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POWER–TYPE INEQUALITIES IN THE FRAMEWORK OF p–CONVEXITY WITH APPLICATIONS TO MATRIX INEQUALITIES

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Abstract. Convexity, log-convexity, and p-convexity are fundamental tools in analysis and matrix theory, serving as powerful techniques for deriving functional and operator inequalities. In this paper, we establish several new refinements and generalizations of classical convex inequalities, extending earlier results of Sababheh, Alzer, Heinz, and Young-type. Our approach is based on convexity-inspired factorizations and power-type extensions, which allow us to derive sharper inequalities for p-convex, convex and log-convex functions. Applications are provided to a wide range of mathematical means, including arithmetic, geometric, harmonic, power, and Heinz means, where we obtain improved bounds and new inequalities valid for arbitrary integer powers. Furthermore, we apply these results to matrix analysis, yielding refinements of determinant inequalities, inequalities for unitarily invariant norms, and Hölder-type inequalities. Additional applications are given to generalized numerical radius inequalities for operators. These improvements not only unify and extend several known results but also provide new insights into the interplay between convexity, p-convexity, matrix inequalities, and operator theory.

1. Introduction

Convexity is a fundamental concept in mathematics, with significant applications in analysis, optimization, and functional inequalities. A function is convex if its value at any point on a line segment does not exceed the linear interpolation of its values at the endpoints. This property allows the derivation of inequalities that describe the behavior of functions under linear combinations. Convex functions have been extensively studied both for their intrinsic mathematical interest and for applications in fields such as economics, probability, and numerical analysis.

In recent decades, convexity has been extended from scalar functions to operators and matrices, giving rise to operator convex functions and various forms of matrix convexity. These generalizations facilitate the formulation of inequalities involving matrices, determinants, and norms, which are essential tools in matrix analysis and linear algebra. Such inequalities provide insight into spectral properties, the behavior of matrix functions, and relationships between different matrix means.

Formally, a function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on an interval I if, for all $a,b \in I$ and $\alpha \in [0,1]$,

$$f\big((1-\alpha)a+\alpha b\big) \leqslant (1-\alpha)f(a)+\alpha f(b). \tag{1.1}$$

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This condition implies that the graph of a convex function lies below the straight line connecting any two points on it. Convex functions enjoy several important properties, including monotonicity of derivatives (when differentiable) and the validity of Jensen-type inequalities, making them particularly useful in analysis, optimization, and matrix theory.

A function $f: I \subseteq \mathbb{R} \to (0,\infty)$ is called log-convex if its logarithm is convex; that is, $\log f$ is convex on I. Equivalently, f is log-convex if, for all $a,b \in I$ and $\alpha \in [0,1]$,

$$f\big((1-\alpha)a+\alpha b\big) \leqslant \big(f(a)\big)^{1-\alpha}\big(f(b)\big)^{\alpha}. \tag{1.2}$$

Log-convexity is a stronger condition than ordinary convexity, and it provides multiplicative-type bounds. This property is especially useful in the study of positive functions, determinants, and operator means in matrix analysis.

Both convexity and log-convexity serve as fundamental tools for deriving inequalities, and their interplay allows for generalizations of classical results to matrix and operator settings.

The inequality (1.1) has been explored and refined in several ways in the literature, including reversals. One of the most notable improvements is presented in the following theorem [21].

THEOREM 1.1. *Let $f : [0,1] \to [0,\infty)$ be convex. Then*

$$\left(\frac{\alpha}{\beta}\right)^{\lambda} \leqslant \frac{((1-\alpha)f(0)+\alpha f(1))^{\lambda} - f^{\lambda}(\alpha)}{((1-\beta)f(0)+\beta f(1))^{\lambda} - f^{\lambda}(\beta)} \leqslant \left(\frac{1-\alpha}{1-\beta}\right)^{\lambda} \tag{1.3}$$

for $\lambda \geqslant 1$ and $0 < \alpha \leqslant \beta < 1$.

By choosing the function $f(t) = a^{1-t}b^{t}$ in Theorem 1.1, we arrive at a remarkable inequality established by Alzer and collaborators (see [2]). This result can be viewed as a refinement and extension of the classical Young's inequality. In particular, it highlights how the convexity approach provides sharper bounds than the traditional form, thus offering a significant improvement over the classical statement.

For recent works that refine Theorem 1.1 by adding positive terms, readers are encouraged to refer to [9, 10, 11, 13]. For some interesting publications in this direction, and for applications to obtaining operator and matrix inequalities, the reader is encouraged to consult the following papers. [17, 20, 22, 23, 24, 25, 26].

The concept of convexity has been extensively extended in the literature to encompass a wide variety of convexity types. One of the most significant generalizations is the notion of p-convexity, which unifies and generalizes many classical forms of convexity. Recall that a function $f: I \subseteq [0,\infty) \to \mathbb{R}$ is said to be p-convex on the p-set I (see [27]) if for all $a,b \in I$ and $\alpha \in [0,1]$, the following inequality holds:

$$f\left(\left((1-\alpha)a^{p}+\alpha b^{p}\right)^{\frac{1}{p}}\right) \leqslant (1-\alpha)f(a)+\alpha f(b),$$

where $p \in \mathbb{R}\setminus\{0\}$. This framework provides a rich setting that recovers various well-known convexity classes such as standard convexity, harmonic-convexity, geometric-convexity by choosing suitable values of p.

Recently, Theorem 1.1 has been extended to the framework of p-convexity by Ighachane [12], as follows:

THEOREM 1.2. ([12]) *Let* $f:[a,b]\to\mathbb{R}$ *be a positive* p*-convex function, let* $\lambda\geqslant 1$, $p\in\mathbb{R}\setminus\{0\}$, *and* $0\leqslant\alpha\leqslant\beta\leqslant 1$. *Then,*

$$\left(\frac{\alpha}{\beta}\right)^{\lambda}\leqslant\frac{((1-\alpha)f(a)+\alpha f(b))^{\lambda}-f^{\lambda}\left(((1-\alpha)a^{p}+\alpha b^{p})^{1/p}\right)}{((1-\beta)f(a)+\beta f(b))^{\lambda}-f^{\lambda}\left(((1-\beta)a^{p}+\beta b^{p})^{1/p}\right)}\leqslant\left(\frac{1-\alpha}{1-\beta}\right)^{\lambda}.$$

Taking $p=0$ and $f(t)=t$ in Theorem 1.2, we get the following results:

THEOREM 1.3. ([2]) *Let* $a,b>0$. *Then for all* $\lambda\geqslant 1$ *and* $0<\alpha\leqslant\beta<1$, *we have*

$$\left(\frac{\alpha}{\beta}\right)^{\lambda}\leqslant\frac{((1-\alpha)a+\alpha b)^{\lambda}-\left(a^{1-\alpha}b^{\alpha}\right)^{\lambda}}{((1-\beta)a+\beta b)^{\lambda}-\left(a^{1-\beta}b^{\beta}\right)^{\lambda}}\leqslant\left(\frac{1-\alpha}{1-\beta}\right)^{\lambda}\tag{1.4}$$

For some recent refinements of Alzer-type inequalities, the reader is referred to the following papers [19, 28, 29] and for an analogue of Alzer-type inequalities between the arithmetic-harmonic, or arithmetic-power means, as well as similar inequalities between other means, the reader is referred to [15, 18].

Another interesting class of Heinz-type inequalities, established by Huy et al. [5], is presented as follows:

THEOREM 1.4. ([5]) *Let* $f:[0,1]\to I\subset[0,\infty)$ *be a twice differentiable convex functions, and suppose* $0<\alpha\leqslant\beta<1$.

1. *If* $0<\alpha<\beta\leqslant\frac{1}{2}$, *then*

$$\frac{f(0)-f(\alpha)}{f(0)-f(\beta)}\geqslant\frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

2. *If* $\frac{1}{2}\leqslant\alpha<\beta<1$, *then*

$$\frac{f(0)-f(\alpha)}{f(0)-f(\beta)}\leqslant\frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

The Heinz mean of two positive numbers $a,b>0$ with parameter $\alpha\in[0,1]$ is defined by

$$H_{\alpha}(a,b)=\frac{a^{1-\alpha}b^{\alpha}+a^{\alpha}b^{1-\alpha}}{2}.$$

which is convex in α under appropriate conditions. By applying Theorem 1.4 to this function, we obtain the following improvement of the difference between the Heinz mean and the arithmetic mean, established by Kittaneh in [17]. Here, the arithmetic mean is denoted by $a\nabla b=\frac{a+b}{2}$.

THEOREM 1.5. ([17]) *Let* $a,b>0$ *and* $0<\alpha<\beta\leqslant\frac{1}{2}$. *Then*

$$\frac{a\nabla b-H_{\alpha}(a,b)}{a\nabla b-H_{\beta}(a,b)}\geqslant\frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

The inequality is reversed if $\frac{1}{2}\leqslant\alpha<\beta<1$.

An alternative quadratic version of the preceding theorem was established by Kittaneh et al., in [17]. This result provides a refined formulation of the inequality, which can be stated as follows.

THEOREM 1.6. ([17]) *Let* $a,b>0$ *and* $0<\alpha<\beta\leqslant\frac{1}{2}$. *Then*

$$\frac{(a\nabla b)^2-H_{\alpha}^2(a,b)}{(a\nabla b)^2-H_{\beta}^2(a,b)}\geqslant\frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

The inequality is reversed if $\frac{1}{2}\leqslant\alpha<\beta<1$.

A natural question that arises is whether Theorem 1.2 can be further improved by replacing the bounds $\left(\frac{\alpha}{\beta}\right)^{\lambda}$ and $\left(\frac{1-\alpha}{1-\beta}\right)^{\lambda}$ with sharper, refined estimates?

Another interesting question is whether Theorems 1.5 and 1.6 can be extended to the case of a positive integer m, thus providing more general results in the context of Heinz-type inequalities?

The structure of the paper is as follows. In Section 2, we provide a response to the above questions under certain hypotheses on the function f. Section 3 is devoted to the derivation of new means inequalities, while Section 4 focuses on applications, including bounds for determinants and matrix inequalities for unitarily invariant norms and numerical radii.

2. New power-type for p-convex inequalities

In this section, we prove our main results, starting with an improvement of Theorem 1.2 under certain hypotheses on the function f, as detailed in Remark 2.1. We further extend these results to obtain sharper bounds and refinements for related inequalities, illustrating the effectiveness of our approach.

THEOREM 2.1. *Let* $f:[a,b]\subseteq[0,\infty)\to\mathbb{R}$ *be a* p*-convex function, and let* $m\in\mathbb{N}$ *with* $m\geqslant 1$. *Suppose* $0<\alpha\leqslant\beta<1$.

1. If $p\in\mathbb{R}\setminus\{0\}$, *then*

(1). If f *is increasing, then*

$$\frac{\left((1-\alpha)f(a)+\alpha f(b)\right)^m-f^m\left(\left((1-\alpha)a^p+\alpha b^p\right)^{1/p}\right)}{\left((1-\beta)f(a)+\beta f(b)\right)^m-f^m\left(\left((1-\beta)a^p+\beta b^p\right)^{1/p}\right)}\leqslant\frac{1-\alpha}{1-\beta}.$$

(2). If f is decreasing, then

$$\frac{\alpha}{\beta} \leqslant \frac{\big((1-\alpha)f(a)+\alpha f(b)\big)^m - f^m\Big(\big((1-\alpha)a^p+\alpha b^p\big)^{1/p}\Big)}{\big((1-\beta)f(a)+\beta f(b)\big)^m - f^m\Big(\big((1-\beta)a^p+\beta b^p\big)^{1/p}\Big)}.$$

2. *If $p=0$, then*

(1). If f is increasing, then

$$\frac{\big((1-\alpha)f(a)+\alpha f(b)\big)^m - f^m\big(a^{1-\alpha}b^{\alpha}\big)}{\big((1-\beta)f(a)+\beta f(b)\big)^m - f^m\big(a^{1-\beta}b^{\beta}\big)} \leqslant \frac{1-\alpha}{1-\beta}.$$

(2). If f is decreasing, then

$$\frac{\alpha}{\beta} \leqslant \frac{\big((1-\alpha)f(a)+\alpha f(b)\big)^m - f^m\big(a^{1-\alpha}b^{\alpha}\big)}{\big((1-\beta)f(a)+\beta f(b)\big)^m - f^m\big(a^{1-\beta}b^{\beta}\big)}.$$

Proof. For $v \in (0,1)$, factor the powers as

$$\big((1-v)f(a)+vf(b)\big)^m - \big(f((1-v)a^p+vb^p)^{1/p}\big)^m = \Delta(v)\,S(v),$$

where

$$\Delta(v) := (1-v)f(a)+vf(b) - f\Big(\big((1-v)a^p+vb^p\big)^{1/p}\Big) \geqslant 0,$$

and

$$S(v) := \sum_{i=0}^{m-1} ((1-v)f(a)+vf(b))^{m-1-i}\left(f\left(((1-v)a^p+vb^p)^{1/p}\right)\right)^i.$$

Case 1: f is increasing.
Since $f(a) \leqslant f(b)$, the linear combination

$$(1-v)f(a)+vf(b)$$

is an increasing function of v for $0 \leqslant v \leqslant 1$. Consequently, each term in the sum

$$S(v) := \sum_{i=0}^{m-1} \big((1-v)f(a)+vf(b)\big)^{m-1-i}\big(f((1-v)a^p+vb^p)^{1/p}\big)^i$$

is also increasing in v, because it is a product of increasing functions. Therefore, we have

$$S(\alpha) \leqslant S(\beta) \quad \text{and hence} \quad \frac{S(\alpha)}{S(\beta)} \leqslant 1.$$

On the other hand, applying Theorem 1.2 with $\lambda = 1$ yields

$$\frac{\Delta(\alpha)}{\Delta(\beta)} \leqslant \frac{1-\alpha}{1-\beta}.$$

Combining the two estimates, we obtain

$$\frac{\left((1-\alpha)f(a)+\alpha f(b)\right)^m - f^m\left(\left((1-\alpha)a^p+\alpha b^p\right)^{1/p}\right)}{\left((1-\beta)f(a)+\beta f(b)\right)^m - f^m\left(\left((1-\beta)a^p+\beta b^p\right)^{1/p}\right)} = \frac{\Delta(\alpha)}{\Delta(\beta)}\cdot\frac{S(\alpha)}{S(\beta)} \leqslant \frac{1-\alpha}{1-\beta},$$

which gives the desired inequality.

Case 2: f is decreasing.
If f is p-convex and decreasing on the interval $[a,b]$, then the function

$$g(x) := f\left((a^p+b^p-x^p)^{1/p}\right)$$

is also p-convex on the same interval and increasing.

Notice that if $0<\alpha\leqslant\beta<1$, then

$$0<1-\beta\leqslant 1-\alpha<1.$$

Hence, by replacing α,β and f with $1-\beta, 1-\alpha$ and g, respectively, in the first case, we obtain the desired inequality.

Other inequalities follow by a similar argument. This completes the proof. □

REMARK 2.1. The fact that Theorem 2.1 provides a significant improvement and extension of Theorem 1.2 follows from noting that when $0<\alpha\leqslant\beta<1$ and $m\geqslant 1$, one has

$$\left(\frac{\alpha}{\beta}\right)^m \leqslant \frac{\alpha}{\beta} \quad\text{and}\quad \frac{1-\alpha}{1-\beta} \leqslant \left(\frac{1-\alpha}{1-\beta}\right)^m.$$

Notice that this refinement is proved only for integer values of m.

By setting $p=1$ in the previous theorem, we immediately obtain a refinement of Theorem 1.1. This special case not only recovers the classical result but also provides sharper bounds and improved estimates, illustrating the broader applicability of the general p-convex framework.

THEOREM 2.2. *Let $f:[0,1]\to I\subset[0,\infty)$ be convex, and let $m\in\mathbb{N}$ with $m\geqslant 1$. Suppose $0<\alpha\leqslant\beta<1$.*

1. *If f is increasing, then*

$$\frac{\left((1-\alpha)f(0)+\alpha f(1)\right)^m - \left(f(\alpha)\right)^m}{\left((1-\beta)f(0)+\beta f(1)\right)^m - \left(f(\beta)\right)^m} \leqslant \frac{1-\alpha}{1-\beta}.$$

2. *If f is decreasing, then*

$$\frac{\alpha}{\beta} \leqslant \frac{\left((1-\alpha)f(0)+\alpha f(1)\right)^m - \left(f(\alpha)\right)^m}{\left((1-\beta)f(0)+\beta f(1)\right)^m - \left(f(\beta)\right)^m}.$$

By replacing f with $\log f$ in the previous theorem, we obtain the following result:

THEOREM 2.3. *Let* $f:[0,1]\to(0,\infty)$ *be* $\log$*-convex, and let* $m\in\mathbb{N}$, $m\geqslant 1$. *Suppose* $0<\alpha\leqslant\beta<1$.

1. *If* f *is increasing, then*

$$\frac{\left(\log(f(0)^{1-\alpha}f(1)^{\alpha})\right)^m-(\log f(\alpha))^m}{\left(\log(f(0)^{1-\beta}f(1)^{\beta})\right)^m-(\log f(\beta))^m}\leqslant\frac{1-\alpha}{1-\beta}.$$

2. *If* f *is decreasing, then*

$$\frac{\alpha}{\beta}\leqslant\frac{\left(\log(f(0)^{1-\alpha}f(1)^{\alpha})\right)^m-(\log f(\alpha))^m}{\left(\log(f(0)^{1-\beta}f(1)^{\beta})\right)^m-(\log f(\beta))^m}.$$

By taking $p=-1$, we obtain the following theorem, which presents a new improvement of power-type inequalities for harmonic convex functions.

THEOREM 2.4. *Let* $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ *be a harmonic convex function, and let* $m\in\mathbb{N}$ *with* $m\geqslant 1$. *Suppose* $0<\alpha\leqslant\beta<1$.

1. *If* f *is increasing, then*

$$\frac{\left((1-\alpha)f(a)+\alpha f(b)\right)^m-f^m\left(\left((1-\alpha)a^{-1}+\alpha b^{-1}\right)^{-1}\right)}{\left((1-\beta)f(a)+\beta f(b)\right)^m-f^m\left(\left((1-\beta)a^{-1}+\beta b^{-1}\right)^{-1}\right)}\leqslant\frac{1-\alpha}{1-\beta}.$$

2. *If* f *is decreasing, then*

$$\frac{\alpha}{\beta}\leqslant\frac{\left((1-\alpha)f(a)+\alpha f(b)\right)^m-f^m\left(\left((1-\alpha)a^{-1}+\alpha b^{-1}\right)^{-1}\right)}{\left((1-\beta)f(a)+\beta f(b)\right)^m-f^m\left(\left((1-\beta)a^{-1}+\beta b^{-1}\right)^{-1}\right)}.$$

In the next theorem, we present a generalization of Theorem 1.4 to an arbitrary power m, where m is a positive integer. This extension allows us to obtain refined inequalities for higher powers, providing a broader framework that encompasses the original result as a special case.

THEOREM 2.5. *Let* $f:[0,1]\to I\subset[0,\infty)$ *be convex, and let* $m\in\mathbb{N}$ *with* $m\geqslant 1$. *Suppose* $0<\alpha\leqslant\beta<1$.

1. *If* $0<\alpha<\beta\leqslant\frac{1}{2}$, *and* f *is decreasing, then*

$$\frac{f^m(0)-f^m(\alpha)}{f^m(0)-f^m(\beta)}\geqslant\frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

2. *If* $\frac{1}{2} \leqslant \alpha < \beta < 1$, *and* f *is increasing, then*

$$\frac{f^m(0) - f^m(\alpha)}{f^m(0) - f^m(\beta)} \leqslant \frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

Proof. Set

$$A := f(0), \qquad B := f(\alpha), \qquad C := f(\beta).$$

Since f is convex and nonnegative, we can use the factorization

$$A^m - t^m = (A-t)\sum_{k=0}^{m-1} A^{m-1-k}t^k, \quad t \geqslant 0.$$

Thus,

$$\frac{A^m - B^m}{A^m - C^m} = \frac{A-B}{A-C} \cdot \frac{\sum_{k=0}^{m-1} A^{m-1-k}B^k}{\sum_{k=0}^{m-1} A^{m-1-k}C^k} = \frac{A-B}{A-C} \cdot \frac{S(\alpha)}{S(\beta)},$$

where $S(t) := \sum_{k=0}^{m-1} A^{m-1-k}(f(t))^k$.

Case 1*:* $0 < \alpha < \beta \leqslant \frac{1}{2}$, *and* f *decreasing.*

Here $0 \leqslant C \leqslant B \leqslant A$. Note that each term in the sum $S(t)$ is decreasing in $f(t)$ since $A^{m-1-k} \geqslant 0$ and $(f(t))^k \geqslant 0$. Therefore, $S(t)$ is decreasing in t, and we have

$$S(\alpha) \geqslant S(\beta) \quad \Rightarrow \quad \frac{S(\alpha)}{S(\beta)} \geqslant 1.$$

Moreover, by Theorem 1.4 the standard estimate

$$\frac{A-B}{A-C} \geqslant \frac{\alpha(1-\alpha)}{\beta(1-\beta)}$$

holds. Combining these gives

$$\frac{A^m - B^m}{A^m - C^m} = \frac{A-B}{A-C} \cdot \frac{S(\alpha)}{S(\beta)} \geqslant \frac{\alpha(1-\alpha)}{\beta(1-\beta)},$$

which proves the first inequality.

Case 2*:* $\frac{1}{2} \leqslant \alpha < \beta < 1$, *and* f *increasing.*

Here $0 \leqslant A \leqslant B \leqslant C$. Note that each term in the sum $S(t)$ is increasing in $f(t)$ since $A^{m-1-k} \geqslant 0$ and $(f(t))^k \geqslant 0$. Therefore, $S(t)$ is increasing in t, and we have

$$S(\alpha) \leqslant S(\beta) \quad \Rightarrow \quad \frac{S(\alpha)}{S(\beta)} \leqslant 1.$$

Moreover, by Theorem 1.4 the standard estimate

$$\frac{A-B}{A-C} \leqslant \frac{\alpha(1-\alpha)}{\beta(1-\beta)}$$

holds. Combining these gives

$$\frac{A^m-B^m}{A^m-C^m} = \frac{A-B}{A-C}\cdot\frac{S(\alpha)}{S(\beta)} \leqslant \frac{\alpha(1-\alpha)}{\beta(1-\beta)},$$

which complete the proof. □

3. Application to means

The concept of means plays a central role in mathematical analysis and its applications. Given two positive real numbers $a,b>0$, a *mean* $M(a,b)$ is a quantity that lies between them, i.e.,

$$\min\{a,b\} \leqslant M(a,b) \leqslant \max\{a,b\}.$$

Several classical means are of particular importance:

(1). Arithmetic weighted mean

The arithmetic weighted mean (AM) of two positive numbers x and y with weight $\alpha\in[0,1]$ is defined as:

$$x\nabla_\alpha y := (1-\alpha)x+\alpha y.$$

This is the most familiar mean and is linear in both variables. It represents a straight interpolation between x and y.

(2). Geometric mean

The geometric weighted mean (GM) captures multiplicative relationships and is defined (for $x,y>0$) by:

$$x\sharp_\alpha x := x^{1-\alpha}y^\alpha.$$

This mean lies between the harmonic and arithmetic means and is particularly useful in proportional growth and information theory.

(3). Harmonic mean

The harmonic weighted mean is defined as the reciprocal of the arithmetic weighted mean of the reciprocals:

$$x!_\alpha y := \left((1-\alpha)x^{-1}+\alpha y^{-1}\right)^{-1}, \quad x,y>0.$$

It is often used in situations involving rates (e.g., average speed or electrical resistance in parallel).

(4). Power means

More generally, the power weighted mean (or Hölder mean) of order $p \in \mathbb{R}$ is defined by:

$$x\sharp_{\alpha,p}y := ((1-\alpha)x^p + \alpha y^p)^{\frac{1}{p}}, \quad p \neq 0,$$

and for $p = 0$, it is taken as the geometric weighted mean:

$$x\sharp_{\alpha}y := x^{1-\alpha}y^{\alpha}.$$

Special cases include:

- $p = 1$: Arithmetic mean
- $p = 0$: Geometric mean
- $p = -1$: Harmonic mean
- Monotonicity in p: For fixed x, y, α, if $p \leqslant q$, then $x\sharp_{\alpha,p}y \leqslant x\sharp_{\alpha,q}y$.
- $p \to \infty$, $x\sharp_{\alpha,p}y \to \max\{x,y\}$; and $p \to -\infty$, $x\sharp_{\alpha,p}y \to \min\{x,y\}$.

It is well known that the function

$$f(\alpha) = x\sharp_{\alpha,p}y := ((1-\alpha)x^p + \alpha y^p)^{\frac{1}{p}} \quad \text{for p} \leqslant 1$$

is convex.

Other notable examples include the quadratic mean (or root mean square) and the family of power means. Such notions provide powerful tools in establishing inequalities and studying convexity properties in various mathematical settings.

By applying Theorem 2.2 to the function

$$f(\alpha) = x\sharp_{\alpha,p}y := ((1-\alpha)x^p + \alpha y^p)^{\frac{1}{p}}, \quad p \leqslant 1,$$

We obtain the following refinement of the difference between the power mean and the arithmetic mean inequalities.

THEOREM 3.1. *Let $a, b > 0$, $p \leqslant 1$, and $m \in \mathbb{N}$ with $m \geqslant 1$. Then, for $0 < \alpha \leqslant \beta < 1$, we have:*

1. *If $a \leqslant b$,*
$$\frac{((1-\alpha)a + \alpha b)^m - (a\sharp_{\alpha,p}b)^m}{((1-\beta)a + \beta b)^m - (a\sharp_{\beta,p}b)^m} \leqslant \frac{1-\alpha}{1-\beta}.$$

2. *If $a \geqslant b$,*
$$\frac{\alpha}{\beta} \leqslant \frac{((1-\alpha)a + \alpha b)^m - (a\sharp_{\alpha,p}b)^m}{((1-\beta)a + \beta b)^m - (a\sharp_{\beta,p}b)^m}.$$

Before presenting the harmonic mean version, we note that by taking $p=-1$ in the weighted p-power mean, one obtains the weighted harmonic mean. The following theorem provides a refinement of the classical arithmetic-harmonic mean inequalities for any positive integer m.

THEOREM 3.2. *Let $a,b>0$ and $m\in\mathbb{N}$ with $m\geqslant 1$. Then, for $0<\alpha\leqslant\beta<1$, we have:*

1. *If $a\leqslant b$,*

$$\frac{\big((1-\alpha)a+\alpha b\big)^m-\big(a!_\alpha b\big)^m}{\big((1-\beta)a+\beta b\big)^m-\big(a!_\beta b\big)^m}\leqslant\frac{1-\alpha}{1-\beta}.$$

2. *If $a\geqslant b$,*

$$\frac{\alpha}{\beta}\leqslant\frac{\big((1-\alpha)a+\alpha b\big)^m-\big(a!_\alpha b\big)^m}{\big((1-\beta)a+\beta b\big)^m-\big(a!_\beta b\big)^m}.$$

Before presenting the main result, we note that by taking the limit $p\to 0$ in the weighted p-power mean, one obtains the weighted geometric mean. The following theorem provides a refinement of the classical arithmetic-geometric mean inequalities in terms of powers for any positive integer m.

THEOREM 3.3. *Let $a,b>0$ and $m\in\mathbb{N}$ with $m\geqslant 1$. Then, for $0<\alpha\leqslant\beta<1$, we have:*

1. *If $a\leqslant b$,*

$$\frac{\big((1-\alpha)a+\alpha b\big)^m-\big(a\sharp_\alpha b\big)^m}{\big((1-\beta)a+\beta b\big)^m-\big(a\sharp_\beta b\big)^m}\leqslant\frac{1-\alpha}{1-\beta}.$$

2. *If $a\geqslant b$,*

$$\frac{\alpha}{\beta}\leqslant\frac{\big((1-\alpha)a+\alpha b\big)^m-\big(a\sharp_\alpha b\big)^m}{\big((1-\beta)a+\beta b\big)^m-\big(a\sharp_\beta b\big)^m}.$$

The following theorem provides a refinement of the classical arithmetic-Heinz mean inequalities for any positive integer m.

THEOREM 3.4. *Let $a,b>0$ and $m\in\mathbb{N}$ with $m\geqslant 1$. Then, we have*

1. *If $0<\alpha<\beta\leqslant\frac{1}{2}$ and $a\geqslant b$, then*

$$\frac{(a\nabla b)^m-H_\alpha^m(a,b)}{(a\nabla b)^m-H_\beta^m(a,b)}\geqslant\frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

2. *If $\frac{1}{2}\leqslant\alpha<\beta<1$ and $a\leqslant b$, then*

$$\frac{(a\nabla b)^m-H_\alpha^m(a,b)}{(a\nabla b)^m-H_\beta^m(a,b)}\leqslant\frac{\alpha(1-\alpha)}{\beta(1-\beta)}.$$

4. Application to refine some classical matrix inequalities

In this part of the paper, we select suitable convex and log-convex functions to derive refinements of results related to matrices.

We begin by introducing some notations and briefly reviewing relevant concepts related to matrices. Let $\mathbf{M}_n$ denote the algebra of all complex matrices of size $n \times n$, with identity I. A matrix $\mathscr{A} \in \mathbf{M}_n$ is called Hermitian if it satisfies $\mathscr{A} = \mathscr{A}^*$, where $\mathscr{A}^*$ is the adjoint of $\mathscr{A}$. We use the notation $\mathscr{A} \geqslant 0$ ($\mathscr{A} > 0$) to indicate that $\mathscr{A}$ is positive semi-definite (positive definite). If $\mathscr{A}$ and $\mathscr{B}$ are Hermitian and $\mathscr{A} - \mathscr{B}$ is positive semi-definite, we write $\mathscr{A} \geqslant \mathscr{B}$.

The set of all positive semi-definite matrices is denoted by $\mathbf{M}_n^+$, and the set of positive definite matrices within this class is denoted by $\mathbf{M}_n^{++}$. The singular values of a matrix $\mathscr{A} \in \mathbf{M}_n$ are defined as the eigenvalues of the positive semi-definite matrix $|\mathscr{A}| := (\mathscr{A}^*\mathscr{A})^{1/2}$, which we denote by $s_j(\mathscr{A})$ for $j = 1, 2, \ldots, n$, arranged in non-increasing order.

A unitarily invariant norm $\|\cdot\|$ on $\mathbf{M}_n$ is a matrix norm that satisfies the invariance property:

$$\|\mathscr{U}\mathscr{A}\mathscr{V}\| = \|\mathscr{A}\| \quad \text{for all} \quad \mathscr{A} \in \mathbf{M}_n \text{ and unitary matrices } \mathscr{U}, \mathscr{V} \in \mathbf{M}_n.$$

One example of a unitarily invariant norm is the trace norm, given by:

$$\|\mathscr{A}\|_1 := \operatorname{tr}|\mathscr{A}| = \sum_{j=1}^{n} s_j(\mathscr{A}),$$

where tr denotes the trace operator. This norm is unitarily invariant. Another important example is the Hilbert-Schmidt norm $\|\cdot\|_2$, which is defined as:

$$\|\mathscr{A}\|_2 := \sqrt{\operatorname{tr}(\mathscr{A}\mathscr{A}^*)} = \left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2}, \quad \text{for} \quad \mathscr{A} = (a_{i,j}).$$

4.1. Application to some matrix means inequalities

Determinants and matrix means are fundamental concepts in linear algebra, serving as powerful tools for analyzing and comparing matrices. These inequalities provide essential information about the spectral properties of matrices, such as their eigenvalues and norms. In particular, determinant inequalities describe the behavior of matrix functions, while matrix mean inequalities allow comparisons among different types of means, including arithmetic, geometric, and harmonic. Such results find applications in areas like matrix analysis, optimization, and other mathematical fields.

In the following subsection, we present several noteworthy inequalities related to determinants and matrix means.

When referring to the eigenvalues of a Hermitian matrix $\mathscr{A}$, we denote by $\lambda_j(\mathscr{A})$ the j-th eigenvalue arranged in decreasing order.

For matrices $\mathscr{A}, \mathscr{B} \in \mathbf{M}_n^{++}$, a parameter $\alpha \in [0,1]$, and a nonzero real number $p \in \mathbb{R} \setminus \{0\}$, the following matrix means are defined:

- The arithmetic mean:

$$\mathscr{A}\nabla_\alpha\mathscr{B} := (1-\alpha)\mathscr{A} + \alpha\mathscr{B},$$

- The geometric mean:

$$\mathscr{A}\sharp_\alpha\mathscr{B} := \mathscr{A}^{1/2}\left(\mathscr{A}^{-1/2}\mathscr{B}\mathscr{A}^{-1/2}\right)^\alpha \mathscr{A}^{1/2},$$

- The harmonic mean:

$$\mathscr{A}!_\alpha\mathscr{B} := \left((1-\alpha)\mathscr{A}^{-1} + \alpha\mathscr{B}^{-1}\right)^{-1},$$

- The power mean:

$$\mathscr{A}\sharp_{p,\alpha}\mathscr{B} := \mathscr{A}^{1/2}\left((1-\alpha)I + \alpha\left(\mathscr{A}^{-1/2}\mathscr{B}\mathscr{A}^{-1/2}\right)^p\right)^{1/p}\mathscr{A}^{1/2}, \quad p\in\mathbb{R}\setminus\{0\}.$$

As a result, the limit as $p\to 0$ gives the geometric mean, while for $p=1$ and $p=-1$, we obtain the arithmetic and harmonic means, respectively.

To prove a particular determinant version of Theorem 3.1, we need the following known Lemma.

LEMMA 4.1. ([4]) *Let $u=[u_i]$ and $v=[v_i]$, $i=1,2,\ldots,n$, be positive real numbers. Then*

$$\left(\prod_{i=1}^n u_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n v_i\right)^{\frac{1}{n}} \leqslant \left(\prod_{i=1}^n (u_i+v_i)\right)^{\frac{1}{n}}.$$

THEOREM 4.1. *Let $\mathscr{A},\mathscr{B}\in\mathbf{M}_n^{++}(\mathbb{C})$, $0<\alpha\leqslant\beta<1$, $p\leqslant 1$, and $m\geqslant 1$.*

1. *If $\mathscr{B}\leqslant\mathscr{A}$, then*

$$\det(\mathscr{A}\nabla_\alpha\mathscr{B})^{\frac{m}{n}} - \det(\mathscr{A}\sharp_{p,\alpha}\mathscr{B})^{\frac{m}{n}} \geqslant \frac{\alpha}{\beta}\cdot\det(\mathscr{A}\nabla_\beta\mathscr{B} - \mathscr{A}\sharp_{p,\beta}\mathscr{B})^{\frac{m}{n}}. \tag{4.1}$$

2. *If $\mathscr{B}\geqslant\mathscr{A}$, then*

$$\det(\mathscr{A}\nabla_\alpha\mathscr{B})^{m} - \det(\mathscr{A}\sharp_{p,\alpha}\mathscr{B})^{m} \leqslant \frac{1-\alpha}{1-\beta}\cdot\det(\mathscr{A}\nabla_\beta\mathscr{B} - \mathscr{A}\sharp_{p,\beta}\mathscr{B})^{\frac{m}{n}}. \tag{4.2}$$

Proof. Let $\mathscr{A},\mathscr{B}\in\mathbf{M}_n^{++}(\mathbb{C})$ with $\mathscr{A}\geqslant\mathscr{B}$, and define $\mathscr{T}=\mathscr{A}^{-\frac{1}{2}}\mathscr{B}\mathscr{A}^{-\frac{1}{2}}$. The determinant of a positive definite matrix is the product of its singular values.

By the second inequality of Theorem 3.1, for $m\geqslant 1$ and $i=1,\cdots,n$, the singular values of $\mathscr{T}$ satisfy

$$\frac{(1\nabla_\alpha s_i(\mathscr{T}))^m - (1\sharp_{p,\alpha}s_i(\mathscr{T}))^m}{(1\nabla_\beta s_i(\mathscr{T}))^m - (1\sharp_{p,\beta}s_i(\mathscr{T}))^m} \geqslant \frac{\alpha}{\beta}, \tag{4.3}$$

since $s_i(\mathscr{T}) \leqslant 1$.

Using the inequalities for singular values and the determinant formula, we get

$$\det(I\nabla_\alpha \mathscr{T})^{\frac{m}{n}} = \left(\prod_{i=1}^{n} 1\nabla_\alpha s_i(\mathscr{T})\right)^{\frac{m}{n}}$$

$$\geqslant \left(\prod_{i=1}^{n}\left[\frac{\alpha}{\beta}\Big((1\nabla_\beta s_i(\mathscr{T}))^m - (1\sharp_{p,\beta} s_i(\mathscr{T}))^m\Big) + (1\sharp_{p,\alpha} s_i(\mathscr{T}))^m\right]\right)^{\frac{1}{n}} \quad \text{(by (4.3))}$$

$$\geqslant \prod_{i=1}^{n}\left[\frac{\alpha}{\beta}\Big((1\nabla_\beta s_i(\mathscr{T}))^m - (1\sharp_{p,\beta} s_i(\mathscr{T}))^m\Big)\right]^{\frac{1}{n}} + \prod_{i=1}^{n}(1\sharp_{p,\alpha} s_i(\mathscr{T}))^{\frac{m}{n}} \quad \text{(by Lemma 4.1)}$$

$$\geqslant \frac{\alpha}{\beta}\prod_{i=1}^{n}(1\nabla_\beta s_i(\mathscr{T}) - 1\sharp_{p,\beta} s_i(\mathscr{T}))^{\frac{m}{n}} + \prod_{i=1}^{n}(1\sharp_{p,\alpha} s_i(\mathscr{T}))^{\frac{m}{n}}$$

$$= \frac{\alpha}{\beta}\det(I\nabla_\beta \mathscr{T} - I\sharp_{p,\beta}\mathscr{T})^{\frac{m}{n}} + \det(I\sharp_{p,\alpha}\mathscr{T})^{\frac{m}{n}},$$

where we used $1\nabla_\beta s_i(\mathscr{T}) \geqslant 1\sharp_{p,\beta} s_i(\mathscr{T})$ and $(a-b)^m \leqslant a^m - b^m$ for $a \geqslant b$ and $m \geqslant 1$.

Multiplying both sides by $\det(\mathscr{A}^{1/2})^{\frac{m}{n}}$ completes the proof.

Notice that if $0 < \alpha \leqslant \beta < 1$, then $0 < 1-\beta \leqslant 1-\alpha < 1$. The other inequality can then be obtained by interchanging $\mathscr{A}$ and $\mathscr{B}$, as well as replacing α with $1-\beta$ and β with $1-\alpha$. $\square$

4.2. Matrix norm inequalities via log-convexity

For all $\mathscr{X}, \mathscr{A} \in \mathbf{M}_n$, any real number $r > 0$, and every unitarily invariant norm $\|\cdot\|$, Horn and Mathias [7, 8] derived the following matrix Cauchy-Schwarz inequality:

$$\|\,|\mathscr{X}\mathscr{A}^*|^r\,\|^2 \leqslant \|(\mathscr{X}^*\mathscr{X})^r\| \cdot \|(\mathscr{A}^*\mathscr{A})^r\|. \tag{4.4}$$

Bhatia and Davis established a more general form of the Cauchy-Schwarz inequality for $\mathscr{A}, X, \mathscr{X} \in \mathbf{M}_n$, and $r > 0$,

$$\|\,|\mathscr{X}^*X\mathscr{A}|^r\,||| \leqslant |||\,|\mathscr{X}\mathscr{X}^*X|^r\,\| \cdot \|\,|X\mathscr{A}\mathscr{A}^*|^r\,\|. \tag{4.5}$$

When $\mathscr{A}, \mathscr{X} \in \mathbf{M}_n^+$, (4.5) can be written as

$$\|\,|\mathscr{X}^{\frac{1}{2}}X\mathscr{A}^{\frac{1}{2}}|^r\,\|^2 \leqslant \|\,|\mathscr{X}X|^r\,\| \cdot \|\,|X\mathscr{A}|^r\,\|. \tag{4.6}$$

For $\mathscr{A}, \mathscr{B} \in \mathbf{M}_n^+$ and $\alpha \in [0,1]$, the following Hölder-type inequality holds [14]

$$||\,|\mathscr{A}^{1-\alpha}X\mathscr{B}^{\alpha}|^r\,|| \leqslant ||\,|\mathscr{A}X|^r\,||^{1-\alpha} \cdot ||\,|X\mathscr{B}|^r\,||^{\alpha}. \tag{4.7}$$

For every $\mathscr{A}, \mathscr{B} \in \mathbf{M}_n^{++}$ and $\mathscr{X} \in \mathbf{M}_n$, the function $f(\alpha) = ||\,|\mathscr{A}^{\alpha}\mathscr{X}\mathscr{B}^{\alpha}|^r\,||$ is log-convex. Furthermore, if $\mathscr{A}$ and $\mathscr{B}$ are expansive, meaning that $\mathscr{A}, \mathscr{B} \geqslant \mathscr{I}$, then the function is increasing. On the other hand, if $\mathscr{A}$ and $\mathscr{B}$ are contractive, meaning that $\mathscr{A},\mathscr{B} \leqslant \mathscr{I}$, then f is decreasing. For further details, refer to [14, 16, 20] and [26, Section 3]. An application of Theorem 2.2 is the following sequence of inequalities, which refine the classical Hölder-type inequality for unitarily invariant norms.

THEOREM 4.2. *Let $\mathscr{A},\mathscr{B} \in \mathbf{M}_n^{++}$, $\mathscr{X} \in \mathbf{M}_n$, and let $0<\alpha \leqslant \beta <1$. For a positive integer m, the following inequalities hold:*

1. If $\mathscr{A}$ and $\mathscr{B}$ are expansive, then

$$\frac{\Big((1-\alpha)\log \|\,|\mathscr{X}|^r\,\| + \alpha \log \|\,|\mathscr{A}\mathscr{X}\mathscr{B}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\alpha \mathscr{X}\mathscr{B}^\alpha|^r\,\|\big)^m}{\Big((1-\beta)\log \|\,|\mathscr{X}|^r\,\| + \beta \log \|\,|\mathscr{A}\mathscr{X}\mathscr{B}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\beta \mathscr{X}\mathscr{B}^\beta|^r\,\|\big)^m} \leqslant \frac{1-\alpha}{1-\beta}.$$

2. If $\mathscr{A}$ and $\mathscr{B}$ are contractive, then

$$\frac{\alpha}{\beta} \leqslant \frac{\Big((1-\alpha)\log \|\,|\mathscr{X}|^r\,\| + \alpha \log \|\,|\mathscr{A}\mathscr{X}\mathscr{B}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\alpha \mathscr{X}\mathscr{B}^\alpha|^r\,\|\big)^m}{\Big((1-\beta)\log \|\,|\mathscr{X}|^r\,\| + \beta \log \|\,|\mathscr{A}\mathscr{X}\mathscr{B}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\beta \mathscr{X}\mathscr{B}^\beta|^r\,\|\big)^m}.$$

For every $\mathscr{A},\mathscr{B} \in \mathbf{M}_n^{++}$ and $\mathscr{X} \in \mathbf{M}_n$, the function $f(\alpha) = \|\,|\mathscr{A}^\alpha \mathscr{X}\mathscr{B}^{1-\alpha}|^r\,\|$ is log-convex. Moreover, if $\mathscr{A}$ is expansive and $\mathscr{B}$ is contractive, the function is increasing, while if $\mathscr{A}$ is contractive and $\mathscr{B}$ is expansive, the function is decreasing. By applying Theorem 2.2 once more, we obtain the following inequalities.

THEOREM 4.3. *Let $\mathscr{A},\mathscr{B} \in \mathbf{M}_n^{++}$, $\mathscr{X} \in \mathbf{M}_n$, and let $0<\alpha \leqslant \beta <1$. For a positive integer m,*

1. If $\mathscr{A}$ and $\mathscr{B}$ are expansive, then

$$\frac{\Big((1-\alpha)\log \|\,|\mathscr{X}\mathscr{B}|^r\,\| + \alpha \log \|\,|\mathscr{A}\mathscr{X}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\alpha \mathscr{X}\mathscr{B}^{1-\alpha}|^r\,\|\big)^m}{\Big((1-\beta)\log \|\,|\mathscr{X}\mathscr{B}|^r\,\| + \beta \log \|\,|\mathscr{A}\mathscr{X}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\beta \mathscr{X}\mathscr{B}^{1-\beta}|^r\,\|\big)^m} \leqslant \frac{1-\alpha}{1-\beta}.$$

2. If $\mathscr{A}$ and $\mathscr{B}$ are contractive, then

$$\frac{\alpha}{\beta} \leqslant \frac{\Big((1-\alpha)\log \|\,|\mathscr{X}\mathscr{B}|^r\,\| + \alpha \log \|\,|\mathscr{A}\mathscr{X}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\alpha \mathscr{X}\mathscr{B}^{1-\alpha}|^r\,\|\big)^m}{\Big((1-\beta)\log \|\,|\mathscr{X}\mathscr{B}|^r\,\| + \beta \log \|\,|\mathscr{A}\mathscr{X}|^r\,\|\Big)^m - \big(\log \|\,|\mathscr{A}^\beta \mathscr{X}\mathscr{B}^{1-\beta}|^r\,\|\big)^m}.$$

In a similar manner, we have the following theorem.

THEOREM 4.4. *Let $\mathscr{A},\mathscr{B} \in \mathbf{M}_n^{++}$, $\mathscr{X} \in \mathbf{M}_n$, and let $0<\alpha \leqslant \beta <1$. For a positive integer m,*

1. *If $\mathscr{A}$ is contractive and $\mathscr{B}$ is expansive, then*

$$\frac{\Big((1-\alpha)\log\|\mathscr{A}\mathscr{X}\|+\alpha\log\|\mathscr{X}\mathscr{B}\|\Big)^m-\big(\log\|\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{B}^{\alpha}\|\big)^m}{\Big((1-\beta)\log\|\mathscr{A}\mathscr{X}\|+\beta\log\|\mathscr{X}\mathscr{B}\|\Big)^m-\big(\log\|\mathscr{A}^{1-\beta}\mathscr{X}\mathscr{B}^{\beta}\|\big)^m}\leqslant\frac{1-\alpha}{1-\beta}.$$

2. *If both $\mathscr{A}$ and $\mathscr{B}$ are contractive, then*

$$\frac{\alpha}{\beta}\leqslant\frac{\Big((1-\alpha)\log\|\mathscr{A}\mathscr{X}\|+\alpha\log\|\mathscr{X}\mathscr{B}\|\Big)^m-\big(\log\|\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{B}^{\alpha}\|\big)^m}{\Big((1-\beta)\log\|\mathscr{A}\mathscr{X}\|+\beta\log\|\mathscr{X}\mathscr{B}\|\Big)^m-\big(\log\|\mathscr{A}^{1-\beta}\mathscr{X}\mathscr{B}^{\beta}\|\big)^m}.$$

REMARK 4.1. Note that Theorems 4.2, 4.3, and 4.4 present another significant refinement of the classical Hölder's type inequality for unitarily invariant norms. These refinements are both clear and similar to those obtained in [11, 13], offering further insights into the topic.

4.3. Operators inequalities for generalized numerical radius

Let $N(\cdot)$ be a norm on $\mathbf{M}_n$. The generalized numerical radius, induced by $N(\cdot)$, of a matrix $\mathscr{A}\in\mathbf{M}_n$, denoted as $w_N(\mathscr{A})$, is defined as the supremum of the norm over the real parts of all rotations of $\mathscr{A}$, that is:

$$w_N(\mathscr{A})=\sup_{\theta\in\mathbb{R}}N\left(\mathrm{Re}\left(e^{i\theta}\mathscr{A}\right)\right).$$

Here, $\mathscr{X}=\mathrm{Re}(\mathscr{X})+i\,\mathrm{Im}(\mathscr{X})$ is the Cartesian decomposition of $\mathscr{X}\in\mathbf{M}_n$, where $\mathrm{Re}(\mathscr{X})=\frac{\mathscr{X}+\mathscr{X}^*}{2}$ and $\mathrm{Im}(\mathscr{X})=\frac{\mathscr{X}-\mathscr{X}^*}{2i}$, with $\mathscr{X}^*$ being the conjugate transpose of $\mathscr{X}$. A straightforward computation shows that when N represents the usual operator norm (or the spectral norm), inherited from the inner product on $\mathbb{C}^n$, $w_N(\cdot)$ coincides with the standard numerical radius norm $w(\cdot)$, which is given by:

$$w(\mathscr{A})=\sup_{\|x\|=1}|\langle\mathscr{A}x,x\rangle|.$$

On the other hand, it is known that if $\mathscr{A}\in\mathbf{M}_n^+$ and $\mathscr{X}\in\mathbf{M}_n$, then the function $f(\alpha)=w_N\left(\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{A}^{\alpha}+\mathscr{A}^{\alpha}\mathscr{X}\mathscr{A}^{1-\alpha}\right)$ is convex on $[0,1]$, decreasing on $[0,\frac{1}{2}]$, and increasing on $[\frac{1}{2},1]$ for any unitarily invariant norm $N(\cdot)$ on $\mathbf{M}_n$, [1]. Consequently, we can apply Theorem 2.2 using this function to derive the following Heinz-type inequalities for the numerical radius. In the sequel, $N(\cdot)$ represents an arbitrary unitarily invariant norm on $\mathbf{M}_n$.

THEOREM 4.5. *Let $\mathscr{A}\in\mathbf{M}_n^+$ and $\mathscr{X}\in\mathbf{M}_n$, $0\leqslant\alpha\leqslant\beta<1$ and $m\geqslant 1$.*

1. *If $\alpha\in[\frac{1}{2},1]$, then*

$$\frac{w_N^m(\mathscr{A}\mathscr{X}+\mathscr{X}\mathscr{A}))-w_N^m(\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{A}^{\alpha}+\mathscr{A}^{\alpha}\mathscr{X}\mathscr{A}^{1-\alpha})}{w_N^m(\mathscr{A}\mathscr{X}+\mathscr{X}\mathscr{A}))-w_N^m(\mathscr{A}^{1-\beta}\mathscr{X}\mathscr{A}^{\beta}+\mathscr{A}^{\beta}\mathscr{X}\mathscr{A}^{1-\beta})}\leqslant\frac{1-\alpha}{1-\beta}.$$

2. *If* $\beta \in [0, \frac{1}{2}]$, *then*

$$\frac{\alpha}{\beta} \leqslant \frac{w_N^m(\mathscr{A}\mathscr{X} + \mathscr{X}\mathscr{A})) - w_N^m(\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{A}^{\alpha} + \mathscr{A}^{\alpha}\mathscr{X}\mathscr{A}^{1-\alpha})}{w_N^m(\mathscr{A}\mathscr{X} + \mathscr{X}\mathscr{A})) - w_N^m(\mathscr{A}^{1-\beta}\mathscr{X}\mathscr{A}^{\beta} + \mathscr{A}^{\beta}\mathscr{X}\mathscr{A}^{1-\beta})}.$$

REMARK 4.2. The above theorem presents an extended matrix version of the scalar inequalities established in [2, 28, 29].

Alternatively, it is known that if $\mathscr{A} \in \mathbf{M}_n^+$ and $\mathscr{X} \in \mathbf{M}_n$, the function

$$f(\alpha) = w_N\left(\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{A}^{\alpha} - \mathscr{A}^{\alpha}\mathscr{X}\mathscr{A}^{1-\alpha}\right)$$

is convex on $[0,1]$, decreasing on $[0, \frac{1}{2}]$, and increasing on $[\frac{1}{2}, 1]$ for any unitarily invariant norm $N(\cdot)$ on $\mathbf{M}_n$, [1]. Therefore, we can apply Theorem 2.2 with this function to derive the following Heinz-type inequalities for the numerical radius.

THEOREM 4.6. *Let* $\mathscr{A} \in \mathbf{M}_n^+$ *and* $\mathscr{X} \in \mathbf{M}_n$, $0 \leqslant \alpha \leqslant \beta < 1$ *and* $m \geqslant 1$.

1. *If* $\alpha \in [\frac{1}{2}, 1]$, *then*

$$\frac{((1-\alpha)w_N(\mathscr{A}\mathscr{X} - \mathscr{X}\mathscr{A}) + \alpha w_N(\mathscr{X}\mathscr{A} - \mathscr{A}\mathscr{X}))^m - w_N^m(\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{A}^{\alpha} - \mathscr{A}^{\alpha}\mathscr{X}\mathscr{A}^{1-\alpha})}{((1-\beta)w_N(\mathscr{A}\mathscr{X} - \mathscr{X}\mathscr{A}) + \beta w_N(\mathscr{X}\mathscr{A} - \mathscr{A}\mathscr{X}))^m - w_N^m(\mathscr{A}^{1-\beta}\mathscr{X}\mathscr{A}^{\beta} - \mathscr{A}^{\beta}\mathscr{X}\mathscr{A}^{1-\beta})} \leqslant \frac{1-\alpha}{1-\beta}.$$

2. *If* $\beta \in [0, \frac{1}{2}]$, *then*

$$\frac{\alpha}{\beta} \leqslant \frac{((1-\alpha)w_N(\mathscr{A}\mathscr{X} - \mathscr{X}\mathscr{A}) + \alpha w_N(\mathscr{X}\mathscr{A} - \mathscr{A}\mathscr{X}))^m - w_N^m(\mathscr{A}^{1-\alpha}\mathscr{X}\mathscr{A}^{\alpha} - \mathscr{A}^{\alpha}\mathscr{X}\mathscr{A}^{1-\alpha})}{((1-\beta)w_N(\mathscr{A}\mathscr{X} - \mathscr{X}\mathscr{A}) + \beta w_N(\mathscr{X}\mathscr{A} - \mathscr{A}\mathscr{X}))^m - w_N^m(\mathscr{A}^{1-\beta}\mathscr{X}\mathscr{A}^{\beta} - \mathscr{A}^{\beta}\mathscr{X}\mathscr{A}^{1-\beta})}.$$

Concluding remarks

In this paper, we have developed a unified framework for power-type inequalities within the setting of p-convexity, extending and refining several classical results associated with convex, log-convex, and harmonic convex functions. By employing convexity-inspired factorizations and integer power techniques, we obtained sharper bounds that improve and generalize earlier inequalities of Alzer, Heinz, Young, and Sababheh type.

Our results provide systematic refinements of inequalities involving various mathematical means, including arithmetic, geometric, harmonic, power, and Heinz means, valid for arbitrary positive integer powers. Furthermore, we derived meaningful applications in matrix analysis, leading to refined determinant inequalities, inequalities for unitarily invariant norms, and new Heinz-type inequalities for generalized numerical radii of operators.

These contributions unify and extend a wide range of known inequalities and offer deeper insight into the interplay between p-convexity, power-type refinements, and operator inequalities. Future research directions may include extensions to non-integer powers, investigations within other generalized convexity frameworks, and further applications to unbounded operators and functional calculus.

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REFERENCES

[1] H. ABBAS, S. HARB AND H. ISSA, *Convexity and inequalities of some generalized numerical radius functions*, Filomat **36** (2022), 1649–1662.

[2] H. ALZER, C. M. FONSECA AND A. KOVAČES, *Young-type inequalities and their matrix analogues*, Linear Multilinear Alebra **63** (2015), 622–635.

[3] R. BHATIA AND C. DAVIS, *A Cauchy-Schwarz inequality for operators with applications*, Linear Algebra Appl. **223/224** (1995), 119–129.

[4] A. BURQAN AND M. KHANDAQJI, *Reverses of Young type inequalities*, J. Math. Inequal. **9** (2015), 113–120.

[5] D. Q. HUY AND D. T. T. VAN, *Some inequalities for convex functions and applications to operator inequalities*, Asian-European J. Math. **16** (2023).

[6] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge University Press, (2012).

[7] R. A. HORN AND R. MATHIAS, *Cauchy-Schwarz inequalities associated with positive semi-definite matrices*, Linear Algebra Appl. **142** (1990) 63–82.

[8] R. A. HORN AND R. MATHIAS, *An analog of the Cauchy-Schwarz inequality for Hadamard products and unitarily invariant norms*, SIAM J. Matrix Anal. Appl. **11** (1990), 481–498.

[9] M. A. IGHACHANE, M. AKKOUCHI AND M. SABABHEH, *Power inequalities for log-convex functions with applications*, Filomat **37** (13) (2023), 4425–4441.

[10] M. A. IGHACHANE, M. AKKOUCHI AND M. SABABHEH, *New inequalities for positive convex functions*, J. Appl. Anal. Comput. **14** (2024), 703–716.

[11] M. A. IGHACHANE AND M. BOUCHANGOUR, *Some refinements of real power form inequalities for convex functions via weak sub-majorization*, Oper and Matrices **17** (2023), 213–233.

[12] M. A. IGHACHANE AND M. BOUCHANGOUR, *New inequalities for (p,h)-convex functions for τ-measurable operators*, Filomat **37** (2023), 5259–5271.

[13] M. A. IGHACHANE, D. Q. HUY, D. T. THUY VAN AND M. BOUCHANGOUR, *Further refinements of real power form inequalities for convex functions via weak sub-majorization*, Rend. Circ. Mat. Palermo, II. Ser 73 (2024), 1101–1138.

[14] Y. KAPIL, R. PAL, M. SINGH AND J. S. AUJLA, *Some norm inequalities for operators*, Adv. Oper. Theory **5** (2020) 627–639.
[15] M. KHOSRAVI, *Some matrix inequalities for weighted power mean*, Ann. Func. Anal. **7** (2016), 348–357.
[16] F. KITTANEH, *Norm inequalities for fractional powers of positive operators*, Lett. Math. Phys. **27** (1993) 279–285.
[17] F. KITTANEH, M. S. MOSLEHIAN AND M. SABABHEH, *Quadratic interpolation of the Heinz means*, Math. Inequal. Appl. **21** (2018), 739–757.
[18] W. LIAO AND J. WU, *Matrix inequalities for the difference between arithmetic mean and harmonic mean*, Ann. Func. Anal. **6** (2015), 191–202.
[19] Y. REN, H. ZUO AND X. NIU, *Some Alzer-Fonseca-Kovačec type inequalities*, J. Math. Inequal. **19** (2025), 275–285.
[20] M. SABABHEH, *Log and Harmonically log-convex functions related to matrix norms*, Oper. Matrices. **10** (2016), 453–465.
[21] M. SABABHEH, *Convexity and matrix means*, Linear Algebra App. **506** (2016), 588–602.
[22] M. SABABHEH AND D. CHOI, *A complete refinement of Young's inequality*, J. Math. Anal. Appl. **440** (2016), 379–393.
[23] M. SABABHEH, *Means refinements via convexity*, Mediterr. J. Math. **14** (2017), 125.
[24] M. SABABHEH, *Improved Jensen's inequality*, Math. Inequal. Appl. **20** (2017), 389–403.
[25] M. SABABHEH, *Extrapolation of convex functions*, Filomat **32** (2018), 127–139.
[26] M. SABABHEH, H. R. MORADI AND S. FURUICHI, *Operator inequalities via geometric convexity*, Math. Inequal. Appl. **22** (2019), 1215–1231.
[27] S. SEZER, Z. EKEN, G. TINAZTEPE, AND G. ADILOV, *p-convex functions and some of their properties*, Numer. Funct. Anal. Optim. **42** (2021), 443–459.
[28] C. YANG AND Z. WANG, *Some new improvements of Young's inequalities*, J. Math. Inequal. **17** (2023), 205–217.
[29] X. YANG, C. YANG AND H. LI, *Further improvements for Young's inequalities on the arithmetic, geometric, and harmonic mean*, J. Math. Inequal. **18** (2024), 315–328.

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