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SHARP INEQUALITIES FOR HERMITIAN–TOEPLITZ DETERMINANTS OF STRONGLY OZAKI CLOSE–TO–CONVEX FUNCTIONS

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Abstract. Sharp lower and upper bounds are found of the second and third order Hermitian-Toeplitz determinants for the class $\mathscr{F}_O(\lambda,\beta)$ of strongly Ozaki close-to-convex functions.

1. Introduction

Let $\mathscr{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 := 0$. Then $f \in \mathscr{A}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathscr{S}$ denote the subclass of all univalent (i.e., one-to-one) functions in $\mathscr{A}$.

Denote by $\mathscr{S}^*$ the subclass of $\mathscr{S}$ consisting of starlike functions, i.e., functions $f \in \mathscr{S}$ which map $\mathbb{D}$ onto a set which is star-shaped with respect to the origin. Then it is well-known that a function $f \in \mathscr{A}$ belongs to $\mathscr{S}^*$ if, and only if,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

Next denote by $\mathscr{K}$ the subclass of $\mathscr{A}$ consisting of functions which are close-to-convex, i.e., a function $f \in \mathscr{A}$ belongs to $\mathscr{K}$ if, and only if, there exist $\delta \in (-\pi/2, \pi/2)$ and $g \in \mathscr{S}^*$ such that for

$$\operatorname{Re} \left\{ e^{i\delta} \frac{zf'(z)}{g(z)} \right\} > 0, \quad z \in \mathbb{D},$$

and it is well known that $\mathscr{K} \subset \mathscr{S}$.

The class $\mathscr{C}(\alpha)$ for $\alpha \in [0,1)$, of convex functions of order α consisting of functions $f \in \mathscr{A}$ satisfying

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad z \in \mathbb{D},$$

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is well-known (Robertson [9]), and has been widely studied, and it is also well known that $\mathscr{C}(\alpha) \subset \mathscr{S}$ for every $\alpha \in [0,1)$. When $\alpha = 0$, we obtain the class $\mathscr{C} := \mathscr{C}(0)$ of convex functions.

Little attention has been given to the case when $\alpha < 0$, but several authors have considered the class $\mathscr{C}(-1/2)$, consisting of functions satisfying

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in \mathbb{D},$$

who's members are known to be close-to-convex.

The class $\mathscr{F}_O(\lambda)$, defined for $\lambda \in [1/2, 1]$ by

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2} - \lambda, \quad z \in \mathbb{D}, \tag{1.2}$$

was formally introduced in [3], and is also known to be a subclass of the close-convex functions, noting that $\mathscr{F}_O(1/2) = \mathscr{C}$, and $\mathscr{F}_O(1) = \mathscr{C}(-1/2)$. More information concerning the coefficients of functions in $\mathscr{F}_O(\lambda)$ (called Ozaki close-to-convex functions), can also be found in [2] and [3].

In [3], the following notion of strongly Ozaki functions was introduced, and some basic properties obtained:

DEFINITION 1.1. Let $\lambda \in [1/2, 1]$ and $\beta \in (0,1]$. A function $f \in \mathscr{A}$ is called strongly Ozaki close-to-convex if, and only if,

$$\left|\arg\left(\frac{2\lambda - 1}{2\lambda + 1} + \frac{2}{2\lambda + 1}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}. \tag{1.3}$$

We denote this class of functions by $\mathscr{F}_O(\lambda, \beta)$, noting that when $\beta := 1$ this reduces to (1.2), and when $\lambda := 1/2$ we obtain the class $\mathscr{C}_\beta := \mathscr{F}_O(1/2, \beta)$ of strongly convex functions considered in [11] (cf. [6, Vol. I. p. 139]).

In recent decades a great many papers have appeared attempting to obtain sharp upper bounds for the modulus of Hankel determinants, (i.e., square matrices with constant entries along the reverse diagonal) whose elements are coefficients of functions in $\mathscr{A}$, and a selection of the more significant results concerning these problems can be found in [10].

In [1], a similar study of finding bounds for the determinants of Toeplitz matrices, i.e square matrices having constant entries along the diagonal was introduced, and in [2] some sharp bounds were obtained for functions in $\mathscr{F}_O(\lambda)$.

In 2020, Cudna, *et al.* [5] and Kowalczyk, *et al.* [7] started the study of the determinants of Hermitian-Toeplitz matrices $T_{q,n}(f)$ defined below, whose elements are coefficients of functions in $\mathscr{A}$, obtaining some sharp bounds for several subclasses of $\mathscr{S}$. More recently some sharp bounds for the determinants of Hermitian-Toeplitz matrices when $f \in \mathscr{F}_O(\lambda)$ have been obtained [2].

We remark that Hankel, Toeplitz and Hermitian-Toeplitz matrices all have applications in real world problems, and play an important role in functional analysis, applied mathematics as well as in physics and other technical sciences.

DEFINITION 1.2. Given $q,n \in \mathbb{N}$, the matrix $T_{q,n}(f)$ of a function $f \in \mathscr{A}$ of the form (1.1) is defined by

$$T_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \dots & a_n \end{bmatrix},$$

where $\overline{a}_k := \overline{a_k}$. In particular

$$\det T_{2,1}(f) = 1 - |a_2|^2$$

and

$$\det T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} = 1 + 2\operatorname{Re}\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2. \tag{1.4}$$

When a_n is real, then $T_{q,n}(f)$ is the Hermitian-Toeplitz matrix.

In [2], sharp bounds were obtained for $\det T_{2,1}(f)$ and $\det T_{3,1}(f)$ when $f \in \mathscr{F}_O(\lambda)$, and it is the purpose of this paper to extend these results to $\mathscr{F}_O(\lambda,\beta)$.

Denote by $\mathscr{P}$, the class of analytic functions $p : \mathbb{D} \to \mathbb{C}$ with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \tag{1.5}$$

In proving our results, we will use the following lemma (see e.g., [8, p. 166]).

LEMMA 1.3. *If $p \in \mathscr{P}$ and is of the form (1.5), then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta \tag{1.6}$$

for some $\zeta \in \overline{\mathbb{D}}$.

2. Main results

We first recall the following simple observation [7]. Given a compact subclass $\mathscr{F}$ of $\mathscr{A}$, let $A_2(\mathscr{F}) := \max\{|a_2| : f \in \mathscr{F}\}$. Thus if $f \in \mathscr{F}$,

$$\det T_{2,1}(f) = 1 - |a_2|^2,$$

and the result below is clear. Equality for the lower bound is attained by a function in $\mathscr{F}$ which is extremal for $A_2(\mathscr{F})$, and for the upper bound when f is the identity function.

THEOREM 2.1. ([7]) *Let $\mathscr{F}$ be a compact subclass of $\mathscr{A}$. If the identity is an element of $\mathscr{F}$, then*

$$1 - A_2^2(\mathscr{F}) \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

Next let $\lambda \in [1/2,1]$ and $\beta \in (0,1]$. From (2.5) below, it follows that $A_2(\mathscr{F}_O(\lambda,\beta)) = \beta(1+2\lambda)/2$ with extremal function f satisfying

$$1+\frac{zf''(z)}{f'(z)} = \left(\frac{1}{2}+\lambda\right)\left(\frac{1+z}{1-z}\right)^{\beta}+\frac{1}{2}-\lambda, \quad z\in\mathbb{D}, \tag{2.1}$$

and since the identity function belongs to the class $\mathscr{F}_O(\lambda,\beta)$, by Theorem 2.1 we have proved the following.

THEOREM 2.2. *Let $\lambda \in [1/2,1]$ and $\beta \in (0,1]$. If $f \in \mathscr{F}_O(\lambda,\beta)$, then*

$$1-\frac{1}{4}\beta^2(1+2\lambda)^2 \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

We next determine the sharp lower and upper bounds of $\det T_{3,1}(f)$, which require significant more analysis. We prove the following.

THEOREM 2.3. *Let $\lambda \in [1/2,1]$ and $\beta \in (0,1]$. If $f \in \mathscr{F}_O(\lambda,\beta)$, then*

$$\det T_{3,1}(f) \leqslant \begin{cases} 1, & (1+5\lambda+4\lambda^2)\beta^2 \leqslant 9, \\ 1+\dfrac{1}{18}\beta^2(1+2\lambda)^2\left[-9+(1+5\lambda+4\lambda^2)\beta^2\right], & \text{otherwise} \end{cases} \tag{2.2}$$

and

$$\det T_{3,1}(f) \geqslant \begin{cases} 1-\dfrac{\beta^2(1+2\lambda)^2\left[28+4(2\lambda-1)\beta+(2\lambda+1)^2\beta^2\right]}{16\left[-1+(-1+2\lambda)\beta+2(1+5\lambda+4\lambda^2)\beta^2\right]}, & (\beta,\lambda)\in\Omega, \\ 1+\dfrac{1}{18}\beta^2(1+2\lambda)^2\left[-9+(1+5\lambda+4\lambda^2)\beta^2\right], & \text{otherwise,} \end{cases} \tag{2.3}$$

where

$$\Omega := \left\{(\beta,\lambda)\in(0,1]\times[1/2,1] : 36+(2-4\lambda)\beta-8(1+5\lambda+4\lambda^2)\beta^2<0\right\}.$$

All inequalities are sharp.

Proof. From (1.3) it follows that

$$1+\frac{zf''(z)}{f'(z)} = \left(\frac{1}{2}+\lambda\right)p(z)^{\beta}+\frac{1}{2}-\lambda, \quad z\in\mathbb{D}, \tag{2.4}$$

for some $p\in\mathscr{P}$ of the form (1.5). Thus equating coefficients on both sides of the above equation we have

$$\begin{aligned} a_2 &= \frac{\beta}{4}\Big(1+2\lambda\Big)c_1, \\ a_3 &= \frac{\beta}{12}\Big(1+2\lambda\Big)\Big(c_2-\frac{1}{2}(1-2\beta-2\beta\lambda)c_1^2\Big). \end{aligned} \tag{2.5}$$

First note that both $\mathscr{F}_O(\lambda,\beta)$ and $\det T_{3,1}(f)$ are rotationally invariant, and so since $|c_1|\leqslant 2$, without loss in generality we can assume that $c:=c_1\in[0,2]$ ([4]). Thus using (1.4) and 1.6 we get

$$T_{3,1}(f)-1=\frac{\beta^2(1+2\lambda)^2}{576}\Phi(c^2,\mathrm{Re}(\zeta),|\zeta|), \tag{2.6}$$

for some $\zeta\in\overline{\mathbb{D}}$, where for $x\in[0,4]$, $s\in[-1,1]$ and $t\in[0,1]$,

$$\Phi(x,s,t):=2(1+5\lambda+4\lambda^2)\beta x^2-72x+\beta(2\lambda-1)x(4-x)s-(4-x)^2t^2.$$

Since $\beta(2\lambda-1)x(4-x)\geqslant 0$ for all $x\in[0,4]$, we see that

$$\begin{aligned}L(c^2,|\zeta|)=\Phi(c^2,-|\zeta|,|\zeta|)&\leqslant\Phi(c^2,\mathrm{Re}(\zeta),|\zeta|)\\&\leqslant\Phi(c^2,|\zeta|,|\zeta|)=R(c^2,|\zeta|),\quad c\in[0,2],\ \zeta\in\overline{\mathbb{D}},\end{aligned} \tag{2.7}$$

where L and R are defined on $[0,4]\times[0,1]$ by

$$L(x,y):=2(1+5\lambda+4\lambda^2)\beta x^2-72x-\beta(2\lambda-1)x(4-x)y-(4-x)^2y^2$$

and

$$R(x,y):=2(1+5\lambda+4\lambda^2)\beta x^2-72x+\beta(2\lambda-1)x(4-x)y-(4-x)^2y^2.$$

We consider two cases.

A. We show that (2.2) holds. To see that the function R has no maximum in $(0,4)\times(0,1)$, we argue as follows. Solving the system of equations $(\partial R)/(\partial x)=(\partial R)/(\partial y)=0$, we obtain solutions (x_1,y_1) and (x_2,y_2), where

$$x_1:=4,\quad y_1:=\frac{2(-9+2\beta^2+10\lambda\beta^2+8\lambda^2\beta^2)}{(2\lambda-1)\beta},$$

and

$$x_2:=\frac{16}{(1+2\lambda)^2\beta^2},\quad y_2:=\frac{2\beta(-1+2\lambda)}{-4+\beta^2+4\lambda\beta^2+4\lambda^2\beta^2}.$$

Clearly, $(x_1,y_1)\notin(0,4)\times(0,1)$. For $\beta=1$ and $\lambda=1/2$, we have $x_2=4$, so $(x_2,y_2)\notin(0,4)\times(0,1)$. When $\beta\neq 1$ or $\lambda\neq 1/2$, we have

$$\left(\frac{\partial^2 R}{\partial x^2}\frac{\partial^2 R}{\partial y^2}-\left(\frac{\partial^2 R}{\partial x\partial y}\right)^2\right)(x_2,y_2)=-\frac{144[-4+\beta^2(2\lambda+1)^2]^2}{\beta^2(1+2\lambda)^2}<0.$$

Thus (x_2,y_2) is either a saddle point in $(0,4)\times(0,1)$ of R or $(x_2,y_2)\notin(0,4)\times(0,1)$, and so the maximum of R occurs on the boundary of $[0,4]\times[0,1]$.

Next note that

$$R(4,y)=32\left[-9+(1+5\lambda+4\lambda^2)\beta^2\right],\quad y\in[0,1], \tag{2.8}$$

$$R(0,y) = -16y^2 \leqslant 0, \quad y \in [0,1] \tag{2.9}$$

and

$$\begin{aligned} R(x,0) &= 2x(-36 + (1+5\lambda+4\lambda^2)\beta^2 x) \\ &\leqslant \max\left\{0, 32\left[-9 + (1+5\lambda+4\lambda^2)\beta^2\right]\right\}, \quad x \in [0,4]. \end{aligned} \tag{2.10}$$

(Inequality (2.10) follows from the fact that $[0,4] \ni x \mapsto R(x,0)$ is a convex function).

Further

$$R(x,1) = -16 + 2b_1 x + b_2 x^2 =: g(x), \quad x \in [0,4], \tag{2.11}$$

where

$$b_1 := -32 + 2(2\lambda - 1)\beta, \quad b_2 := -1 + (1-2\lambda)\beta + 2(1+5\lambda+4\lambda^2)\beta^2,$$

and clearly $b_1 < 0$.

We now consider two further cases.

A(a) When $b_2 \leqslant 0$, the function g is decreasing, and so

$$g(x) \leqslant g(0) = -16 < 0, \quad x \in [0,4].$$

Therefore from (2.8), (2.9), (2.10) and (2.11) we obtain

$$R(x,y) \leqslant \max\{0, 32[-9 + (1+5\lambda+4\lambda^2)\beta^2]\}, \quad (x,y) \in [0,4] \times [0,1]. \tag{2.12}$$

and from (2.6), (2.7) and (2.12), we obtain the inequality (2.2).

A(b) When $b_2 > 0$, the function g is convex, and since $g(0) < 0$ we have

$$g(x) \leqslant \max\{g(0), g(4)\} \leqslant \max\{0, 32[-9 + (1+5\lambda+4\lambda^2)\beta^2]\}, \quad x \in [0,4].$$

Thus from (2.11) a similar argument as in A(a) gives inequality (2.2) again.

To see that (2.2) is sharp, we note that when $(1+5\lambda+4\lambda^2)\beta^2 \leqslant 9$, we choose the identity function, and when $(1+5\lambda+4\lambda^2)\beta^2 > 9$, we choose f given by (2.1) with coefficients

$$a_2 = \frac{1}{2}\beta(1+2\lambda) \quad \text{and} \quad a_3 = \frac{1}{3}\beta^2(1+2\lambda)(1+\lambda).$$

B. We now show that (2.3) holds. First note that

$$L(x,y) \geqslant L(x,1) = -16 + 2d_1 x + d_2 x^2 =: h(x), \quad x \in [0,4], \tag{2.13}$$

where

$$d_1 := -32 + 2(1-2\lambda)\beta, \quad d_2 := -1 + (-1+2\lambda)\beta + 2(1+5\lambda+4\lambda^2)\beta^2.$$

We again distinguish two cases.

B(a) When $d_2 \leqslant 0$, we easily obtain

$$h(x) \geqslant h(4) = 32\left[-9 + (1+5\lambda+4\lambda^2)\beta^2\right], \quad x \in [0,4]. \tag{2.14}$$

B(b) When $d_2 > 0$, h has a unique local minimum at $x = -d_1/d_2 =: \tau$, and clearly $\tau > 0$. Moreover $\tau < 4$ only when

$$d_1 + 4d_2 = -36 + (-2+4\lambda)\beta + 8(1+5\lambda+4\lambda^2)\beta^2 > 0,$$

i.e., when $(\beta,\lambda) \in \Omega$.

Thus when $(\beta,\lambda) \in \Omega$, we obtain

$$h(x) \geqslant h(\tau) = h\left(-\frac{d_1}{d_2}\right) = \frac{-36[28+(-4+8\lambda)\beta+(2\lambda+1)^2\beta^2]}{-1+(-1+2\lambda)\beta+2(1+5\lambda+4\lambda^2)\beta^2}. \tag{2.15}$$

When $(\beta,\lambda) \in ((0,1]\times[1/2,1]) \setminus \Omega$, then $\tau \geqslant 4$, and since $d_2 > 0$, we obtain (2.14) again. Thus when $x \in [0,4]$

$$h(x) \geqslant \begin{cases} \dfrac{-36\left[28+(-4+8\lambda)\beta+(2\lambda+1)^2\beta^2\right]}{-1+(-1+2\lambda)\beta+2(1+5\lambda+4\lambda^2)\beta^2}, & \text{when } (\beta,\lambda)\in\Omega, \\ 32[-9+(1+5\lambda+4\lambda^2)\beta^2], & \text{otherwise,} \end{cases} \tag{2.16}$$

and from (2.6), (2.7), (2.13) and (2.16) we obtain (2.3).

It remains to show that (2.3) is sharp. When $(\beta,\lambda) \in ((0,1]\times[1/2,1]) \setminus \Omega$, it is clear that equality in (2.3) holds for the function f defined by (2.1).

When $(\beta,\lambda) \in \Omega$, let f be defined by (2.4) with

$$p(z) := \frac{1-z^2}{1-bz+z^2}, \quad z \in \mathbb{D}$$

and

$$b := \sqrt{\frac{32+(-2+4\lambda)\beta}{-1+(-1+2\lambda)\beta+2(1+5\lambda+4\lambda^2)\beta^2}}. \tag{2.17}$$

Then $f \in \mathscr{F}_O(,\lambda,\beta)$ having

$$a_2 = \frac{1}{4}(1+2\lambda)\beta b \quad \text{and} \quad a_3 = \frac{1}{24}(1+2\lambda)\beta\left[-4+(1+2(1+\lambda)\beta)b^2\right].$$

Thus

$$\begin{aligned} T_{3,1}(f) =& 1 + \frac{\beta^2(1+2\lambda)^2}{576}\left[-16+(-64+(4-8\lambda)\beta)b^2\right. \\ &\left.+(-1+(-1+2\lambda)\beta+2(1+5\lambda+4\lambda^2)\beta^2)b^4\right]. \end{aligned} \tag{2.18}$$

Finally substituting b in (2.17) into (2.18), we have equality in (2.3). $\square$

Observe now that for $\beta = 1$, the inequality defining Ω is as follow

$$30 - 40\lambda - 32\lambda^2 = -2(2\lambda-1)(8\lambda+5) < 0$$

and holds for $\lambda \in [0,1/2)$. Then the first inequality in (2.3) holds with $\beta = 1$. Moreover for $\beta = 1$ and $\lambda = 1/2$ we have $\det T_{3,1}(f) \geqslant 0$. Therefore for $\beta = 1$, i.e., for the class $\mathscr{F}_O(\lambda)$ we get the following result proved in [2].

COROLLARY 2.4. *Let* $\lambda \in [1/2,1]$. *If* $f \in \mathscr{F}_O(\lambda)$, *then*

$$\det T_{3,1}(f) \leqslant \begin{cases} 1, & \lambda \in \left[1/2,(3\sqrt{17}-5)/8\right], \\ 1+\dfrac{1}{18}(1+2\lambda)^2\left(-8+5\lambda+4\lambda^2\right), & \lambda \in \left((3\sqrt{17}-5)/8,1\right] \end{cases}$$

and

$$\det T_{3,1}(f) \geqslant -\frac{(2\lambda-1)^2(2\lambda+5)^2}{64\lambda(2\lambda+3)}.$$

All inequalities are sharp.

By selecting $\lambda = 1/2$ and $\lambda = 1$ we get from Theorem 2.3 the following results, respectively.

COROLLARY 2.5. *Let* $\beta \in (0,1]$. *If* $f \in \mathscr{C}_\beta$, *then*

$$\det T_{3,1}(f) \leqslant 1,$$

and

$$\det T_{3,1}(f) \geqslant (1-\beta^2)^2.$$

All inequalities are sharp.

COROLLARY 2.6. *Let* $\beta \in (0,1]$. *If* $f \in \mathscr{F}_O(1,\beta)$, *then*

$$\det T_{3,1}(f) \leqslant \begin{cases} 1, & \beta \in \left(0,3/\sqrt{10}\right], \\ 1-\dfrac{9}{2}\beta^2+5\beta^4, & \beta \in \left(3/\sqrt{10},1\right], \end{cases}$$

and

$$\det T_{3,1}(f) \geqslant \begin{cases} 1-\dfrac{9\beta^2\left[28+4\beta+9\beta^2\right]}{16\left[-1+\beta+20\beta^2\right]}, & \beta \in \left((\sqrt{2881}-1)/80,1\right], \\ 1-\dfrac{9}{2}\beta^2+5\beta^4, & \beta \in \left(0,(\sqrt{2881}-1)/80\right]. \end{cases}$$

All inequalities are sharp.

At the end, substituting $\beta = 1$ in Corollaries 2.5 and 2.6 we obtain the following inequalities. The first one was proved in [5] and the second one in [2].

COROLLARY 2.7. 1. *If* $f \in \mathscr{C}$, *then*

$$0 \leqslant \det T_{3,1}(f) \leqslant 1.$$

2. *If* $f \in \mathscr{F}_O(1,1)$, *then*

$$-\frac{49}{320} \leqslant \det T_{3,1}(f) \leqslant \frac{3}{2}.$$

All inequalities are sharp.

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