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DIFFERENCE OF DIFFERENTIATION COMPOSITION OPERATORS FROM THE BESOV SPACE B^1 TO BLOCH–TYPE SPACES AND LITTLE BLOCH–TYPE SPACES

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Abstract. We characterize bounded and compact differences of differentiation composition operators from the Besov space B^1 to Bloch-type spaces utilizing the difference of the values of derivatives of the conformal automorphisms of the unit disk at the respective symbols of the these operators. We also derive asymptotic estimates for the operator norms, providing significant insights into their behavior. In addition we characterize boundedness and compactness of $C_\varphi D^n - C_\psi D^n$ from the minimal Möbius invariant space B^1 to little Bloch-type spaces. Our characterization relies on the analysis of the derivatives of conformal automorphisms of $\mathbb{D}$, revealing intricate relationships between conformal automorphisms of $\mathbb{D}$ and differentiation composition operators between these spaces.

1. Introduction

Let $\mathbb{N}_0$ denote the set of non-negative integers, and let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The space $H(\mathbb{D})$ comprises all analytic functions in $\mathbb{D}$, and $S(\mathbb{D})$ is the subset of self-maps of $\mathbb{D}$.

The study of composition operators on analytic function spaces has long been a central theme in functional analysis. For $\varphi \in S(\mathbb{D})$ and an integer $n \in \mathbb{N}_0$, the *differentiation composition operator*, $C_\varphi D^n$, is defined for $f \in H(\mathbb{D})$ as:

$$C_\varphi D^n f = f^{(n)} \circ \varphi.$$

In the base case where $n = 0$, this simplifies to the standard *composition operator*, $C_\varphi f = f \circ \varphi$. The operator is a special case of the weighted differentiation composition operator (or generalized weighted composition operator), defined as follows

$$D^k_{\varphi,u}(f)(z) = u(z) f^{(k)}(\varphi(z))$$

which, along with its n-dimensional counterparts, has been studied a lot (see, e.g., [18, 19, 33, 35, 36, 38, 44, 45] and the related references therein).

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A significant area of investigation has been the study of the difference between two such operators, $C_\varphi - C_\psi$, which provides deep insights into the topological structure of the space of composition operators. A pivotal moment in this field was the work of MacCluer, Ohno, and Zhao, who first characterized the compact difference of composition operators on the space H^∞. Their characterization was fundamentally linked to the *pseudo-hyperbolic distance* between the symbols φ and ψ. This distance is defined by $\rho(z,\zeta) = |\alpha_\zeta(z)|$, where α_ζ is the conformal automorphism of $\mathbb{D}$ that interchanges the points 0 and ζ:

$$\alpha_\zeta(z) = \frac{\zeta - z}{1 - \overline{\zeta} z}, \quad z \in \mathbb{D}. \tag{1}$$

Inspired by this connection, numerous authors (see [1, 6, 8, 9, 16, 17, 21, 23–32, 34, 37, 39, 42, 43], and the related references therein) have investigated the properties of operator differences on various analytic function spaces. Notably, Moorhouse [23] and [24] provided a thorough characterization for the compact difference within the Bergman space, and Choe et al. [8] contributed to this field by characterizing the compact difference of composition operators specifically in the Hardy space. More recently, the third author and S. Ueki estimated the essential norm of $C_\varphi - C_\psi$ from analytic Besov spaces B^p ($p > 1$) to Bloch-type spaces. Motivated by these results, this paper extends the analysis to the difference of differentiation composition operators, $C_\varphi D^n - C_\psi D^n$, at the critical case of $p = 1$. We investigate these operators acting from the minimal Möbius invariant space, B^1, to weighted Bloch-type spaces, $\mathscr{B}_\nu$. Our work reveals a compelling interplay between the operator norm, the derivatives of conformal automorphisms, and the pseudo-hyperbolic distance, highlighting the deep connections between operator theory and the underlying complex geometry of the disk.

We begin by defining the function spaces central to our analysis. A *weight* ν is a positive, continuous, and radial function on $\mathbb{D}$ that monotonically decreases to 0 as $|z| \to 1$.

- The *weighted Bloch-type space*, $\mathscr{B}_\nu$, consists of functions $f \in H(\mathbb{D})$ for which the norm $\|f\|_{\mathscr{B}_\nu} = |f(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f'(z)|$ is finite.
- The *little weighted Bloch-type space*, $\mathscr{B}_{\nu,0}$, is the closed subspace of $\mathscr{B}_\nu$ where $\lim_{|z|\to 1} \nu(z)|f'(z)| = 0$.

When the weight is given by $\nu(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, we denote $\mathscr{B}_\nu$ simply by $\mathscr{B}_\alpha$. Similarly, the space $\mathscr{A}_\nu^\infty$, consists of functions $f \in H(\mathbb{D})$ for which the norm $\|f\|_{\mathscr{A}_\nu^\infty} = \sup_{z\in\mathbb{D}} \nu(z)|f(z)|$ is finite.

The domain space is the *analytic Besov space* B^1, which consists of functions $f \in H(\mathbb{D})$ that can be represented as an absolutely convergent series of conformal automorphisms:

$$f(z) = \sum_{n=0}^{\infty} a_n \alpha_{\lambda_n}(z), \quad \text{for } \{a_n\} \in \ell^1 \text{ and } \{\lambda_n\} \subset \mathbb{D}.$$

The norm is defined as $\|f\|_{B^1} = \inf\{\sum_{n=0}^{\infty} |a_n|\}$ over all such representations. The space B^1 is the minimal Möbius invariant space and admits an equivalent norm based on the function's second derivative: $\|f\|_{B^1} \asymp |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)|\, dA(z)$. A key property established by Blasco [4] is that for any $f \in B^1$ with $f(0) = f'(0) = 0$, there exists a complex Borel measure of bounded variation μ_0 on $\mathbb{D}$ such that:

$$f(z) = \int_{\mathbb{D}} \alpha_\zeta(z)\, d\mu_0(\zeta) \quad \text{and} \quad \|\mu_0\| \lesssim \|f\|_{B^1}.$$

This integral representation will be used frequently in our analysis. For further exploration of these function spaces, including their structures and properties, we refer to [2]–[5], [7], [12]–[15], [40], [41], [46] and [47].

Our primary goal is to characterize the boundedness and compactness of the operators $C_\varphi D^n$ and $C_\varphi D^n - C_\psi D^n$ from B^1 to both $\mathscr{B}_\nu$ and $\mathscr{B}_{\nu,0}$. These characterizations reveal how the geometric properties of the symbols φ and ψ govern the analytic properties of the operators.

The compactness of a closed subset L of $\mathscr{B}_{\nu,0}$ can be established through specific criteria that involve the properties of the weight function and functions in L. By a slight modification of the arguments presented by Madigan and Matheson [22], we can establish the subsequent lemma.

LEMMA 1. *Let L be a closed subset of $\mathscr{B}_{\nu,0}$. Then L is compact if and only if the following two conditions hold:*

1. *L is bounded.*

2. *L obeys the condition*

$$\lim_{|z|\to 1} \sup_{f\in L} \nu(z)\,|\,f'(z)\,| = 0.$$

2. Bounded difference of differentiation composition operators

In this section, we provide a comprehensive characterization of bounded difference of differentiation composition operators that map the analytic Besov space B^1 into Bloch-type spaces. To establish the main results of this paper, the forthcoming lemmas will be crucial to our analysis, see [11]. We shall omit the immediate proof of Lemma 2 for brevity.

LEMMA 2. *If $f \in B^1$ and $z \in \mathbb{D}$, then $|f(z)| \leqslant \|f\|_\infty \leqslant \|f\|_{B^1}$.*

By Lemma 2 and the Cauchy integral formula, there is a constant C such that

$$|\,f^{(n)}(z)\,| \leqslant C\, \frac{\|\,f\,\|_{B^1}}{(1-|\,z\,|^2)^n} \tag{2}$$

for every $z \in \mathbb{D}$ and $f \in B^1$.

LEMMA 3. *Let $f \in B^1$. Then there is a constant $C > 0$ such that*

$$|(1-|\zeta|^2)^{n+1} f^{(n+1)}(\zeta) - (1-|z|^2)^{n+1} f^{(n+1)}(z)| \leqslant C \, \| f \|_{B^1} \, \rho(\zeta, z)$$

for all $\zeta, z \in \mathbb{D}$.

Proof. Let $f \in B^1$ be arbitrary and let $\zeta \in \mathbb{D}$. By (2),

$$(1-|\zeta|^2)^{n+1} |f^{(n+1)}(\zeta)| \leqslant C \|f\|_{B^1}.$$

Thus

$$\|f^{(n)}\|_{\mathscr{B}_{n+1}} \leqslant C \|f\|_{B^1}. \tag{3}$$

For $\beta > 0$, $\zeta, z \in \mathbb{D}$ and $n \in \mathbb{N}_0$, let

$$F(f, n, \beta, \zeta, z) = (1-|\zeta|^2)^{\beta} f^{(n+1)}(\zeta) - (1-|z|^2)^{\beta} f^{(n+1)}(z).$$

Then by Lemma 2.3 in [25], we have

$$|F(f, 0, \beta, \zeta, z)| \leqslant \, C \, \|f\|_{\mathscr{B}_\beta} \rho(\zeta, z)$$

for all $\beta > 0$, $f \in \mathscr{B}_\beta$ and for all $\zeta, z \in \mathbb{D}$. Thus by replacing β by $n+1$ and f by $f^{(n)}$ in the above inequality, we have

$$| \, F(f, n, n+1, \zeta, z) \, | \leqslant C \, \| \, f^{(n)} \, \|_{\mathscr{B}_{n+1}} \, \rho(\zeta, z)$$

Hence by (3) for all $f \in B^1$, we have

$$| \, F(f, n, n+1, \zeta, z) \, | \leqslant C \, \| \, f \, \|_{B^1} \, \rho(\zeta, z).$$

Thus the proof is accomplished. □

The proof of the following lemma can be established through a standard argument involving the normal family, as illustrated in Proposition 3.11 of [12]. In the lemma, we use T to denote either of the operators $C_\varphi D^n$ or $C_\varphi D^n - C_\psi D^n$.

LEMMA 4. *Let $n \in \mathbb{N}_0$. Let $\varphi, \psi \in S(\mathbb{D})$ and let ν be a weight function. Assume that $T : B^1 \to \mathscr{B}_\nu$ is bounded. Then T is compact if and only if for every bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in B^1, which converges uniformly to zero on compact subsets of $\mathbb{D}$, the condition $\lim_{k \to \infty} \|T f_k\|_{\mathscr{B}_\nu} = 0$ holds.*

The next two results characterize bounded and compact differentiation composition operators from B^1 to Bloch-type spaces.

THEOREM 1. *Fix* $n \in \mathbb{N}_0$, *and let* $\varphi \in S(\mathbb{D})$. *Then the following statements are equivalent:*

(i) $C_\varphi D^n : B^1 \to \mathscr{B}_\nu$ *is bounded.*

(ii) $\sup_{z\in\mathbb{D}} \dfrac{\nu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} < \infty.$

(iii) $M := \sup_{z,\zeta\in\mathbb{D}} \nu(z)|\alpha_{\varphi(\zeta)}^{(n+1)}(\varphi(z))\varphi'(z)| < \infty$ *and* $\varphi \in \mathscr{B}_\nu$.

Proof. We first prove the equivalence of (i) and (ii). First suppose that (i) holds. Thus

$$\|(C_\varphi D^n)f\|_{\mathscr{B}_\nu} \lesssim \|f\|_{B^1} \tag{4}$$

for all $f \in B^1$. In particular, since $f(z) = z^{n+1}/(n+1)! \in B^1$, it follows that $\varphi \in \mathscr{B}_\nu$. Consider the function $\alpha_{\varphi(\zeta)}$ defined as in (1). Then $\alpha_{\varphi(\zeta)} \in B^1$ and $\sup_{\zeta\in\mathbb{D}} \|\alpha_{\varphi(\zeta)}\|_{B^1} \lesssim 1$, see [46]. Moreover,

$$\alpha_{\varphi(\zeta)}^{(n+1)}(z) = -(n+1)!\overline{\varphi(\zeta)}^n \frac{1-|\varphi(\zeta)|^2}{(1-\overline{\varphi(\zeta)}z)^{n+2}}.$$

Thus by (4), we have

$$(n+1)!\frac{\nu(\zeta)|\varphi(\zeta)|^n|\varphi'(\zeta)|}{(1-|\varphi(\zeta)|^2)^{n+1}} \leqslant \|(C_\varphi D^n)\alpha_{\varphi(\zeta)}\|_{\mathscr{B}_\nu} \lesssim \|\alpha_{\varphi(\zeta)}\|_{B^1} \lesssim 1$$

for all $\zeta \in \mathbb{D}$. Therefore,

$$\sup_{|\varphi(\zeta)|>1/2} \frac{\nu(\zeta)|\varphi'(\zeta)|}{(1-|\varphi(\zeta)|^2)^{n+1}} < \infty. \tag{5}$$

Since $\varphi \in \mathscr{B}_\nu$, we get

$$\sup_{|\varphi(\zeta)|\leqslant 1/2} \frac{\nu(\zeta)|\varphi'(\zeta)|}{(1-|\varphi(\zeta)|^2)^{n+1}} \leqslant (4/3)^{n+1} \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|\varphi'(\zeta)| < \infty. \tag{6}$$

Combining (5) and (6), we see that (ii) holds.

Conversely, suppose that (ii) holds. Then by (2), we have

$$\begin{aligned}\|(C_\varphi D^n)f\|_{\mathscr{B}_\nu} &= |\, f^{(n)}(\varphi(0))\,| + \sup_{z\in\mathbb{D}} \nu(z)|f^{(n+1)}(\varphi(z))\varphi'(z)| \\ &\leqslant C\ \|\, f\,\|_{B^1} \left(1 + \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\right).\end{aligned}$$

Therefore, (i) holds.

To complete the proof, we need to prove that (ii) and (iii) are equivalent. First assume (ii). Note that it follows easily from (ii) that $\varphi \in \mathscr{B}_\nu$. Moreover,

$$\begin{aligned} \nu(z) \mid \alpha_{\varphi(\zeta)}^{(n+1)}(\varphi(z))\varphi'(z) \mid &\lesssim \frac{\nu(z)(1- \mid \varphi(\zeta) \mid^2) \mid \varphi'(z) \mid}{\mid (1-\overline{\varphi(\zeta)}\varphi(z))^{n+2} \mid} \\ &\leqslant \nu(z) \frac{1- \mid \varphi(\zeta) \mid^2}{\mid 1-\overline{\varphi(\zeta)}\varphi(z) \mid} \frac{\mid \varphi'(z) \mid}{\mid 1-\overline{\varphi(\zeta)}\varphi(z) \mid^{n+1}} \\ &\leqslant C\frac{\nu(z) \mid \varphi'(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n+1}} \end{aligned}$$

and thus we obtain (iii).

Finally, assuming (iii), the substitution $\zeta = z$ gives

$$\sup_{z\in\mathbb{D}} \frac{\nu(z) \mid \varphi(z) \mid^n \mid \varphi'(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n+1}} < \infty.$$

Therefore,

$$\sup_{|\varphi(z)|>\delta} \frac{\nu(z) \mid \varphi'(z) \mid}{(1-\varphi(z) \mid^2)^{n+1}} < \infty$$

for any $\delta \in (0,1)$. Since $\varphi \in \mathscr{B}_\nu$, we have obtained (ii). The proof is complete. □

THEOREM 2. *Let $n \in \mathbb{N}_0$ and $\varphi \in S(\mathbb{D})$. Assume $C_\varphi D^n : B^1 \to \mathscr{B}_\nu$ is bounded. Then the following statements are equivalent:*

(i) *$C_\varphi D^n : B^1 \to \mathscr{B}_\nu$ is compact.*

(ii) $\displaystyle\lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} = 0.$

(iii) $\displaystyle\lim_{|\varphi(z)|\to 1} \sup_{\zeta\in\mathbb{D}} \nu(z) \mid \alpha_{\varphi(\zeta)}^{(n+1)}(\varphi(z))\varphi'(z) \mid = 0.$

Proof. $(i) \Leftrightarrow (ii)$. Assume that (i) holds. Let (z_j) be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. We remark that if no such sequence exists, then (ii) holds vacuously and the proof is complete. Let $\varepsilon > 0$ be given. Consider the family of functions

$$f_j(z) = \frac{1- \mid \varphi(z_j) \mid^2}{1-\overline{\varphi(z_j)}z}.$$

Then $\| f_j \|_{B^1} \lesssim 1$ and $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 4, there exists $N > 0$ such that $\| (C_\varphi D^n)f_j \|_{\mathscr{B}_\nu} < \varepsilon$ for all $j > N$. Therefore

$$\sup_{z\in\mathbb{D}} \nu(z) \mid f_j^{(n+1)}(\varphi(z))\ \varphi'(z) \mid < \varepsilon$$

for all $j > N$. After a calculation,

$$\frac{\nu(z_j) \mid \varphi(z_j) \mid^{n+1} \mid \varphi'(z_j) \mid}{(1- \mid \varphi(z_j) \mid^2)^{n+1}} < \varepsilon$$

for $j > N$. Since $\mid \varphi(z_j) \mid \to 1$ as $j \to \infty$, this yields (ii).

Next assume (ii) and let f_m be a sequence in B^1 with $\| f_m \|_{\mathscr{B}^1} \leqslant 1$ and $f_m \to 0$ uniformly on compact subsets as $m \to \infty$. To prove (1) it is enough to prove that

$$\| (C_\varphi D^n) f_m \|_{\mathscr{B}_\nu} = \mid f_m^{(n)}(\varphi(0)) \mid + \sup_{z \in \mathbb{D}} \nu(z) \mid f_m^{(n+1)}(\varphi(z))\varphi'(z) \mid \to 0$$

as $m \to \infty$. Since $f_m^{(n)} \to 0$ on compact sets, the first term in the sum above tends to 0 as $m \to \infty$. Let $\varepsilon > 0$ be given. There exists r_0, $0 < r_0 < 1$, with

$$\frac{\nu(z) \mid \varphi'(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n+1}} < \varepsilon$$

for all z with $\mid \varphi(z) \mid > r_0$. The above inequality gives

$$\nu(z) \mid f_m^{(n+1)}(\varphi(z))\varphi'(z) \mid \lesssim \frac{\nu(z) \mid \varphi'(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n+1}} < \varepsilon$$

for all z with $\mid \varphi(z) \mid > r_0$ and for all m.

Since $f_m \to 0$ on the set $\{w : \mid w \mid \leqslant r_0\}$ and since $\varphi \in \mathscr{B}_\nu$, there exists $N_1 > 0$ such that

$$\nu(z) \mid f_m^{(n+1)}(\varphi(z)) \mid \varphi'(z) \mid < \varepsilon$$

for all z with $\mid \varphi(z) \mid \leqslant r_0$ and for all $m > N_1$. The argument shows that $\| (C_\varphi D^n) f_m \|_{\mathscr{B}_\nu}$ $\to 0$ and Lemma 4 completes the proof of (i).

The proof that (ii) $\Leftrightarrow$ (iii) is similar to the argument in Theorem 1. $\square$

In the next result, we will characterize boundedness of the difference of composition operators from B^1 to Bloch-type spaces. We adopt the following notation to minimize redundancy and to focus on the essential components of the proofs of the following results. Let

$$J_z(f, n, \varphi, \psi) = f^{(n+1)}(\varphi(z))\varphi'(z) - f^{(n+1)}(\psi(z))\psi'(z)$$

and

$$T_z(\varphi, \psi) = \frac{\nu(z)\varphi'(z)}{(1- \mid \varphi(z) \mid^2)^{n+1}} - \frac{\nu(z)\psi'(z)}{(1- \mid \psi(z) \mid^2)^{n+1}}.$$

THEOREM 3. *Let $n \in \mathbb{N}_0$, let ν be a weight and assume $\varphi \neq \psi \in S(\mathbb{D})$. Then $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded if and only if $\varphi - \psi \in \mathscr{B}_\nu$ and the family of functions*

$$\{(\alpha_\zeta^{(n+1)} \circ \varphi)\varphi' - (\alpha_\zeta^{(n+1)} \circ \psi)\psi',\ \zeta \in \mathbb{D}\}$$

is bounded in $\mathscr{A}_\nu^\infty$*, that is*

$$M_n := \sup_{\zeta\in\mathbb{D}} \|(\alpha_\zeta^{(n+1)}\circ\varphi)\varphi' - (\alpha_\zeta^{(n+1)}\circ\psi)\psi'\|_{\mathscr{A}_\nu^\infty} < \infty. \tag{7}$$

Moreover, if $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ *is bounded, then*

$$\begin{aligned}\| \varphi-\psi \|_{\mathscr{B}_\nu} + M_n &\lesssim \|C_\varphi D^n - C_\psi D^n\|_{B^1\to\mathscr{B}_\nu} \\ &\lesssim \| \varphi-\psi \|_{\mathscr{B}_\nu} + \frac{1}{(1-|\varphi(0)|^2)^n} + \frac{1}{(1-|\psi(0)|^2)^n} + M_n. \end{aligned} \tag{8}$$

Proof. First suppose that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded. Since $f(z) = z^{n+1}/(n+1)! \in B^1$, it is clear that $\varphi-\psi \in \mathscr{B}_\nu$ and

$$\| \varphi-\psi \|_{\mathscr{B}_\nu} \lesssim \| C_\varphi D^n - C_\psi D^n \|_{B^1\to\mathscr{B}_\nu}.$$

Consider the family of functions: $\alpha_\zeta(z)$, $\zeta\in\mathbb{D}$. By ([46], p. 53), $\alpha_\zeta \in B^1$ and $\sup_{\zeta\in\mathbb{D}}\|\alpha_\zeta\|_{B^1} \lesssim 1$. Since $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded,

$$\|(C_\varphi D^n - C_\psi D^n)\alpha_\zeta\|_{\mathscr{B}_\nu} \lesssim \|C_\varphi D^n - C_\psi D^n\|_{B^1\to\mathscr{B}_\nu}$$

for every $\zeta\in\mathbb{D}$. Thus the family of functions $\{(\alpha_\zeta^{(n+1)}\circ\varphi)\varphi' - (\alpha_\zeta^{(n+1)}\circ\psi)\psi', \zeta\in\mathbb{D}\}$ is bounded in $\mathscr{A}_\nu^\infty$ and

$$\begin{aligned} M_n = \sup_{\zeta\in\mathbb{D}} \|(\alpha_\zeta^{(n+1)}\circ\varphi)\varphi' - (\alpha_\zeta^{(n+1)}\circ\psi)\psi'\|_{\mathscr{A}_\nu^\infty} &\leqslant \sup_{\zeta\in\mathbb{D}} \|\alpha_\zeta^{(n)}\circ\varphi - \alpha_\zeta^{(n)}\circ\psi\|_{\mathscr{B}_\nu} \\ &\lesssim \|C_\varphi D^n - C_\psi D^n\|_{B^1\to\mathscr{B}_\nu} \end{aligned}$$

Therefore,

$$\| \varphi-\psi \|_{\mathscr{B}_\nu} + M_n \lesssim \|C_\varphi D^n - C_\psi D^n\|_{B^1\to\mathscr{B}_\nu}. \tag{9}$$

Conversely, suppose that $\varphi-\psi\in\mathscr{B}_\nu$ and (7) holds. Let $f\in B^1$. We must show that

$$| f^{(n)}(\varphi(0)) - f^{(n)}(\psi(0)) | + \sup_{z\in\mathbb{D}} \nu(z) \,| J_z(f,n,\varphi,\psi) | \leqslant C \| f \|_{B^1} \tag{10}$$

for a constant C independent of f. By (2),

$$| f^{(n)}(\varphi(0)) - f^{(n)}(\psi(0)) | \lesssim \| f \|_{B^1} \left(\frac{1}{(1-|\varphi(0)|^2)^n} + \frac{1}{(1-|\psi(0)|^2)^n} \right). \tag{11}$$

By Blasco's result stated in the introduction, there is a measure μ with $\| \mu \| \lesssim \| f \|_{B^1}$ and such that

$$f(z) - f(0) - f'(0)z = \int_{\mathbb{D}} \alpha_\zeta(z)\, d\mu(\zeta). \tag{12}$$

If $n \geqslant 1$, then it follows that

$$f^{(n+1)}(z) = \int_{\mathbb{D}} \alpha_\zeta^{(n+1)}(z)\, d\mu(\zeta).$$

Therefore

$$\begin{aligned} \nu(z) \mid J_z(f,n,\varphi,\psi) \mid = & \ \nu(z) \ \left| \int_{\mathbb{D}} J_z(\alpha_\zeta,n,\varphi,\psi) d\mu(\zeta) \right| \\ \leqslant & \int_{\mathbb{D}} \mid \nu(z) J_z(\alpha_\zeta,n,\varphi,\psi) \mid d \mid \mu \mid (\zeta) \\ \leqslant & \sup_{\zeta\in\mathbb{D}} \nu(z) \ \mid J_z(\alpha_\zeta,n,\varphi,\psi) \mid \| \mu \| . \end{aligned} \tag{13}$$

It now follows that

$$\sup_{z\in\mathbb{D}} \nu(z) \mid J_z(f,n,\varphi,\psi) \mid \lesssim M_n \| f \|_{B^1} .$$

We conclude

$$\|(C_\varphi D^n - C_\psi D^n) f\|_{\mathscr{B}_\nu} \lesssim \left\{ \frac{1}{(1-|\varphi(0)|^2)^n} + \frac{1}{(1-|\psi(0)|^2)^n} + M_n \right\} \|f\|_{B^1}. \tag{14}$$

For convenience in the argument for $n = 0$, we define $g(z) = f(z) - f(0) - f'(0)z$. It follows that $f'(z) = g'(z) + f'(0)$ and

$$\begin{aligned} \nu(z) \mid J_z(f,0,\varphi,\psi) \mid &= \nu(z) \mid (g'(\varphi(z)) + f'(0))\ \varphi'(z) - (g'(\psi(z)) + f'(0))\ \psi'(z) \mid \\ &\leqslant \ \nu(z) \mid J_z(g,0,\varphi,\psi) \mid \ + \nu(z) \mid f'(0) \mid \mid \varphi'(z) - \psi'(z) \mid . \end{aligned}$$

An argument as in the case for $n \geqslant 1$ yields

$$\sup_{z\in\mathbb{D}} \nu(z) \mid ((C_\varphi - C_\psi) f)'(z) \mid \lesssim (\ \| \varphi - \psi \|_{\mathscr{B}_\nu} \ + M_0 \ ) \| f \|_{B^1} . \tag{15}$$

An application of (11), (14) and (15), we have that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded and

$$\|C_\varphi D^n - C_\psi D^n\|_{B^1\to\mathscr{B}_\nu} \lesssim \| \varphi - \psi \|_{\mathscr{B}_\nu} + \left\{ \frac{1}{(1-|\varphi(0)|^2)^n} + \frac{1}{(1-|\psi(0)|^2)^n} + M_n \right\}. \tag{16}$$

Combining (9) and (16), we conclude (8) and the theorem is established. □

3. Compact difference of differentiation composition operators

Next we characterize the compactness of the difference of differentiation composition operators from the space B^1 to Bloch-type spaces.

Before stating Theorem 4, we note that if $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is compact and $\psi \in \mathscr{B}_\nu$, then $\varphi \in \mathscr{B}_\nu$. Our proof below will rely on $\varphi, \psi \in \mathscr{B}_\nu$. We remark that there are other equivalent ways to state this hypothesis.

THEOREM 4. *Let* $n \in \mathbb{N}_0, \varphi \neq \psi \in S(\mathbb{D})$ *and let* ν *be a weight. Assume that* $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ *is compact and* $\psi \in \mathscr{B}_\nu$. *Then*

$$(i) \quad \lim_{|\varphi(z)|\to 1} \frac{\nu(z) \mid \varphi'(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n+1}} \rho(\varphi(z), \psi(z)) = 0$$

$$(ii) \quad \lim_{|\psi(z)|\to 1} \frac{\nu(z) \mid \psi'(z) \mid}{(1- \mid \psi(z) \mid^2)^{n+1}} \rho(\varphi(z), \psi(z)) = 0 \text{ and}$$

$$(iii) \quad \lim_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \mid T_z(\varphi, \psi) \mid = 0.$$

Proof. We assume that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is compact. As in [32], we may assume that $\| \varphi \|_\infty = 1$ and $\| \psi \|_\infty = 1$. By way of contradiction, suppose that (i) does not hold. Therefore there exists $\delta > 0$ and a sequence $z_j \in \mathbb{D}$ with $\mid \varphi(z_j) \mid \to 1$ as $j \to \infty$ such that

$$\frac{\nu(z_j) \mid \varphi'(z_j) \mid}{(1- \mid \varphi(z_j) \mid^2)^{n+1}} \rho(\varphi(z_j), \psi(z_j)) \geqslant \delta \tag{17}$$

for $j = 1, 2, \ldots$. Note that we may assume that $\mid \varphi(z_j) \mid > 1/2$ for all j. Let

$$g_j(z) = \frac{(1- \mid \varphi(z_j) \mid^2)}{(n+1)!\, \overline{\varphi(z_j)}^{n+1}} \frac{1}{(1 - \overline{\varphi(z_j)} z)}.$$

Then $\| g_j \|_{B^1} \lesssim C$ and $g_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. By a calculation,

$$g_j^{(n+1)}(z) = \frac{1- \mid \varphi(z_j) \mid^2}{(1 - \overline{\varphi(z_j)} z)^{n+2}}.$$

The assumption of compactness yields $J_1 > 0$ such that

$$\nu(z_j)|J_{z_j}(g_j, n, \varphi, \psi)|$$

$$= \left| \frac{\nu(z_j)\varphi'(z_j)}{(1- \mid \varphi(z_j) \mid^2)^{n+1}} - \frac{\nu(z_j)(1- \mid \varphi(z_j) \mid^2)\psi'(z_j)}{(1 - \overline{\varphi(z_j)}\psi(z_j))^{n+2}} \right| < \delta/2$$

for all $j > J_1$. We obtain

$$\left| \frac{\nu(z_j)\varphi'(z_j)}{(1- \mid \varphi(z_j) \mid^2)^{n+1}} \right| - \left| \frac{\nu(z_j)(1- \mid \varphi(z_j) \mid^2)\psi'(z_j)}{(1 - \overline{\varphi(z_j)}\psi(z_j))^{n+2}} \right| < \delta/2 \tag{18}$$

for $j > J_1$. Now let

$$f_j(z) = \frac{(-1)^{n+1}}{\overline{\varphi(z_j)}^{n+2}} \left\{ - \frac{(1- \mid \varphi(z_j) \mid^2)^2}{(n+2)!(1 - \overline{\varphi(z_j)} z)^2} + \frac{(1- \mid \varphi(z_j) \mid^2}{(n+1)!(1 - \overline{\varphi(z_j)} z)} \right\}$$

where, as before, we assume $\mid \varphi(z_j) \mid > 1/2$. Then $\| f_j \|_{B^1} \lesssim C$ and $f_j \to 0$ uniformly on compact subsets. Thus there exists $J_2 > 0$ with

$$\nu(z_j) \mid J_{z_j}(f_j, n, \varphi, \psi) \mid < \delta/2 \tag{19}$$

for $j > J_2$. After a calculation,

$$f_j^{(n+1)}(z) = \frac{(1-|\varphi(z_j)|^2)(-1)^{n+2}(z-\varphi(z_j))}{(1-\overline{\varphi(z_j)}z)^{n+3}}.$$

In particular $f_j^{(n+1)}(\varphi(z_j)) = 0$ and (19) yields

$$\frac{\nu(z_j)\,|\,\psi'(z_j)\,|\,(1-|\varphi(z_j)|^2)}{|\,1-\overline{\varphi(z_j)}\psi(z_j)\,|^{n+2}}\rho(\varphi(z_j),\psi(z_j)) < \delta/2 \tag{20}$$

for $j > J_2$. Equation (18) now yields

$$\left(\frac{\nu(z_j)\,|\,\varphi'(z_j)\,|}{(1-|\varphi(z_j)|^2)^{n+1}} - \frac{\nu(z_j)(1-|\varphi(z_j)|^2)\,|\,\psi'(z_j)\,|}{|\,1-\overline{\varphi(z_j)}\psi(z_j)\,|^{n+2}}\right)\rho(\varphi(z_j),\psi(z_j)) < \delta/2 \tag{21}$$

for $j > J_1$. Addition of equations (20) and (21) yields

$$\frac{\nu(z_j)\,|\,\varphi'(z_j)\,|}{(1-|\varphi(z_j)|^2)^{n+1}}\rho(\varphi(z_j),\psi(z_j)) < \delta$$

for all large j. We have obtained a contradiction to (17) and thus compactness of $C_\varphi D^n - C_\psi D^n$ implies condition (i). A similar argument shows that compactness implies condition (ii).

Finally, we recall that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is compact and we assume that (iii) is false. Thus there exists $\eta > 0$ and $z_j \in \mathbb{D}$ with $\min\{|\,\varphi(z_j)\,|, |\,\psi(z_j)\,|\} \to 1$ as $j \to \infty$ and

$$\nu(z_j)\left|\frac{\varphi'(z_j)}{(1-|\varphi(z_j)|^2)^{n+1}} - \frac{\psi'(z_j)}{(1-|\psi(z_j)|^2)^{n+1}}\right| > \eta \tag{22}$$

for $j = 1,2,\ldots$.

We first note that

$$\|\,(C_\varphi D^n - C_\psi D^n)g_j\,\|_{\mathscr{B}_\nu} \geqslant \left|\frac{\nu(z_j)\varphi'(z_j)}{(1-|\varphi(z_j)|^2)^{n+1}} - \frac{\nu(z_j)\psi'(z_j)(1-|\varphi(z_j)|^2)}{(1-\overline{\varphi(z_j)}\psi(z_j))^{n+2}}\right|.$$

After subtracting and adding the term $\nu(z_j)\psi'(z_j)/(1-|\psi(z_j)|^2)^{n+1}$ we have

$$\begin{aligned}\|\,(C_\varphi D^n - C_\psi D^n)g_j\,\|_{\mathscr{B}_\nu} &\geqslant \left|\frac{\nu(z_j)\varphi'(z_j)}{(1-|\varphi(z_j)|^2)^{n+1}} - \frac{\nu(z_j)\psi'(z_j)}{(1-|\psi(z_j)|^2)^{n+1}}\right| \\ &- \frac{\nu(z_j)\,|\,\psi'(z_j)\,|}{(1-|\psi(z_j)|^2)^{n+1}}\left|1 - \frac{(1-|\varphi(z_j)|^2)(1-|\psi(z_j)|^2)^{n+1}}{(1-\overline{\varphi(z_j)}\psi(z_j))^{n+2}}\right|.\end{aligned} \tag{23}$$

After calculation and an application of Lemma 3,

$$\begin{aligned}&\left|1 - \frac{(1-|\varphi(z_j)|^2)(1-|\psi(z_j)|^2)^{n+1}}{(1-\overline{\varphi(z_j)}\psi(z_j))^{n+2}}\right| \\ &= |\,F(g_j,n,n+1,\varphi(z_j),\psi(z_j))\,| \\ &\leqslant C\,\|\,g_j\,\|_{B^1}\;\rho(\varphi(z_j),\psi(z_j)).\end{aligned}$$

Substitution into (23) yields

$$\| (C_\varphi D^n - C_\psi D^n) g_j \|_{\mathscr{B}_\nu} + C \frac{\nu(z_j) \mid \psi'(z_j) \mid}{(1- \mid \psi(z_j) \mid^2)^{n+1}} \| g_j \|_{B^1} \rho(\varphi(z_j), \psi(z_j))$$

$$\geqslant \left| \frac{\nu(z_j)\varphi'(z_j)}{(1- \mid \varphi(z_j) \mid^2)^{n+1}} - \frac{\nu(z_j)\psi'(z_j)}{(1- \mid \psi(z_j) \mid^2)^{n+1}} \right| = |T_{z_j}(\varphi, \psi)|.$$

Recalling that the operator is compact and using (ii), which has already been proven, we obtain a contradiction to (22). Thus, compactness implies condition (iii). Therefore, the proof is complete. □

Next we establish a relationship between the difference of derivatives of the conformal automorphisms and compact differences of the differentiation composition operators from the Besov space B^1 to Bloch-type spaces.

THEOREM 5. *Let $n \in \mathbb{N}_0$, ν be a weight and let $\varphi \neq \psi \in S(\mathbb{D})$. Assume that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded and $\psi \in \mathscr{B}_\nu$. Then $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is compact if and only if the conditions (i) and (ii) in the hypothesis of Theorem 4 are satisfied, along with the following condition:*

$$\lim_{\min\{|\varphi(z)|,|\psi(z)|\} \to 1} \sup_{\zeta \in \mathbb{D}} \nu(z) \mid \alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z) - \alpha_\zeta^{(n+1)}(\psi(z))\psi'(z) \mid = 0.$$

Proof. We prove the result for $n \geqslant 1$. A brief argument will be given for the case $n = 0$.

We first prove that the three conditions imply compactness of the operator $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$. Let $\{f_k\}$ denote a sequence in B^1 that fulfills the necessary conditions for the application of Lemma 4. Our argument will show that

$$\| (C_\varphi D^n - C_\psi D^n) f_k \|_{\mathscr{B}_\nu} \to 0 \quad \text{as} \quad k \to \infty.$$

Since $\mid f_k^{(n)}(\varphi(0)) - f_k^{(n)}(\psi(0)) \mid \ \to 0$ as $k \to \infty$, it is enough to prove that $\sup_{z\in\mathbb{D}} \nu(z) \mid J_z(f_k, n, \varphi, \psi) \mid \ \to 0$. For each k, Blasco's result provides a measure μ_k with $\| \mu_k \| \lesssim \| f_k \|_{B^1} \leqslant C$ and

$$f_k(z) = f_k(0) + f_k'(0)z + \int_{\mathbb{D}} \alpha_\zeta(z)\, d\mu_k(\zeta).$$

Therefore

$$\nu(z) \mid J_z(f_k, n, \varphi, \psi) \mid \leqslant C \sup_{\zeta \in \mathbb{D}} \nu(z) \mid J_z(\alpha_\zeta, n, \varphi, \psi) \mid.$$

Now let $\varepsilon > 0$. The hypothesis yields $r, r \in (0,1)$, such that

$$\sup_{\zeta \in \mathbb{D}} \nu(z) \mid J_z(\alpha_\zeta, n, \varphi, \psi) \mid < \varepsilon / C$$

if $\min\{\mid \varphi(z) \mid, \mid \psi(z) \mid\} > r$. We conclude

$$\sup_{\min\{|\varphi(z)|,|\psi(z)|\} > r} \nu(z) \mid J_z(f_k, n, \varphi, \psi) \mid < \varepsilon.$$

We next consider

$$\sup_{\max\{|\varphi(z)|,|\psi(z)|\}\leqslant r} \nu(z) \mid J_z(f_k,n,\varphi,\psi) \mid$$

$$\leqslant (\| \varphi \|_{\mathscr{B}_\nu} + \| \psi \|_{\mathscr{B}_\nu})) \max_{|w|\leqslant r} | f_k^{(n+1)}(w) | \to 0 \quad \text{as} \quad k \to \infty.$$

As a final case we consider $z \in \mathbb{D}$ with $| \varphi(z) | \leqslant r < | \psi(z) |$. The parallel case $| \psi(z) | \leqslant r < | \varphi(z) |$ will be omitted. Again

$$\nu(z) \mid J_z(f_k,n,\varphi,\psi) \mid =$$

$$\left| \left(T_z(\varphi,\psi) + \frac{\nu(z)\psi'(z)}{(1-|\psi(z)|^2)^{n+1}} \right) (1-|\varphi(z)|^2)^{n+1} f_k^{(n+1)}(\varphi(z)) - f_k^{(n+1)}(\psi(z))\psi'(z) \right|$$

$$\leqslant | T_z(\varphi,\psi) | \, | f_k^{(n+1)}(\varphi(z)) | + \frac{\nu(z) \mid \psi'(z) \mid}{(1-|\psi(z)|^2)^{n+1}} | F(f_k,n,n+1,\varphi(z),\psi(z)) | .$$

We now refer the reader to the proof of Theorem 4. After minor adjustments, that proof establishes that if $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded, then $N := \sup_{z\in\mathbb{D}} | T_z(\varphi,\psi) | < \infty$. We use Lemma 3 with this fact to conclude that

$$\sup_{|\varphi(z)| \leqslant r<|\psi(z)|} \nu(z) \mid J_z(f_k,n,\varphi,\psi) \mid$$

$$\leqslant N \max_{|w|\leqslant r} | f_k^{(n+1)}(w) | + \frac{\nu(z) \mid \psi'(z) \mid}{(1-|\psi(z)|^2)^{n+1}} \| f_k \|_{B^1} \rho(\varphi(z),\psi(z)).$$

Since $f_k^{(n+1)} \to 0$ uniformly on $\{w : | w | \leqslant r\}$ and using (ii) of Theorem 4, we conclude that

$$\sup_{|\varphi(z)|\leqslant r<|\psi(z)|} \nu(z) \mid J_z(f_k,n,\varphi,\psi) \mid \to 0$$

as $k \to \infty$. Therefore the operator is compact.

We now assume that the operator is compact. Conditions (i) and (ii) have been established in Theorem 4. It remains to prove the third condition in the theorem. As in [32], we may assume that $\| \varphi \|_\infty = 1$ and $\| \psi \|_\infty = 1$. Calculations similar to those with f_k above yield

$$\nu(z) \mid J_z(\alpha_\zeta,n,\varphi,\psi) \mid \leqslant | T_z(\varphi,\psi) | \ (1-|\varphi(z)|^2)^{n+1} | \alpha_\zeta^{(n+1)}(\varphi(z)) |$$
$$+ \frac{\nu(z) \mid \psi'(z) \mid}{(1-|\psi(z)|^2)^{n+1}} | F(\alpha_\zeta,n,n+1,\varphi(z),\psi(z)) | .$$

We now apply (2) and Lemma 3 together with the fact that $\| \alpha_\zeta \|_{B^1} \lesssim 1$. We obtain

$$\nu(z) \mid J_z(\alpha_\zeta,n,\varphi,\psi) \mid$$
$$\leqslant | T_z(\varphi,\psi) | \ C \| \alpha_\zeta \|_{B^1} + \frac{\nu(z) \mid \psi'(z) \mid}{(1-|\psi(z)|^2)^{n+1}} C \| \alpha_\zeta \|_{B^1} \rho(\varphi(z),\psi(z))$$
$$\lesssim | T_z(\varphi,\psi) | + \frac{\nu(z) \mid \psi'(z) \mid}{(1-|\psi(z)|^2)^{n+1}} \rho(\varphi(z),\psi(z)).$$

Finally we invoke conditions (ii) and (iii) in Theorem 4. Given $\varepsilon > 0$, there exists r, $0 < r < 1$, such that

$$\frac{\nu(z) \mid \psi'(z) \mid}{(1- \mid \psi(z) \mid^2)^{n+1}} \, \rho(\varphi(z), \psi(z)) \; < \; \frac{\varepsilon}{2}$$

and

$$\mid T_z(\varphi, \psi) \mid < \frac{\varepsilon}{2}$$

for z with $\min\{\mid \varphi(z) \mid, \mid \psi(z) \mid\} > r$. We conclude that

$$\sup_{\zeta \in \mathbb{D}} \nu(z) \mid J_z(\alpha_\zeta, n, \varphi, \psi) \mid < \varepsilon$$

for all $z \in \mathbb{D}$ with $\min\{\mid \varphi(z) \mid, \mid \psi(z) \mid\} \; > \; r$. The proof for $n \geqslant 1$ is complete.

We present an outline of the proof for $n = 0$. First we assume the three conditions. By Blasco's result,

$$f_k'(z) = f_k'(0) + \int_{\mathbb{D}} \alpha_\zeta'(z) \, d\mu_k(\zeta)$$

and thus

$$\begin{aligned} & \nu(z) \mid J(f_k, 0, \varphi(z), \psi(z)) \mid \\ & \leqslant \mid f_k'(0) \mid \nu(z) \, (\mid \varphi'(z) - \psi'(z) \mid) + \nu(z) \mid \int_{\mathbb{D}} J_z(\alpha_\zeta, 0, \varphi, \psi) d\mu_k(\zeta) \mid \\ & \leqslant \mid f_k'(0) \mid (\| \varphi \|_{\mathscr{B}_\nu} + \| \psi \|_{\mathscr{B}_\nu}) + \; \| \mu_k \| \; \nu(z) \sup_{\zeta \in \mathbb{D}} \mid J_z(\alpha_\zeta, 0, \varphi, \psi) \mid \end{aligned} \tag{24}$$

Because $\varphi, \psi \in \mathscr{B}_\nu$ and because $f_k' \to 0$ uniformly on compact subsets, the first summand in (24) tends to 0 as $k \to \infty$. The proof that the second summand tends to 0 uses an argument similar to the argument for $n \geqslant 1$. We conclude that the operator is compact. The proof is complete. □

4. Bounded and compact operators from B^1 to $\mathscr{B}_{\nu,0}$

In this section, we characterize the boundedness and compactness of $C_\varphi D^n$ and $C_\varphi D^n - C_\psi D^n$ from the minimal Möbius invariant space B^1 to little Bloch-type spaces.

THEOREM 6. *Let ν be a weight, let $n \in \mathbb{N}_0$ and let $\varphi \in S(\mathbb{D})$.*

1. *Let $n \geqslant 1$. Then $C_\varphi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is bounded if and only if $C_\varphi D^n : B^1 \to \mathscr{B}_\nu$ is bounded and $\alpha_\zeta^{(n)} \circ \varphi \in \mathscr{B}_{\nu,0}$ for every $\zeta \in \mathbb{D}$.*

2. *Let $n = 0$. Then $C_\varphi : B^1 \to \mathscr{B}_{\nu,0}$ is bounded if and only if $\varphi \in \mathscr{B}_{\nu,0}$, $C_\varphi : B^1 \to \mathscr{B}_\nu$ is bounded, and $\alpha_\zeta \circ \varphi \in \mathscr{B}_{\nu,0}$ for every $\zeta \in \mathbb{D}$.*

Proof. We first consider the case $n \geqslant 1$.

Assume that $C_\varphi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is bounded. It is immediate that $C_\varphi D^n : B^1 \to \mathscr{B}_\nu$ is bounded. By considering functions $\alpha_\zeta \in B^1,\ \zeta \in \mathbb{D}$ we obtain $\alpha_\zeta^{(n)} \circ \varphi \in \mathscr{B}_{\nu,0}$ for every $\zeta \in \mathbb{D}$.

For the converse, it is enough to prove $(C_\varphi D^n)(f) \in \mathscr{B}_{\nu,0}$ for every $f \in B^1$. By Blasco's result there is a complex Borel measure of bounded variation such that

$$f(z) = f(0) + f'(0)z + \int_{\mathbb{D}} \alpha_\zeta(z)\, d\mu(\zeta).$$

Therefore, we have

$$\nu(z) \mid (C_\varphi D^n f)'(z) \mid \leqslant \int_{\mathbb{D}} \nu(z) \mid \alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z) \mid d \mid \mu \mid (\zeta).$$

By hypothesis the integrand above tends to 0 as $\mid z \mid \to 1$ for every $\zeta \in \mathbb{D}$. Furthermore the integrand is bounded by M, where M is as defined in Theorem 1. The Dominated Convergence Theorem now implies

$$\lim_{|z|\to 1} \nu(z) \mid (C_\varphi D^n f)'(z) \mid =\ 0$$

and the proof for $n \geqslant 1$ is complete.

We briefly outline the case $n = 0$. First, assume $C_\varphi : B^1 \to \mathscr{B}_{\nu,0}$ is bounded and let $f(z) = z$. Since $f \in B^1$ we obtain $\varphi \in \mathscr{B}_{\nu,0}$. The remainder of the argument proceeds just as in the case for $n \geqslant 1$. For the opposite direction, it is enough to prove that $C_\varphi(f) \in \mathscr{B}_{\nu,0}$ for $f \in B^1$. An argument using Blasco's result yields

$$\nu(z) \mid (C_\varphi f)'(z) \mid \lesssim\ \nu(z) \mid \varphi'(z) \mid\ +\ \int_{\mathbb{D}} \nu(z) \mid \alpha_\zeta'(\varphi(z))\varphi'(z) \mid\ d \mid \mu \mid (\zeta)$$

and the hypotheses now yield

$$\lim_{|z|\to 1} \nu(z) \mid (C_\varphi f)'(z) \mid = 0. \quad \square$$

THEOREM 7. *Let ν be a weight, $n \in \mathbb{N}_0$ and $\varphi \in S(\mathbb{D})$. Assume that $C_\varphi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is bounded. Then $C_\varphi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \sup_{\zeta\in\mathbb{D}} \nu(z) \mid \alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z) \mid =\ 0. \tag{25}$$

Proof. Assume that $C_\varphi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is compact. We use the test functions α_ζ. Note that $\sup_\zeta \parallel \alpha_\zeta \parallel_{B^1} <\ \infty$. Lemma 1 now implies (25).

For the opposite implication, let f be in the closed unit ball of B^1. As already noted, there is a measure μ with $\parallel \mu \parallel \lesssim \parallel f \parallel_{B^1}$ and

$$f(z) = f(0) + f'(0)z + \int_{\mathbb{D}} \alpha_\zeta(z) d\mu(\zeta).$$

We consider the case $n \geqslant 1$. We obtain

$$\begin{aligned} \nu(z) \mid (C_\varphi D^n f)'(z) \mid &\leqslant \mid \int_{\mathbb{D}} \nu(z)\alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z)d\mu(\zeta) \mid \\ &\leqslant \| \mu \| \sup_{\zeta\in\mathbb{D}} \nu(z) \mid \alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z) \mid \\ &\lesssim \| f \|_{B^1} \sup_{\zeta\in\mathbb{D}} \nu(z) \mid \alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z) \mid \end{aligned}$$

and therefore

$$\sup_{\|f\|_{B^1}\leqslant 1} \nu(z) \mid f^{(n+1)}(\varphi(z))\varphi'(z) \mid \lesssim \sup_{\zeta\in\mathbb{D}} \nu(z) \mid \alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z) \mid .$$

The hypothesis now implies

$$\lim_{|z|\to 1} \sup_{\|f\|_{B^1}\leqslant 1} \nu(z) \mid (C_\varphi D^n f)'(z) \mid = 0$$

and an application of Lemma 1 completes the proof for $n \geqslant 1$.

The proof for $n = 0$ is omitted. □

THEOREM 8. *Let ν be a weight, $n \in \mathbb{N}_0$ and $\varphi \neq \psi \in S(\mathbb{D})$. Then $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is bounded if and only if $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded, $\varphi - \psi \in \mathscr{B}_{\nu,0}$ and $\alpha_\zeta^{(n)} \circ \varphi - \alpha_\zeta^{(n)} \circ \psi \in \mathscr{B}_{\nu,0}$ for every $\zeta \in \mathbb{D}$.*

Proof. Suppose that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is bounded. It is clear that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_\nu$ is bounded. After considering the function $f(z) = z^{n+1}/(n+1)!$ and the family of test functions $\alpha_\zeta(z)$, $\zeta \in \mathbb{D}$ in B^1, we have that $\varphi - \psi \in \mathscr{B}_{\nu,0}$ and $(C_\varphi D^n - C_\psi D^n)\alpha_\zeta \in \mathscr{B}_{\nu,0} = \alpha_\zeta^{(n)} \circ \varphi - \alpha_\zeta^{(n)} \circ \psi \in \mathscr{B}_{\nu,0}$ for every $\zeta \in \mathbb{D}$.

For the converse, let $f \in B^1$ and define μ as in Theorem 3. If $n \geqslant 1$, then by an argument as in the proof of Theorem 3,

$$\nu(z) \mid J_z(f,n,\varphi,\psi) \mid \leqslant \int_{\mathbb{D}} \nu(z) \mid J_z(\alpha_\zeta,n,\varphi,\psi) \mid d \mid \mu \mid (\zeta).$$

Also if $n = 0$, then

$$\begin{aligned} &\nu(z) \mid J_z(\alpha_\zeta,0,\varphi,\psi) \mid \\ &\leqslant \int_{\mathbb{D}} \nu(z) \mid J_z(\alpha_\zeta,0,\varphi,\psi) \mid d \mid \mu \mid (\zeta) + \nu(z) \mid \varphi'(z) - \psi'(z) \mid \mid f'(0) \mid . \end{aligned}$$

The integrand in the above equations tends to zero as $\mid z \mid \to 1$ for every $\zeta \in \mathbb{D}$. Furthermore, the integrand is bounded by M_n, where M_n is as in Theorem 3.

The Lebesgue Dominated Convergence Theorem now implies

$$\lim_{|z|\to 1} \nu(z)|(C_\varphi D^n - C_\psi D^n)f)'(z)| = 0.$$

Therefore, for every $f \in B^1$, we have that $(C_\varphi D^n - C_\psi D^n)f \in \mathscr{B}_{\nu,0}$, from which the boundedness of $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_{\nu,0}$ follows. The proof is complete. □

THEOREM 9. *Let ν be a weight, $n \in \mathbb{N}_0$ and $\varphi \neq \psi \in S(\mathbb{D})$. Assume that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is bounded. Then $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \sup_{\zeta\in\mathbb{D}} \nu(z)\left|\alpha_\zeta^{(n+1)}(\varphi(z))\varphi'(z) - \alpha_\zeta^{(n+1)}(\psi(z))\psi'(z)\right| = 0. \tag{26}$$

Proof. By Lemma 1, the set $\{(C_\varphi D^n - C_\psi D^n)f : f \in B^1, \| f \|_{B^1} \leqslant 1\}$ has compact closure if and only of

$$\lim_{|z|\to 1} \sup_{\|f\|_{B^1}\leqslant 1} \nu(z) \mid ((C_\varphi D^n - C_\psi D^n)f)'(z) \mid = 0. \tag{27}$$

First suppose that (26) holds. Assume that $n \geqslant 1$ and $\| f \|_{B^1} \leqslant 1$. Then there exists a complex Borel measure μ of bounded variation on $\mathbb{D}$ such that $\|\mu\| \lesssim \|f\|_{B^1}$ and

$$f(z) = f(0) + zf'(0) + \int_{\mathbb{T}} \alpha_\zeta(z) d\mu(\zeta).$$

It follows easily that

$$\begin{aligned} \nu(z)|((C_\varphi D^n - C_\psi D^n)f)'(z)| &\leqslant \nu(z) \int_{\mathbb{D}} \left|J_z(\alpha_\zeta, n, \varphi, \psi)\right| d|\mu|(\zeta) \\ &\leqslant \|\mu\| \sup_{\zeta\in\mathbb{D}} \nu(z)\left|J_z(\alpha_\zeta, n, \varphi, \psi)\right| \\ &\lesssim \sup_{\zeta\in\mathbb{D}} \nu(z)\left|J_z(\alpha_\zeta, n, \varphi, \psi)\right|. \end{aligned}$$

Using (26), we see that (27) holds. We conclude that the operator is compact in the case $n \geqslant 1$. The proof for $n = 0$ is similar and is omitted.

Conversely, suppose that $C_\varphi D^n - C_\psi D^n : B^1 \to \mathscr{B}_{\nu,0}$ is compact. Taking the test functions α_ζ, $\zeta \in \mathbb{D}$ in (27), we can easily obtain that (26) follows from 27. This completes the proof. □

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