Journal of
Mathematical
Inequalities

Volume 20, Number 1 (2026), 267–282 doi:10.7153/jmi-2026-20-17

NOTE ON NORM ESTIMATE OF A LINEAR OPERATOR FROM THE CLASSICAL WEIGHTED BERGMAN SPACE TO THE mTH WEIGHTED–TYPE SPACE

STEVO STEVIĆ

(*Communicated by M. Krnić*)

Abstract. We give an upper and a lower bound for the norm of a linear operator of a polynomial differentiation composition type from the classical weighted Bergman space to the mth weighted-type space and the little mth weighted-type space on the open unit disk in the complex plane, which were introduced by us about twenty years ago. We also give some sufficient conditions and some necessary conditions for the boundedness of the operator.

1. Introduction

Throughout the paper by $\mathbb{N}$ we denote the set of positive integers, by $\mathbb{N}_0$ the set of nonnegative integers, that is, the set $\mathbb{N}\cup\{0\}$, and by $\mathbb{C}$ the set of complex numbers. If s and t are two elements of $\mathbb{N}_0$, $s\leqslant t$, then $j=\overline{s,t}$ is an abbreviation for the expression: $s\leqslant j\leqslant t$, $j\in\mathbb{N}_0$.

Instead of using the notation

$$\binom{n}{k}:=\frac{n(n-1)\cdots(n-k+1)}{k!},$$

where $0\leqslant k\leqslant n$, for the binomial coefficients, here we use the notation C^n_k, which we regard as less robust and more elegant (a part of the literature use the notation C^k_n, which, in our opinion, is less suggestive than C^n_k).

By $\mathbb{D}$ we denote the open unit disk in $\mathbb{C}$, that is, the set $\{z\in\mathbb{C}:|z|<1\}$, whereas by $dm(z)$ we denote the Lebesgue area measure on $\mathbb{D}$ (for some basic information on the Lebesgue measure and measure theory consult, for instance, [24]).

Let

$$dm_\alpha(z):=c_\alpha(1-|z|^2)^\alpha dm(z),$$

with $\alpha>-1$, be the normalized weighted Lebesgue measure such that the constant c_α satisfies the relation

$$m_\alpha(\mathbb{D})=\int_{\mathbb{D}}c_\alpha(1-|z|^2)^\alpha dm(z)=1$$

Mathematics subject classification (2020): Primary 47B38.

Keywords and phrases: Operator norm, polynomial differentiation composition operator, classical weighted Bergman space, mth weighted-type space, bounded operator.

© ELEMENT, Zagreb

(the value of c_α, that is, $(\int_{\mathbb{D}}(1-|z|^2)^\alpha dm(z))^{-1}$, can be easily calculated, but its value is not relevant in the paper, because of which we do not give the value).

If $\Omega \subseteq \mathbb{C}$ is a domain, that is, an open and connected set in $\mathbb{C}$, then by $H(\Omega)$ we denote the family of all holomorphic functions on Ω. Some basics on holomorphic functions of one variable can be found, for instance, in the standard textbooks [1] and [24]. By $S(\mathbb{D})$ we denote the family of all holomorphic self-maps of $\mathbb{D}$. The family of all positive and continuous functions on $\mathbb{D}$ we denote by $W(\mathbb{D})$, and their elements we call weights or weight functions. If X is a normed space, then by B_X we denote the unit ball in space.

The (classical) weighted Bergman space on the disk $\mathbb{D}$

$$A^1_\alpha(\mathbb{D}) = A_\alpha(\mathbb{D}) = A_\alpha,$$

where $\alpha > -1$, consists of all $f \in H(\mathbb{D})$ satisfying the following condition

$$\|f\|_{A_\alpha} = \int_{\mathbb{D}} |f(z)| dm_\alpha(z) < +\infty.$$

It is easy to see that the quantity $\|\cdot\|_{A_\alpha}$ is a norm on the weighted Bergman space. Besides, with the norm $\|\cdot\|_{A_\alpha}$, it is a Banach space. Since the domain of definition of the functions considered in the paper is only the open unit disk, from now on, we will only use the notation A_α for the space instead of using any of the other two notations given above. Let us also mention that the space is a special case of the weighted Bergman space $A^p_\alpha(\mathbb{D})$, where $p > 0$ and $\alpha > -1$, which has been investigated a lot. For some information on the space see [54].

The space consisting of $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \omega(z)|f^{(m)}(z)| < +\infty,$$

for some fixed $m \in \mathbb{N}_0$ and $\omega \in W(\mathbb{D})$, is called the mth weighted-type space on $\mathbb{D}$ and was introduced by us in [33]. We denote the space by $\mathscr{W}^{(m)}_\omega(\mathbb{D})$, or simply by $\mathscr{W}^{(m)}_\omega$.

The space consisting of $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \omega(z)|f^{(m)}(z)| = 0,$$

for some fixed $m \in \mathbb{N}_0$ and $\omega \in W(\mathbb{D})$, is called the little mth weighted-type space on $\mathbb{D}$ and was also introduced by us in [33]. We denote the space by $\mathscr{W}^{(m)}_{\omega,0}(\mathbb{D})$, or simply by $\mathscr{W}^{(m)}_{\omega,0}$.

The quantity

$$\|f\|_{\mathscr{W}^{(m)}_\omega} := \sum_{k=0}^{m-1} |f^{(k)}(0)| + \sup_{z\in\mathbb{D}} \omega(z)|f^{(m)}(z)|,$$

is a norm on the mth weighted-type space, as well as on the little mth weighted-type space. Besides, with the norm they both are Banach spaces.

Beside paper [33], the spaces and operators from or to them are investigated in several other papers such as [13,35,52,53]. Note that for $m=0$ the mth weighted-type space reduces to the weighted-type space, for $m=1$ to the Bloch-type space, whereas for $m=2$ it reduces to the Zygmund-type space. For these values of the parameter m the little mth weighted-type space reduces to the corresponding little weighted-type space, little Bloch-type space and little Zygmund-type space. For some previous studies of these and related spaces and operators on them see, for example, [3,4,21,23] and the related references therein. These facts, among other ones, show the importance of these two spaces.

Let $\varphi \in S(\Omega)$. Then by

$$C_\varphi f(z) = (f \circ \varphi)(z),$$

where $f \in H(\Omega)$ and $z \in \Omega$, is defined a composition operator (the composition operator with symbol φ).

Let $\psi \in H(\Omega)$. Then by

$$M_\psi f(z) = \psi(z) f(z),$$

where $f \in H(\Omega)$ and $z \in \Omega$, is defined a multiplication operator (the multiplication operator with symbol ψ).

By

$$Df(z) = f'(z)$$

where $f \in H(\Omega)$ and $z \in \Omega$, is defined the standard differentiation operator. The iterated differentiation operator D^n, where $n \in \mathbb{N}$, is defined as follows

$$D^n f(z) = f^{(n)}(z),$$

where $f \in H(\Omega)$ and $z \in \Omega$.

We can also allow that $n=0$ and for this value of the parameter n we obtain the identity operator, that is,

$$D^0 f(z) := f(z),$$

where $f \in H(\Omega)$ and $z \in \Omega$.

All these operators are linear, as well as many integral-type operators (see, for example, the ones in [15–17, 31] and the related references therein). Their products are also linear, and many of them have been investigated on spaces of holomorphic functions a lot. One can consult, for instance, [5,8,10,13,19,27–29,35,36,42,43,52] and the related references therein.

The operators

$$D \circ C_\varphi \quad \text{and} \quad C_\varphi \circ D \tag{1.1}$$

are among the basic product-type ones containing the differentiation operator. For some results on the operators in (1.1) on some spaces of holomorphic functions on $\mathbb{D}$ see, for instance, [8,19,35,46] and the related references therein.

Motivated by some of the investigations of the operators in (1.1), some authors started investigating the following product-type operator on spaces of holomorphic functions

$$D^n_{\varphi,\psi}f = \psi(f^{(n)} \circ \varphi), \tag{1.2}$$

the, so-called, weighted differentiation composition operator or generalized weighted composition operators, where $n \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi \in H(\mathbb{D})$ (see, for instance, [11, 18, 47, 49–51] and the related references therein). An n-dimensional analog of the operator was introduced in [37] and studied later in a series of papers.

The first sum of the operators in (1.2) investigated in the literature, was the following

$$T_{\varphi,\psi_1,\psi_2}f(z) = \psi_1(z)f(\varphi(z)) + \psi_2(z)f'(\varphi(z)), \tag{1.3}$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $z \in \mathbb{D}$ (see [42]). The operator (1.3) was later investigated between some spaces of holomorphic functions on some other domains in the complex plane. For example, in [41] it was studied between some spaces of holomorphic functions defined in the upper half-plane.

Operator (1.3) motivated us to investigate the other sums of the operators in (1.2). For example, the following natural generalization of the operator in (1.3)

$$T^n_{\varphi,\psi_1,\psi_2}f(z) = \psi_1(z)f^{(n)}(\varphi(z)) + \psi_2(z)f^{(n+1)}(\varphi(z)), \tag{1.4}$$

where $n \in \mathbb{N}_0$, $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $z \in \mathbb{D}$, was later investigated in [43].

For some other investigations of the operators in (1.3) and (1.4), as well as their extensions and close relatives see, for instance, [2, 7, 9, 12, 38, 40, 48].

Having published [43] we realized that a big part of the results in the literature can be relatively easily extended for the case of the following sum type operator

$$T^n_{\varphi,\vec{\psi}}f(z) = \sum_{j=0}^{n} \psi_j(z)f^{(j)}(\varphi(z)), \quad z \in \mathbb{D}, \tag{1.5}$$

where $n \in \mathbb{N}_0$, $\psi_j \in H(\mathbb{D})$, $j = \overline{0,n}$, and $\varphi \in S(\mathbb{D})$. Hence, the author suggested to some colleagues working in the research area to investigate the operator in (1.5).

In [38, 40] can be found some of the n-dimensional relatives of the operator, where the differentiation operator is replaced by the radial differentiation operator

$$\Re f(z) = \sum_{j=1}^{n} z_j D_j f(z),$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and

$$D_j f(z) = \frac{\partial f}{\partial z_j}(z), \quad j = \overline{1,n},$$

(see, e.g., [25]).

A more challenging problem is investigation of the following sum-type operator

$$T^n_{\vec{\varphi},\vec{\psi}}f(z)=\sum_{j=0}^{n}\psi_j(z)f^{(j)}(\varphi_j(z)),\quad z\in\mathbb{D}, \tag{1.6}$$

where $n\in\mathbb{N}_0$, $\psi_j\in H(\mathbb{D})$, $j=\overline{0,n}$, and $\varphi_j\in S(\mathbb{D})$, $j=\overline{0,n}$. The sum-type operator defined in (1.6) we call the general polynomial differentiation composition.

One of the main obstacles in investigating operator (1.6) is appearance of different self-maps $\varphi_j\in S(\mathbb{D})$, $j=\overline{0,n}$, therein, since the appearance produces many technical obstacles.

First, as usual, it should be investigated the boundedness of the operator and tried to estimate or, if possible, calculate the norm of the operator acting between two spaces of holomorphic functions on $\mathbb{D}$. For some classical results on calculating and estimating norms of linear operators see, for example, [6, 22, 24, 26, 44, 45] and the related references therein. For some recent results on concrete linear operators on spaces of holomorphic functions see, for instance, [14, 20, 30, 32, 34, 39] and the related references therein.

Here we investigate the operators

$$T^n_{\vec{\varphi},\vec{\psi}}:A_\alpha\to\mathscr{W}^{(m)}_{\omega}\ (\text{or } \mathscr{W}^{(m)}_{\omega,0}). \tag{1.7}$$

The purpose of the paper is to present some sufficient conditions for the boundedness of the operators and to give some upper and lower bounds for the norm of this operators.

By C we denote some constants, which may be different from one appearance to another. If we write $a\lesssim b$ (resp. $a\gtrsim b$), then $a\leqslant Cb$ (resp. $a\geqslant Cb$) for some $C>0$. We write $a\asymp b$, if $a\lesssim b$ and $a\gtrsim b$.

2. Main results

Our main results are stated and proved in this section.

2.1. An auxiliary result

First, we quote the following lemma which is very useful in dealing with the mth weighted-type spaces and the little mth weighted-type spaces (see, for example, [35]).

LEMMA 1. *Let Ω be a domain in $\mathbb{C}$, $n\in\mathbb{N}_0$, $\psi,f\in H(\Omega)$ and $\varphi\in S(\Omega)$. Then*

$$(\psi(f\circ\varphi))^{(n)}(z)=\sum_{k=0}^{n}f^{(k)}(\varphi(z))\sum_{l=k}^{n}C^n_l\psi^{(n-l)}(z)B_{l,k}(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)),$$

for $z\in\Omega$, where

$$B_{l,k}(\varphi'(z),\varphi''(z),\dots,\varphi^{(l-k+1)}(z))=\sum_{k_1,k_2,\dots,k_l}\frac{l!}{k_1!k_2!\cdots k_l!}\prod_{j=1}^{l}\left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j},$$

and the sum is taken over all nonnegative integers $k_1, k_2, \dots, k_l$ *such that* $k = k_1 + k_2 + \cdots + k_l$ *and* $k_1 + 2k_2 + \cdots + lk_l = l$.

Let

$$\widehat{B}_{l,k}(\varphi(z)) := B_{l,k}(\varphi'(z), \varphi''(z), \dots, \varphi^{(l-k+1)}(z)),$$

where $0 \leqslant k \leqslant l \leqslant n$ and $\varphi \in H(\mathbb{D})$.

2.2. Bounds for the operator norm

Our first result gives some sufficient conditions and some necessary conditions for the boundedness of the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_\omega$ and gives an upper and a lower bound for its operator norm.

THEOREM 1. *Suppose that* $m, n \in \mathbb{N}_0$, $\alpha > -1$, $\psi_j \in H(\mathbb{D})$, $\varphi_j \in S(\mathbb{D})$, $j = \overline{0,n}$, *and* $\omega \in W(\mathbb{D})$. *Let*

$$F(w) := \sum_{k=0}^{m-1} \left| \sum_{j=0}^{n} \sum_{i=0}^{k} \frac{(\overline{w})^{i+j} \prod_{l=0}^{i+j-1}(\alpha+l+2)}{(1-\overline{w}\varphi_j(0))^{\alpha+i+j+2}} \sum_{l=i}^{k} C_l^k \psi_j^{(k-l)}(0) \widehat{B}_{l,i}(\varphi_j(0)) \right|, \tag{2.1}$$

$$G(z,w) := \left| \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{\omega(z)(\overline{w})^{j+k} \prod_{i=0}^{j+k-1}(\alpha+i+2)}{(1-\overline{w}\varphi_j(z))^{\alpha+j+k+2}} \sum_{l=k}^{m} C_l^m \psi_j^{(m-l)}(z) \widehat{B}_{l,k}(\varphi_j(z)) \right|, \tag{2.2}$$

$$\widetilde{F}(w) := \sum_{k=0}^{m-1} \left| \sum_{j=0}^{n} \sum_{i=0}^{k} \frac{(\overline{w})^{i+j} \prod_{s=0}^{i+j-1}(2\alpha+4+s)}{(1-\overline{w}\varphi_j(0))^{2\alpha+i+j+4}} \sum_{l=i}^{k} C_l^k \psi_j^{(k-l)}(0) \widehat{B}_{l,i}(\varphi_j(0)) \right| (1-|w|^2)^{\alpha+2}, \tag{2.3}$$

$$\begin{aligned}\widetilde{G}(z,w) := &\left| \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{\omega(z) \prod_{i=0}^{j+k-1}(2\alpha+4+i)(\overline{w})^{j+k}}{(1-\overline{w}\varphi_j(z))^{2\alpha+j+k+4}} \sum_{l=k}^{m} C_l^m \psi_j^{(m-l)}(z) \widehat{B}_{l,k}(\varphi_j(z)) \right| \\ &\times (1-|w|^2)^{\alpha+2},\end{aligned} \tag{2.4}$$

$$M := \sup_{w\in\mathbb{D}} \sup_{z\in\mathbb{D}} \big(F(w) + G(z,w)\big), \tag{2.5}$$

and

$$\widetilde{M} := \sup_{w\in\mathbb{D}} \sup_{z\in\mathbb{D}} \big(\widetilde{F}(w) + \widetilde{G}(z,w)\big). \tag{2.6}$$

Then, the following statements are true.

(a) If the quantity M *is finite. Then, the operator* $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_\omega$ *is bounded and*

$$\|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha \to \mathscr{W}^{(m)}_\omega} \leqslant M. \tag{2.7}$$

(b) If the operator $T^n_{\vec{\varphi},\vec{\psi}}: A_\alpha \to \mathscr{W}_\omega^{(m)}$ *is bounded, then the quantity* $\widetilde{M}$ *is finite and*

$$\widetilde{M} \leqslant \|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha \to \mathscr{W}_\omega^{(m)}}. \tag{2.8}$$

Proof. *(a)* First note that for any function $f \in A_\alpha$ the following integral representation

$$f(z) = \int_{\mathbb{D}} \frac{f(w)dm_\alpha(w)}{(1-\overline{w}z)^{\alpha+2}}, \tag{2.9}$$

holds for every $z \in \mathbb{D}$ (the representation is well known, see, e.g., [54]).

By using the differentiation under the integral sign, we easily obtain the formula

$$f^{(j)}(z) = \prod_{i=0}^{j-1}(\alpha+i+2)\int_{\mathbb{D}} \frac{(\overline{w})^j f(w)dm_\alpha(w)}{(1-\overline{w}z)^{\alpha+j+2}}, \tag{2.10}$$

for any $z \in \mathbb{D}$ and $j \in \mathbb{N}_0$.

Let

$$H_m(z) := \omega(z)|(T^n_{\vec{\varphi},\vec{\psi}} f)^{(m)}(z)|,$$

for $z \in \mathbb{D}$.

Employing the formula presented in Lemma 1, we have that the relation holds

$$\begin{aligned} H_m(z) &= \omega(z)\left|\sum_{j=0}^n (\psi_j(z) f^{(j)}(\varphi_j(z)))^{(m)}\right| \\ &= \omega(z)\left|\sum_{j=0}^n \sum_{k=0}^m f^{(j+k)}(\varphi_j(z)) \sum_{l=k}^m C_l^m \psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right|, \end{aligned} \tag{2.11}$$

for $z \in \mathbb{D}$.

If in (2.11) we use the positivity of the weight function ω and the relation obtained from (2.10) when the variable z is replaced by the self-map $\varphi_j(z)$ it follows that

$$\begin{aligned} H_m(z) &= \omega(z)\left|\sum_{j=0}^n \sum_{k=0}^m f^{(j+k)}(\varphi_j(z)) \sum_{l=k}^m C_l^m \psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right| \\ &= \left|\sum_{j=0}^n \sum_{k=0}^m \int_{\mathbb{D}} \frac{\omega(z)\prod_{i=0}^{j+k-1}(\alpha+i+2)(\overline{w})^{j+k} f(w)dm_\alpha(w)}{(1-\overline{w}\varphi_j(z))^{\alpha+j+k+2}} \sum_{l=k}^m C_l^m \psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right| \end{aligned} \tag{2.12}$$

for $z \in \mathbb{D}$.

From (2.12), a well-known inequality for integrals and the definition of the quantity $G(z,w)$, it follows that

$$\begin{aligned} &H_m(z) \\ &\leqslant \int_{\mathbb{D}} \left|\sum_{j=0}^n \sum_{k=0}^m \frac{\omega(z)(\overline{w})^{j+k}\prod_{i=0}^{j+k-1}(\alpha+i+2)}{(1-\overline{w}\varphi_j(z))^{\alpha+j+k+2}} \sum_{l=k}^m C_l^m \psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right| |f(w)|dm_\alpha(w) \\ &= \int_{\mathbb{D}} G(z,w)|f(w)|dm_\alpha(w) \leqslant \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} G(z,w)|f(w)|dm_\alpha(w). \end{aligned} \tag{2.13}$$

for $z \in \mathbb{D}$.

Further, employing some similar arguments which have lead to obtaining the inequality in (2.13), as well as the definition of the quantity in (2.1), we have

$$\begin{aligned}
&\sum_{k=0}^{m-1}\left|(T^n_{\vec{\varphi},\vec{\psi}}f)^{(k)}(0)\right| = \sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}(\psi_j(z)f^{(j)}(\varphi_j(z)))^{(k)}\right|_{z=0}\\
&=\sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{i=0}^{k}f^{(i+j)}(\varphi_j(0))\sum_{l=i}^{k}C_l^k\psi_j^{(k-l)}(0)\widehat{B}_{l,i}(\varphi_j(0))\right|\\
&=\sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{i=0}^{k}\int_{\mathbb{D}}\frac{(\overline{w})^{i+j}\prod_{s=0}^{i+j-1}(\alpha+s+2)f(w)dm_\alpha(w)}{(1-\overline{w}\varphi_j(0))^{\alpha+i+j+2}}\sum_{l=i}^{k}C_l^k\psi_j^{(k-l)}(0)\widehat{B}_{l,i}(\varphi_j(0))\right|\\
&\leqslant\int_{\mathbb{D}}\sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{i=0}^{k}\frac{(\overline{w})^{i+j}\prod_{s=0}^{i+j-1}(\alpha+s+2)}{(1-\overline{w}\varphi_j(0))^{\alpha+i+j+2}}\sum_{l=i}^{k}C_l^k\psi_j^{(k-l)}(0)\widehat{B}_{l,i}(\varphi_j(0))\right||f(w)|dm_\alpha(w)\\
&=\int_{\mathbb{D}}F(w)|f(w)|dm_\alpha(w). \qquad (2.14)
\end{aligned}$$

Now by combining the inequalities (2.13), (2.14) and using the finiteness of M, it follows that

$$\|T^n_{\vec{\varphi},\vec{\psi}}f\|_{\mathscr{W}_\omega^{(m)}} \leqslant \int_{\mathbb{D}}\big(F(w)+\sup_{z\in\mathbb{D}}G(z,w)\big)|f(w)|dm_\alpha(w) \leqslant M\|f\|_{A_\alpha}, \qquad (2.15)$$

for every $f \in A_\alpha$.

If we take the supremum in inequality (2.15) over the unit ball B_{A_α}, we obtain the boundedness of the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}_\omega^{(m)}$. Moreover, we obtain the following estimate

$$\sup_{f\in B_{A_\alpha}}\|T^n_{\vec{\varphi},\vec{\psi}}f\|_{A_\alpha\to\mathscr{W}_\omega^{(m)}} \leqslant M,$$

which is, in fact, inequality (2.7).

(b) Since the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}_\omega^{(m)}$ is bounded, we have

$$\|T^n_{\vec{\varphi},\vec{\psi}}f\|_{\mathscr{W}_\omega^{(m)}} \leqslant \|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha\to\mathscr{W}_\omega^{(m)}}\|f\|_{A_\alpha} \qquad (2.16)$$

for every $f \in A_\alpha$.

Let $w \in \mathbb{D}$ and

$$f_w(z) = \frac{(1-|w|^2)^{\alpha+2}}{(1-\overline{w}z)^{2\alpha+4}}, \quad z \in \mathbb{D}. \qquad (2.17)$$

It is well known that

$$\|f_w\|_{A_\alpha} = 1, \qquad (2.18)$$

for each $w \in \mathbb{D}$.

We also have

$$f_w^{(j)}(z)=\prod_{i=0}^{j-1}(2\alpha+4+i)\frac{\overline{w}^j(1-|w|^2)^{\alpha+2}}{(1-\overline{w}z)^{2\alpha+4+j}},\quad z\in\mathbb{D},\tag{2.19}$$

for $j\in\mathbb{N}_0$.

Let

$$L_{m,w}(z):=\omega(z)|(T^n_{\vec{\varphi},\vec{\psi}}f_w)^{(m)}(z)|,$$

for $z,w\in\mathbb{D}$

Employing the formula presented in Lemma 1, we have that the relation holds

$$L_{m,w}(z)=\omega(z)\left|\sum_{j=0}^{n}\sum_{k=0}^{m}f_w^{(j+k)}(\varphi_j(z))\sum_{l=k}^{m}C_l^m\psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right|,\tag{2.20}$$

for $z,w\in\mathbb{D}$.

If in (2.20) we use the positivity of the weight function ω and (2.19) when the variable z is replaced by the self-map $\varphi_j(z)$, it follows that

$$\begin{aligned}L_{m,w}(z)&=\omega(z)\left|\sum_{j=0}^{n}\sum_{k=0}^{m}f_w^{(j+k)}(\varphi_j(z))\sum_{l=k}^{m}C_l^m\psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right|\\&=\left|\sum_{j=0}^{n}\sum_{k=0}^{m}\frac{\omega(z)\prod_{i=0}^{j+k-1}(2\alpha+4+i)(\overline{w})^{j+k}}{(1-\overline{w}\varphi_j(z))^{2\alpha+j+k+4}}\sum_{l=k}^{m}C_l^m\psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right|(1-|w|^2)^{\alpha+2}\\&=\widetilde{G}(z,w),\end{aligned}\tag{2.21}$$

for $z,w\in\mathbb{D}$.

Further, by using (2.19), as well as the definition of the quantity in (2.3), we have

$$\begin{aligned}\sum_{k=0}^{m-1}\left|(T^n_{\vec{\varphi},\vec{\psi}}f_w)^{(k)}(0)\right|&=\sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}(\psi_j(z)f_w^{(j)}(\varphi_j(z)))^{(k)}\right|_{z=0}\\&=\sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{i=0}^{k}f_w^{(i+j)}(\varphi_j(0))\sum_{l=i}^{k}C_l^k\psi_j^{(k-l)}(0)\widehat{B}_{l,i}(\varphi_j(0))\right|\\&=\sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{i=0}^{k}\frac{(\overline{w})^{i+j}\prod_{s=0}^{i+j-1}(2\alpha+4+s)}{(1-\overline{w}\varphi_j(0))^{2\alpha+i+j+4}}\sum_{l=i}^{k}C_l^k\psi_j^{(k-l)}(0)\widehat{B}_{l,i}(\varphi_j(0))\right|(1-|w|^2)^{\alpha+2}\\&=\widetilde{F}(w).\end{aligned}\tag{2.22}$$

Now using the relations (2.21), (2.22), (2.16), the definition of the mth weighted-type space, and the relation (2.18), it follows that

$$\widetilde{F}(w)+\sup_{z\in\mathbb{D}}\widetilde{G}(z,w)\leqslant\|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha\to\mathscr{W}_\omega^{(m)}}\|f_w\|_{A_\alpha}=\|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha\to\mathscr{W}_\omega^{(m)}},\tag{2.23}$$

for every $w\in\mathbb{D}$.

If we take the supremum in inequality (2.23) over the disk $\mathbb{D}$ and note that

$$\widetilde{F}(w)+\sup_{z\in\mathbb{D}}\widetilde{G}(z,w)=\sup_{z\in\mathbb{D}}(\widetilde{F}(w)+\widetilde{G}(z,w)),$$

the relation (2.8) immediately follows. □

2.3. Some remarks related to the boundedness of the operator

Since unlike the case in [32, 34, 39], Theorem 1 does not give a formula for the norm of the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_\omega$, but only gives an upper and a lower bound for the norm, it is of some interest to conduct an analysis concerning the problem, which we present in this subsection.

Assume that $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_\omega$ is bounded. Define a class of test functions as follows

$$f_w(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}}, \quad z \in \mathbb{D}, \tag{2.24}$$

where $w \in \mathbb{D}$.

If we take the function

$$\widehat{f_0}(z) \equiv 1,$$

in the integral representation given in (2.9) we obtain the following relation

$$\int_{\mathbb{D}} \frac{dm_\alpha(w)}{(1-\overline{w}z)^{\alpha+2}} = 1,$$

for $z \in \mathbb{D}$, from which by taking the complex conjugation it follows that

$$\int_{\mathbb{D}} \frac{dm_\alpha(w)}{(1-w\overline{z})^{\alpha+2}} = 1.$$

Hence

$$\|f_w\|_{A_\alpha} \geqslant \left|\int_{\mathbb{D}} \frac{dm_\alpha(z)}{(1-\overline{w}z)^{\alpha+2}}\right| = \int_{\mathbb{D}} \frac{dm_\alpha(w)}{(1-w\overline{z})^{\alpha+2}} = 1,$$

for any $w \in \mathbb{D}$.

By some simple calculations we have

$$f_w^{(j)}(z) = \frac{(\overline{w})^j \prod_{i=0}^{j-1}(\alpha+i+2)}{(1-\overline{w}z)^{\alpha+j+2}},$$

for each $j \in \mathbb{N}_0$, and every $w, z \in \mathbb{D}$.

The boundedness of $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_\omega$ implies

$$\begin{aligned}&\sum_{k=0}^{m-1}\left|(T^n_{\vec{\varphi},\vec{\psi}} f_w)^{(k)}(0)\right| + \sup_{z\in\mathbb{D}} \omega(z)|(T^n_{\vec{\varphi},\vec{\psi}} f_w)^{(m)}(z)|\\ &= \|T^n_{\vec{\varphi},\vec{\psi}} f_w\|_{\mathscr{W}^{(m)}_\omega} \leqslant \|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha\to\mathscr{W}^{(m)}_\omega}\|f_w\|_{A_\alpha},\end{aligned} \tag{2.25}$$

for every $w \in \mathbb{D}$.

On the other hand, we have

$$\begin{aligned}&\sum_{k=0}^{m-1}\left|(T^n_{\vec{\varphi},\vec{\psi}} f_w)^{(k)}(0)\right|\\ &= \sum_{k=0}^{m-1}\left|\sum_{j=0}^{n}\sum_{i=0}^{k} \frac{(\overline{w})^{i+j}\prod_{s=0}^{i+j-1}(\alpha+s+2)}{(1-\overline{w}\varphi_j(0))^{\alpha+i+j+2}} \sum_{l=i}^{k} C_l^k \psi_j^{(k-l)}(0)\widehat{B}_{l,i}(\varphi_j(0))\right|\end{aligned} \tag{2.26}$$

for each $w \in \mathbb{D}$, and

$$\begin{aligned}&|(T^n_{\vec{\varphi},\vec{\psi}} f_w)^{(m)}(z)|\\ &=\left|\sum_{j=0}^{n}\sum_{k=0}^{m}\frac{(\overline{w})^{j+k}\prod_{i=0}^{j+k-1}(\alpha+i+2)}{(1-\overline{w}\varphi_j(z))^{\alpha+j+k+2}}\sum_{l=k}^{m}C_l^m\psi_j^{(m-l)}(z)\widehat{B}_{l,k}(\varphi_j(z))\right|,\end{aligned} \tag{2.27}$$

for all $z,w \in \mathbb{D}$.

From the relations in (2.25)–(2.27), we have that for every $z,w \in \mathbb{D}$, the following inequality holds

$$F(w)+G(z,w) \leqslant \|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha \to \mathscr{W}^{(m)}_\omega}\|f_w\|_{A_\alpha}. \tag{2.28}$$

By taking the supremmum in inequality (2.28) over $z \in \mathbb{D}$, we obtain the estimate

$$\sup_{z\in\mathbb{D}}(F(w)+G(z,w)) \leqslant \|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha \to \mathscr{W}^{(m)}_\omega}\|f_w\|_{A_\alpha}, \tag{2.29}$$

for every $w \in \mathbb{D}$.

By Proposition 1.4.10 in [25], we have

$$\|f_w\|_{A_\alpha} \asymp \ln\frac{e}{1-|w|^2}. \tag{2.30}$$

Using (2.30) in (2.29) we get

$$\sup_{z\in\mathbb{D}}(F(w)+G(z,w)) \lesssim \|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha \to \mathscr{W}^{(m)}_\omega}\ln\frac{e}{1-|w|^2}, \tag{2.31}$$

for every $w \in \mathbb{D}$.

Since

$$\lim_{|w|\to 1}\ln\frac{e}{1-|w|^2} = +\infty$$

taking the supremum in (2.31) over the set $w \in \mathbb{D}$, does not produce a finite bound of the quantity M defined in (2.5). So, (2.31) gives only an upper bound for the function

$$g(w) := \sup_{z\in\mathbb{D}}(F(w)+G(z,w)),$$

for $w \in \mathbb{D}$.

2.4. On the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}$

In this subsection we give some sufficient conditions and some necessary conditions for the boundedness of the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}$.

THEOREM 2. *Suppose that* $m,n \in \mathbb{N}_0$, $\alpha > -1$, $\psi_j \in H(\mathbb{D})$, $\varphi_j \in S(\mathbb{D})$, $j = \overline{0,n}$, $\omega \in W(\mathbb{D})$. *Then the following statements are true.*

(a) If the quantity M *defined in* (2.5) *is finite and*

$$\lim_{|z|\to 1} G(z,w) = 0, \tag{2.32}$$

for almost all $w \in \mathbb{D}$, *where* $G(z,w)$ *is defined in* (2.2). *Then, the operator* $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}$ *is bounded.*

(b) If the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}$ *is bounded, then the quantity* $\widetilde{M}$ *defined in* (2.6) *is finite and*

$$\lim_{|z|\to 1} \widetilde{G}(z,w) = 0, \tag{2.33}$$

for $w \in \mathbb{D}$, *where* $\widetilde{G}(z,w)$ *is defined in* (2.4).

Proof. (a) First note that by Theorem 1 the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega}$ is bounded. Let $f \in A_\alpha$. Then, the integral representation in (2.9) holds.

Therefore, by an inequality in (2.13), we have

$$\omega(z)|(T^n_{\vec{\varphi},\vec{\psi}} f)^{(m)}(z)| \leqslant \int_{\mathbb{D}} G(z,w)|f(w)|dm_\alpha(w). \tag{2.34}$$

for $z \in \mathbb{D}$.

Note that condition (2.32), along with the integrability of the function f, implies that

$$\lim_{|z|\to 1} G(z,w)|f(w)| = 0, \tag{2.35}$$

for almost all $w \in \mathbb{D}$.

We show that

$$\lim_{|z|\to 1} \omega(z)|(T^n_{\vec{\varphi},\vec{\psi}} f)^{(m)}(z)| = 0. \tag{2.36}$$

Assume to the contrary that this is not true. Then, there would be a sequence $(z_n)_{n\in\mathbb{N}}$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \omega(z_n)|(T^n_{\vec{\varphi},\vec{\psi}} f)^{(m)}(z_n)| = l > 0. \tag{2.37}$$

Let

$$g_n(w) := G(z_n,w)|f(w)|$$

for $w \in \mathbb{D}$ and $n \in \mathbb{N}$.

The functions g_n are measurable and the function

$$h(w) := M|f(w)|$$

is their Lebesgue integrable dominant (with respect to the weighted measure $dm_\alpha(z)$).

Besides, (2.35) implies

$$\lim_{n\to\infty} g_n(w) = 0, \tag{2.38}$$

for almost all $w \in \mathbb{D}$.

Hence, employing the Lebesgue dominated convergence theorem it follows that

$$\lim_{n\to\infty} \int_{\mathbb{D}} G(z_n,w)|f(w)|dm_\alpha(w) = 0. \tag{2.39}$$

From (2.39) and (2.34) we would get

$$\lim_{n\to\infty} \omega(z_n)|(T^n_{\vec{\varphi},\vec{\psi}}f)^{(m)}(z_n)| = 0,$$

which contradicts (2.37). Hence, relation (2.36) really holds.

Thus, $T^n_{\vec{\varphi},\vec{\psi}}f \in \mathscr{W}^{(m)}_{\omega,0}$, for any $f \in A_\alpha$, which means that the following relation

$$T^n_{\vec{\varphi},\vec{\psi}}(A_\alpha) \subseteq \mathscr{W}^{(m)}_{\omega,0}, \tag{2.40}$$

holds.

The boundedness of the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega}$, together with (2.40), implies the boundedness of the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}$, finishing the proof of the statement.

(b) Since the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}$ is bounded, we have that the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega}$ is also bounded. Hence, by Theorem 1 we have that the quantity $\widetilde{M}$ defined in (2.6) is finite.

By using the test functions in (2.17), which belong to A_α, the relation (2.21), and the fact that $T^n_{\vec{\varphi},\vec{\psi}}(f_w) \in \mathscr{W}^{(m)}_{\omega,0}$, for each $w \in \mathbb{D}$, the relation (2.33) follows for each $w \in \mathbb{D}$, finishing the proof of the theorem. □

Using Theorem 1 and Theorem 2 it is not difficult to see that the following corollary holds.

COROLLARY 1. *Assume that* $m,n \in \mathbb{N}_0$, $\alpha > -1$, $\psi_j \in H(\mathbb{D})$, $\varphi_j \in S(\mathbb{D})$, $j = \overline{0,n}$, *and* $\omega \in W(\mathbb{D})$. *Then the following statements are true.*

(a) If the quantity M *defined in* (2.5) *is finite, and* (2.32) *holds for almost all* $w \in \mathbb{D}$, *then the following inequality holds*

$$\|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}} \leqslant M.$$

(b) If the operator $T^n_{\vec{\varphi},\vec{\psi}} : A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}$ *is bounded, then the quantity* $\widetilde{M}$ *defined in* (2.6) *is finite,* (2.33) *holds for* $w \in \mathbb{D}$, *and the following inequality holds*

$$\widetilde{M} \leqslant \|T^n_{\vec{\varphi},\vec{\psi}}\|_{A_\alpha \to \mathscr{W}^{(m)}_{\omega,0}}.$$

Conclusion

Here we present some sufficient conditions and some necessary conditions for the boundedness of a linear sum-type operator, the, so-called, general polynomial differentiation composition operator acting from the classical weighted Bergman space to the mth weighted-type space and the little mth weighted-type space on the open unit disk in the complex plane, which were introduced by us almost twenty years ago. We also give an upper bound and a lower bound for the norm of these operators. The class of operators should be of some interest to the experts interested in studying concrete operators. The methods and ideas presented here can be applied in some related studies, so should be also of some interest to the experts. The interested reader could try to calculate the norm of the operators in this paper.

Authors contributions. This work is based on some ideas by the author of this paper, who has done all the research and writing by himself.

Use of AI tools declaration. The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest. This work does not have any conflict of interest.

REFERENCES

[1] L. V. AHLFORS, *Complex Analysis*, Third Edition, McGraw-Hill, Inc. New York et al., 1979.

[2] M. S. AL GHAFRI, J. S. MANHAS, *On Stević-Sharma operators from weighted Bergman spaces to weighted-type spaces*, Math. Inequal. Appl. **23** (3) (2020), 1051–1077.

[3] K. AVETISYAN, *Hardy Bloch-type spaces and lacunary series on the polydisk*, Glasgow J. Math. **49** (2007), 345–356.

[4] K. D. BIERSTEDT, W. H. SUMMERS, *Biduals of weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. A **54** (1993), 70–79.

[5] J. DU, S. LI, Y. ZHANG, *Essential norm of generalized composition operators on Zygmund type spaces and Bloch type spaces*, Annal. Polon. Math. **119** (2) (2017), 107–119.

[6] N. DUNFORD, J. T. SCHWARTZ, *Linear Operators I*, Interscience Publishers, Jon Willey and Sons, New York, 1958.

[7] Z. GUO, L. LIU, Y. SHU, *On Stević-Sharma operators from the mixed-norm spaces to Zygmund-type spaces*, Math. Inequal. Appl. **24** (2) (2021), 445–461.

[8] R. A. HIBSCHWEILER, N. PORTNOY, *Composition followed by differentiation between Bergman and Hardy spaces*, Rocky Mountain J. Math. **35** (3) (2005), 843–855.

[9] Q. HU, L. HU, S. LI, *On a Stević-Sharma type operator from weighted-type spaces into Bloch-type spaces*, Math. Inequal. Appl. **26** (1) (2023), 205–232.

[10] Q. HU, S. LI, *Products of differentiation and composition operators from the Bloch space and weighted Dirichlet spaces to Morrey type spaces*, J. Korean Math. Soc. **54** (2) (2017), 627–645.

[11] Q. HU, X. ZHU, *Compact generalized weighted composition operators on the Bergman space*, Opuscula Math. **37** (2) (2017), 303–312.

[12] C. S. HUANG, Z. J. JIANG, *Product-type operators from weighted Bergman-Orlicz spaces to weighted-type spaces on the unit ball*, J. Math. Anal. Appl. **519** (1), Article No. 126739, (2023).

[13] H. LI, Z. GUO, *Note on a Li-Stević integral-type operator from mixed-norm spaces to nth weighted spaces*, J. Math. Inequal. **11** (1) (2017), 77–85.

[14] H. LI, S. LI, *Norm of an integral operator on some analytic function spaces on the unit disk*, J. Inequal. Math. **2013**, (2013), 342.

[15] S. LI, S. STEVIĆ, *Integral type operators from mixed-norm spaces to α-Bloch spaces*, Integral Transforms Spec. Funct. **18** (7) (2007), 485–493.

[16] S. LI, S. STEVIĆ, *Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$*, Bull. Belg. Math. Soc. Simon Stevin **14** (4) (2007), 621–628.

[17] S. LI, S. STEVIĆ, *Riemann-Stieltjes operators between different weighted Bergman spaces*, Bull. Belg. Math. Soc. Simon Stevin **15** (4) (2008), 677–686.

[18] S. LI, S. STEVIĆ, *Generalized weighted composition operators from α-Bloch spaces into weighted-type spaces*, J. Inequal. Appl. Vol. 2015, Article No. 265, (2015), 12 pages.

[19] X. LIU, S. LI, *Boundedness from below of composition followed by differentiation on Bloch-type spaces*, J. Comput. Anal. Appl. **21** (3) (2016), 545–551.

[20] X. LIU, S. LI, *Norm and essential norm of a weighted composition operator on the Bloch space*, Integral Equations Operator Theory **87** (3) (2017), 309–325.

[21] W. LUSKY, *On the isomorphism classes of weighted spaces of harmonic and holomorphic functions*, Studia Math. **175** (1) (2006), 19–45.

[22] A. PELCZYNSKI, *Norms of classical operators in function spaces*, Astérisque bf 131 (1985), 137–162.

[23] L. A. RUBEL, A. L. SHIELDS, *The second duals of certain spaces of analytic functions*, J. Austral. Math. Soc. **11** (1970), 276–280.

[24] W. RUDIN, *Real and Complex Analysis*, Higher Mathematics Series, 3rd ed., McGraw-Hill Education, London, New York, Sidney, 1976.

[25] W. RUDIN, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Grundlehren der Mathematischen Wissenschaften, New York, Berlin, Springer-Verlag, 1980.

[26] W. RUDIN, *Functional Analysis*, 2nd ed., McGraw-Hill, Inc., New York, 1991.

[27] A. K. SHARMA, *Generalized weighted composition operators on the Bergman space*, Demonstr. Math. **44** (2) (2011), 359–372.

[28] A. K. SHARMA, *Products of multiplication, composition and differentiation between weighted Bergman-Nevanlinna and Bloch-type spaces*, Turkish. J. Math. **35** (2011), 275–291.

[29] A. K. SHARMA, *Generalized composition operators between Hardy and weighted Bergman spaces*, Acta Sci. Math. (Szeged) **78** (1–2) (2012), 187–211.

[30] A. K. SHARMA, *Norm of a composition operator from the space of Cauchy transforms into Zygmund-type spaces*, Ukrainian Math. J. **71** (12) (2020), 1945–1958.

[31] S. STEVIĆ, *Boundedness and compactness of an integral operator on a weighted space on the polydisc*, Indian J. Pure Appl. Math. **37** (6) (2006), 343–355.

[32] S. STEVIĆ, *Norms of some operators from Bergman spaces to weighted and Bloch-type space*, Util. Math. **76** (2008), 59–64.

[33] S. STEVIĆ, *Composition operators from the weighted Bergman space to the nth weighted spaces on the unit disc*, Discrete Dyn. Nat. Soc. Vol. 2009, Article ID 742019, (2009), 11 pages.

[34] S. STEVIĆ, *Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces*, Appl. Math. Comput. **215** (2009), 818–820.

[35] S. STEVIĆ, *Composition followed by differentiation from H^∞ and the Bloch space to nth weighted-type spaces on the unit disk*, Appl. Math. Comput. **216** (2010), 3450–3458.

[36] S. STEVIĆ, *On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on unit ball*, Appl. Math. Comput. **215** (2010), 4248–4255.

[37] S. STEVIĆ, *Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball*, Abstr. Appl. Anal. Vol. 2010, Article ID 801264, (2010), 14 pages.

[38] S. STEVIĆ, *On a new product-type operator on the unit ball*, J. Math. Inequal. **16** (4) (2022), 1675–1692.

[39] S. STEVIĆ, *Norm of the general polynomial differentiation composition operator from the space of Cauchy transforms to the mth weighted-type space on the unit disk*, Math. Methods Appl. Sci. **47** (2024), 3893–3902.

[40] S. STEVIĆ, C. S. HUANG, Z. J. JIANG, *Sum of some product-type operators from Hardy spaces to weighted-type spaces on the unit ball*, Math. Methods Appl. Sci. **45** (17) (2022), 11581–11600.

[41] S. STEVIĆ, A. K. SHARMA, *On a product-type operator between Hardy and α-Bloch spaces of the upper half-plane*, J. Inequal. Appl. Vol. 2018, Article No. 273, (2018), 18 pages.

[42] S. STEVIĆ, A. K. SHARMA, A. BHAT, *Essential norm of products of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput. **218** (2011), 2386–2397.

[43] S. STEVIĆ, A. K. SHARMA AND R. KRISHAN, *Boundedness and compactness of a new product-type operator from a general space to Bloch-type spaces*, J. Inequal. Appl. Vol. 2016, Article No. 219, (2016), 32 pages.

[44] V. A. TRENOGIN, *Funktsional'niy Analiz*, Nauka, Moskva, 1980, (in Russian).

[45] V. A. TRENOGIN, B. M. PISAREVSKIY, T. S. SOBOLEVA, *Zadachi i Uprazhneniya po Funktsional'nomu Analizu*, Nauka, Moskva, 1984, (in Russian).

[46] W. YANG, *Products of composition and differentiation operators from $Q_K(p,q)$ spaces to Bloch-type spaces*, Abstr. Appl. Anal. Vol. 2009, Article ID 741920, (2009), 14 pages.

[47] W. YANG, W. YAN, *Generalized weighted composition operators from area Nevanlinna spaces to weighted-type spaces*, Bull. Korean Math. Soc. **48** (6) (2011), 1195–1205.

[48] F. ZHANG, Y. LIU, *On the compactness of the Stević-Sharma operator on the logarithmic Bloch spaces*, Math. Inequal. Appl. **19** (2) (2016), 625–642.

[49] X. ZHU, *Products of differentiation, composition and multiplication from Bergman type spaces to Bers type space*, Integ. Tran. Spec. Funct. **18** (2007), 223–231.

[50] X. ZHU, *A new characterization of the generalized weighted composition operator from H^∞ into the Zygmund space*, Math. Inequal. Appl. **18** (3) (2015), 1135–1142.

[51] X. ZHU, *Generalized weighted composition operators on weighted Bergman spaces II*, Math. Inequal. Appl. **22** (2019) 1055–1066.

[52] X. ZHU, *Weighted composition operators from Dirichlet-type spaces into Stević-type spaces*, Math. Inequal. Appl. **23** (4) (2020), 1311–1323.

[53] X. ZHU, J. DU, *Weighted composition operators from weighted Bergman spaces to Stević-type spaces*, Math. Inequal. Appl. **22** (1) (2019), 361–376.

[54] K. ZHU, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, 2005.

(Received September 19, 2025)

Stevo Stević
Mathematical Institute of the
Serbian Academy of Sciences and Arts
Knez Mihailova 36/III, 11000 Beograd, Serbia
and
Serbian Academy of Sciences and Arts
Knez Mihailova 35, 11000 Beograd, Serbia
and
Department of Medical Research
China Medical University Hospital
China Medical University
Taichung 40402, Taiwan, Republic of China
e-mail: `sscite1@gmail.com`
ORCID: https://orcid.org/0000-0002-7202-9764