cdot)}(\Omega)$, the function ρ satisfy the conditions (13)–(15). If the condition (16) is fulfilled, then the operators M_ρ and I_ρ are bounded from $L_\varphi^{p(\cdot)}(\Omega)$ to $L_\varphi^{q(\cdot)}(\Omega)$.*

Now we consider commutators of the generalized fractional integral operator defined by

$$[b, I_\rho]f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} (b(x) - b(y)) f(y) dy.$$

Given a measurable function b the operator $|b, I_\rho|$ is defined by

$$|b, I_\rho|f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} |b(x) - b(y)| |f(y)| dy.$$

THEOREM 13. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $1 < p(x) < q(x) < \infty$, $\varphi \in A_{p(\cdot)}(\Omega)$, $\varphi \in A_{p(\cdot), q(\cdot)}(\Omega)$, the function ρ satisfy the conditions (13)–(15). If the condition (16) is fulfilled, $b \in BMO(\Omega)$, then*

$$\begin{aligned} \|[b, I_\rho]f\|_{L_\varphi^{q(\cdot)}(\tilde{B}(x, t))} &\leq C \|b\|_* \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \\ &\quad \times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x, s))} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s}, \end{aligned} \quad (22)$$

where C does not depend on f , x and t .

Proof. We represent f as (8) and have

$$|b, I_\rho|f(x) \leq |b, I_\rho|f_1(x) + |b, I_\rho|f_2(x).$$

First we estimate $|b, I_\rho|f_1(x)$. By using Hölder's inequality we have

$$\begin{aligned} |b, I_\rho|f_1(x) &= \int_{\tilde{B}(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |b(y) - b(x)| |f(y)| dy \\ &\leq |b(x) - b_{\tilde{B}(x,t)}| \int_{\tilde{B}(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy \\ &\quad + \int_{\tilde{B}(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |b(y) - b_{\tilde{B}(x,t)}| |f(y)| dy = F_1(x) + F_2(x). \end{aligned}$$

By using inequality (18) we get

$$\begin{aligned} F_1(x) &= |b(x) - b_{\tilde{B}(x,t)}| \int_{\tilde{B}(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy \leq CM_b \chi_{\tilde{B}(x,t)}(x) |I_\rho f_1(x)| \\ &\leq CM_b \chi_{\tilde{B}(x,t)}(x) \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} \|\varphi\|_{L_{q(\cdot)}(\tilde{B}(x,t))}^{-1}. \end{aligned} \tag{23}$$

With the help of Hölder's inequality we obtain

$$\begin{aligned} F_2(x) &= \int_{\tilde{B}(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |b(y) - b_{\tilde{B}(x,t)}| |f(y)| dy \\ &\leq \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} \int_{\tilde{B}(x,2^{j+1}t) \setminus \tilde{B}(x,2^j t)} |b(y) - b_{\tilde{B}(x,t)}| |f(y)| dy \\ &\leq C \|b\|_* \sum_{j=-\infty}^{-1} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,2^{j+1}t) \setminus \tilde{B}(x,2^j t))} \|\varphi\|_{L_q(\tilde{B}(x,2^{j+1}t) \setminus \tilde{B}(x,2^j t))}^{-1} \\ &\leq C \|b\|_* \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} \|\varphi\|_{L_{q(\cdot)}(\tilde{B}(x,t))}^{-1}. \end{aligned} \tag{24}$$

Then from inequalities (23), (24) and Corollary 8 we get

$$\| |b, I_\rho|f_1 \|_{L_\varphi^{q(\cdot)}(\tilde{B}(x,t))} \leq C \|b\|_* \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,2t))},$$

where the constant C is independent of f . Taking into account that, we get

$$\begin{aligned} \| |b, I_\rho|f_1 \|_{L_\varphi^{q(\cdot)}(B(x,t))} &\leq C \|b\|_{BMO} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \\ &\quad \times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L_\varphi^{p(\cdot)}(B(x,s))} ds}{\|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,s))} s}. \end{aligned} \tag{25}$$

When $|x - z| \leq t$, $|z - y| \geq 2t$, we have $\frac{1}{2}|z - y| \leq |x - y| \leq \frac{3}{2}|z - y|$. Therefore we get

$$\begin{aligned} |b, I_\rho|f_2(z) &\leq \int_{\Omega \setminus B(x, 2t)} \frac{\rho(|z - y|)}{|z - y|^n} |b(y) - b(z)| |f(y)| dy \\ &\leq C \int_{\Omega \setminus B(x, 2t)} \frac{\rho(|x - y|)}{|x - y|^n} |b(y) - b(z)| |f(y)| dy. \end{aligned}$$

We obtain

$$\begin{aligned} &\int_{\Omega \setminus \tilde{B}(x, 2t)} \frac{\rho(|x - y|)}{|x - y|^n} |b(y) - b(z)| |f(y)| dy \\ &\leq C \int_{2t}^\infty \frac{\rho(s)}{s^n} \left(\int_{\{y \in \Omega: 2t \leq |x - y| \leq s\}} |b(y) - b_{\tilde{B}(x, t)}| |f(y)| dy \right) \frac{ds}{s} \\ &\quad + C |b(z) - b_{\tilde{B}(x, t)}| \int_{2t}^\infty \frac{\rho(s)}{s^n} \left(\int_{\{y \in \Omega: 2t \leq |x - y| \leq s\}} |f(y)| dy \right) \frac{ds}{s} = J_1 + J_2. \end{aligned}$$

To estimate J_1 :

$$\begin{aligned} J_1 &= C \int_{2t}^\infty \frac{\rho(s)}{s^n} \left(\int_{\{y \in \Omega: 2t \leq |x - y| \leq s\}} |b(y) - b_{\tilde{B}(x, t)}| |f(y)| dy \right) \frac{ds}{s} \\ &\leq C \|b\|_* \int_t^\infty \frac{\rho(s)}{s^n} \|\varphi^{-1}\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s} \\ &\quad + C \|b\|_* \int_t^\infty \frac{\rho(s)}{s^n} \ln \frac{s}{t} \|\varphi^{-1}\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s} \\ &\leq C \|b\|_* \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\rho(s)}{s^{n_{p(x, r)} - n_{q(x, r)}}} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s}. \end{aligned} \tag{26}$$

To estimate J_2 :

$$\begin{aligned} J_2 &= C |b(z) - b_{\tilde{B}(x, t)}| \int_{2t}^\infty \frac{\rho(s)}{s^n} \left(\int_{\{y \in \Omega: 2t \leq |x - y| \leq s\}} |f(y)| dy \right) \frac{ds}{s} \\ &\leq C |B(x, t)|^{-1} \int_{\tilde{B}(x, t)} |b(z) - b(y)| dy \int_t^\infty \frac{\rho(s)}{s^n} \|\varphi^{-1}\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s} \\ &\leq CM_b \chi_{\tilde{B}(x, t)}(z) \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\rho(s)}{s^{n_{p(x, r)} - n_{q(x, r)}}} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s}, \end{aligned} \tag{27}$$

where C does not depend on x, t . Then by Corollary 8 and (16), (26), (27) we have

$$\begin{aligned} &\| |b, I_\rho|f_2 \|_{L^{q(\cdot)}(\tilde{B}(x, t))} \\ &\leq \|J_1\|_{L^{q(\cdot)}(\tilde{B}(x, t))} + \|J_2\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \\ &\leq C \|b\|_* \|\chi_{\tilde{B}(x, t)}\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s} \\ &\quad + C \|M_b \chi_{\tilde{B}(x, t)}\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s} \\ &\leq C \|b\|_* \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s}. \end{aligned}$$

Hence

$$\begin{aligned} \| |b, I_\rho| f_2 \|_{L_{q, \omega_2}(B(x,t))} &\leq C \|b\|_* \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \\ &\quad \times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} \frac{ds}{s}, \end{aligned}$$

which together with (25) yields (22). \square

In the following theorem we prove the boundedness of commutators of the fractional integral operator $|b, I_\rho|$ from the spaces $GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$ to the spaces $GM^{q(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}(\Omega)$.

THEOREM 14. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $1 < p(x) < q(x) < \infty$, $\varphi \in A_{p(\cdot)}(\Omega)$, $\varphi \in A_{p(\cdot), q(\cdot)}(\Omega)$, the function ρ satisfy the conditions (13)–(15). Let the condition (16) be fulfilled, $b \in BMO(\mathbb{R}^n)$ and $\theta_1(t)$ and $\theta_2(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists a positive number δ such that $\theta_1(t) = \theta_1 = const$, $\theta_2(t) = \theta_2 = const$ for $x > \delta$, and $1 < \theta_1^- \leq \tilde{\theta}_1(t) \leq \theta_2(t) \leq \theta_2^+ < \infty$ for a.a. $x \in (0, \infty)$. If the condition*

$$\begin{aligned} &\sup_{x \in \Omega, t > 0} \int_0^t \omega_2(x, \xi)^{\theta_2(\xi)} \\ &\times \left(\int_t^\infty \left(1 + \ln \frac{r}{t}\right)^{[\tilde{\theta}_1(\xi)]'} \left(\frac{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))}}{r \omega_1(x, r) \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,r))}} \right)^{[\tilde{\theta}_1(\xi)]'} dr \right)^{\frac{\theta_2(\xi)}{[\tilde{\theta}_1(\xi)]'}} d\xi < \infty, \end{aligned} \quad (28)$$

holds, where $\tilde{\theta}_1(\xi) = \inf_{s \in (\xi, \infty)} \theta_1(s)$, then the operator $|b, I_\rho|$ is bounded from $GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$ to $GM^{q(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}(\Omega)$.

Proof. Let $f \in GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$. Hence, by choosing $v(r) = \omega_2(x, r)$, $g(r) = \frac{\omega_1(x, r) \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))}}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))}}$, $u(r) = \left(1 + \ln \frac{r}{t}\right) \frac{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))}}{\omega_1(x, r) r \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,r))}}$ in Lemma 1, and using Theorem 13 we obtain

$$\begin{aligned} &\| |b, I_\rho| f \|_{GM^{q(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}} \\ &= \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \| |b, I_\rho| f \|_{L_\varphi^{q(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \|b\|_* \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \frac{\|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} ds}{\|\varphi\|_{L^{q(\cdot)}(\tilde{B}(x,s))} s} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \|b\|_* \sup_{x \in \Omega} \left\| \frac{\omega_1(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta_1(\cdot)}(0, \infty)} \\ &= C \|b\|_* \|f\|_{GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}}. \quad \square \end{aligned}$$

COROLLARY 4. Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $1 < p(x) < q(x) < \infty$, $\varphi \in A_{p(\cdot)}(\Omega)$, $\varphi \in A_{p(\cdot),q(\cdot)}(\Omega)$, the function ρ satisfy the conditions (13)–(15), the condition (16) be fulfilled, $b \in BMO(\mathbb{R}^n)$. Then the operator $|b, I_\rho|$ is bounded from $L_\varphi^{p(\cdot)}(\Omega)$ to $L_\varphi^{q(\cdot)}(\Omega)$.

5. Commutators of singular integral operators in $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$

In this section we prove the boundedness of singular integral operators and their commutators in the spaces $GM^{p(\cdot),\theta(\cdot),\omega(\cdot),\varphi}(\Omega)$.

Calderón-Zygmund type singular operator is defined by

$$Tf(x) = \int_\Omega K(x,y)f(y) dy,$$

where $K(x,y)$ is a “standard singular kernel”, that is, a function continuous on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying following properties:

$$|K(x,y)| \leq C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C \frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C \frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|x-\xi|.$$

Let

$$T_*f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|,$$

be the maximal singular operator, where $T_\varepsilon f(x)$ is the usual truncation

$$T_\varepsilon f(x) = \int_{|x-y| \geq \varepsilon} K(x,y)f(y)dy.$$

Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [9] states that the commutator operator $[b, T]f = T(bf) - bTf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

The following theorem was proved in [25].

THEOREM 15. Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$ and $\varphi \in A_{p(\cdot)}(\Omega)$. Then

$$\|Tf\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \frac{\|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))}}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))}} \frac{ds}{s} \tag{29}$$

for every $f \in L_\varphi^{p(\cdot)}(\mathbb{R}^n)$, where C does not depend on f, x and t .

The following theorem in the case of $\varphi = 1$, constant $p(x)$ and $\theta(t)$ was proved in [19] (see also [20]).

THEOREM 16. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption (1), $\varphi \in A_{p(\cdot)}(\Omega)$, $\theta_1(t)$ and $\theta_2(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists δ positive number a such that $\theta_1(t) = \theta_1 = const$, $\theta_2(t) = \theta_2 = const$ for $x > \delta$, and $1 < \theta_1^- \leq \tilde{\theta}_1(t) \leq \theta_2(t) \leq \theta_2^+ < \infty$ for a.a. $x \in (0, \infty)$, $(\omega_1(x, r), \omega_2(x, r))$ satisfy the condition (6).*

Then the operators T and T_ are bounded from $GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$ to $GM^{p(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}(\Omega)$.*

Proof. Let $f \in GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$. Hence, by choosing $v(r) = \omega_2(x, r)$, $g(r) = \omega_1(x, r) \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))}$, $u(r) = \omega_1(x, r)^{-1} r^{-1}$ in Lemma 1, and using Theorem 15 we obtain

$$\begin{aligned} & \|Tf\|_{G_{\mathcal{M}^{p(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}}} \\ &= \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|Tf\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))} \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \sup_{x \in \Omega} \left\| \frac{\omega_1(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,r))} \right\|_{L^{\theta_1(\cdot)}(0, \infty)} \\ &= C \|f\|_{G_{\mathcal{M}^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}}} \end{aligned}$$

The boundedness of the operator T_* follows then from the known ([47], p. 34) pointwise estimate

$$T_*f(x) \leq c[M(Tf)(x) + Mf(x)],$$

and the corresponding theorems on the maximal operator. \square

The following theorem was proved in [25].

THEOREM 17. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $b \in BMO(\mathbb{R}^n)$ and $\varphi \in A_{p(\cdot)}(\Omega)$. Then*

$$\|[b, T]f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|b\|_* \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L_\varphi^{p(\cdot)}(\tilde{B}(x,s))}}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x,s))}} \frac{ds}{s} \quad (30)$$

for every $f \in L_\varphi^{p(\cdot)}(\mathbb{R}^n)$, where C does not depend on f, x and t .

THEOREM 18. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $b \in BMO(\mathbb{R}^n)$, $\varphi \in A_{p(\cdot)}(\Omega)$, and $\theta_1(t)$ and $\theta_2(t)$ be measurable functions on $(0, \infty)$. Suppose that there exists δ positive number a such that $\theta_1(t) = \theta_1 = const$, $\theta_2(t) = \theta_2 = const$ for $x > \delta$, and $1 < \theta_1^- \leq \tilde{\theta}_1(t) \leq \theta_2(t) \leq \theta_2^+ < \infty$ for a.a. $x \in (0, \infty)$, $(\omega_1(x, r), \omega_2(x, r))$ satisfy the condition (12). Then the operator $[b, T]$ is bounded from $GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\mathbb{R}^n)$ into $GM^{p(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}(\mathbb{R}^n)$.*

Proof. Let $f \in GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}(\Omega)$. Hence, by choosing $v(r) = \omega_2(x, r)$, $g(r) = \omega_1(x, r) \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, r))}^{-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, r))}$, $u(r) = (1 + \ln \frac{r}{t}) \omega_1(x, r)^{-1} r^{-1}$ in Lemma 1, and using Theorem 17 we obtain

$$\begin{aligned} & \| [b, T]f \|_{GM^{p(\cdot), \theta_2(\cdot), \omega_2(\cdot), \varphi}} \\ &= \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \| [b, T]f \|_{L^{p(\cdot)}(\tilde{B}(x, r))} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \|b\|_* \sup_{x \in \Omega} \left\| \frac{\omega_2(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L^{p(\cdot)}(\tilde{B}(x, s))} ds}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, s))} s} \right\|_{L^{\theta_2(\cdot)}(0, \infty)} \\ &\leq C \|b\|_* \sup_{x \in \Omega} \left\| \frac{\omega_1(x, r)}{\|\varphi\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|f\|_{L^{p(\cdot)}(\tilde{B}(x, r))} \right\|_{L^{\theta_1(\cdot)}(0, \infty)} \\ &= C \|b\|_* \|f\|_{GM^{p(\cdot), \theta_1(\cdot), \omega_1(\cdot), \varphi}}. \quad \square \end{aligned}$$

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