Journal of Mathematical Inequalities
Volume 20, Number 1 (2026), 283–306
doi:10.7153/jmi-2026-20-18

SOME NEW IMPROVEMENTS FOR MULTIPLE–TERM VERSIONS OF ALZER–FONSECA–KOVAČEC'S TYPE INEQUALITIES

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(*Communicated by M. Sababheh*)

Abstract. In this paper, we build on the recent advancements in the theory of weakly submajorizations established by D. Q. Huy et al. [Linear Algebra Appl., doi.org/10.1016/j.laa.2025.10.004] to present innovative real power-type and multiple-term refinements of Alzer-Fonseca-Kovačec-type inequalities. These refinements are accompanied by notable applications, including their extensions to operator versions, unitarily invariant norms, and matrix determinants.

1. Introduction

The classical Young inequality is a simple inequality that has garnered significant attention by many mathematicians. This inequality is stated as follows: for positive real numbers a and b, and $\nu \in [0,1]$:

$$a\nabla_\nu b := (1-\nu)a + \nu b \geqslant a^{1-\nu} b^\nu := a\sharp_\nu b, \tag{1}$$

equality with if and only if $a = b$. This inequality is also known as the weighted arithmetic-geometric mean inequality. Based on the Young inequality (1), one can establish versions of the inequality for the arithmetic and geometric means of positive definite complex matrices A and B of size $n \times n$ in the form:

$$A\nabla_\nu B := (1-\nu)A + \nu B \geqslant A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2} := A\sharp_\nu B \tag{2}$$

or inequalities involving the trace, unitary invariant norm inequalities, and determinant inequalities on the space of $n \times n$ complex matrices, respectively given by:

$$\begin{gathered} \operatorname{tr}(A\nabla_\nu B) \geqslant \operatorname{tr}(|A\sharp_\nu B|), \\ (1-\nu)\,|||AX|||+\nu\,|||XB||| \geqslant |||A^{1-\nu}XB^\nu|||, \\ \det(A\nabla_\nu B) \geqslant \det(|A\sharp_\nu B|), \end{gathered} \tag{3}$$

where the absolute value of a matrix A is defined as $|A| := (A^*A)^{1/2}$. These results remain valid when A and B are positive invertible operators. The practice of transforming

Mathematics subject classification (2020): 26D15, 47A63, 15A60, 47A30, 15A39, 15B48.

Keywords and phrases: Young inequality, Alzer-Fonseca-Kovačec-type inequalities, logarithmic constant, Kantorovich constant, operator inequality, positive operator, arithmetic-geometric mean inequality, weak sub-majorization.

Paper JMI-20-18

known inequalities from the scalar setting to their corresponding operator versions is an effective research direction. This approach serves as the motivation for us to propose refinements and reverses for Young's inequality. In the following discussion, we will present some initial and significant results obtained from this direction.

One of the earliest and pioneering achievements in the development of Young's inequality can be attributed to the following results in 2010 and 2011 ([15], [16]):

$$a\sharp_\nu b + r_0(\nu)(\sqrt{a}-\sqrt{b})^2 \leqslant a\nabla_\nu b \leqslant a\sharp_\nu b + R_0(\nu)(\sqrt{a}-\sqrt{b})^2, \tag{4}$$

where $r_0(\nu) = \min\{\nu, 1-\nu\}$, $R_0(\nu) = \max\{\nu, 1-\nu\}$. The authors proposed refined and reverse versions of Young's inequality by adding certain non-negative terms to its right-hand side. In this approach, the quantities $r_0(\nu)$ and $R_0(\nu)$ play a crucial role. If their general properties can be determined, the resulting findings would be transformative and highly impactful. This was further refined by the group of authors M. Sababheh and D. Choi in the year 2016 as follows ([3], [22]):

THEOREM 1.1. ([3, 22]) *Let $a,b>0$, $0 \leqslant \nu \leqslant 1$ and N be a positive integer. Then, we have*

$$a\nabla_\nu b \geqslant a\sharp_\nu b + \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} f_{n,k}(a,b)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu), \tag{5}$$

and

$$a\nabla_\nu b \leqslant a\sharp_\nu b + R_0(\nu)(\sqrt{a}-\sqrt{b})^2 - \sum_{n=1}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} f_{n,k}(b,a)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu), \tag{6}$$

where

$$f_{n,k}(a,b) = \left(\sqrt{a\sharp_{(k-1)/2^n} b} - \sqrt{a\sharp_{k/2^n} b}\right)^2, \tag{7}$$

χ is the characteristic function of the interval I, which is equal to 1 if $\nu \in I$ and 0 otherwise and

$$\begin{aligned} r_0(\nu) &= \min\{\nu, 1-\nu\}, \quad R_0(\nu) = \max\{\nu, 1-\nu\}, \\ r_n(\nu) &= \min\{2r_{n-1}(\nu), 1-2r_{n-1}(\nu)\}, \ n = 1,2,3\ldots \end{aligned} \tag{8}$$

An important observation is that multiplying the term $a\sharp_\nu b$ in (5) by a factor no less than 1 leads to improved results. The most notable factors explored in this direction include the Kantorovich constant, the Specht ratio, and logarithmic coefficients. The key results in this line of research are summarized as follows.

THEOREM 1.2. ([3]) *Let $a,b>0$, $0 \leqslant \nu \leqslant 1$ and N be a positive integer. Then, we have*

$$a\nabla_\nu b \geqslant K\left(\sqrt[2^N]{\frac{b}{a}}, 2\right)^{r_N(\nu)} a\sharp_\nu b + \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} f_{n,k}(a,b)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu), \tag{9}$$

and

$$a\nabla_\nu b \leqslant K\left(\sqrt[2^N]{\frac{b}{a}},2\right)^{-r_N(\nu)} a\sharp_\nu b + R_0(\nu)(\sqrt{a}-\sqrt{b})^2 - \sum_{n=1}^{N-1} r_n(\nu)\sum_{k=1}^{2^n} f_{n,k}(b,a)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu), \tag{10}$$

where χ *is the characteristic function of the interval* I*;* $f_{n,k}(a,b)$ *is defined as in* (7)*;* $r_0(\nu)$*,* $R_0(\nu)$*,* $r_n(\nu)$*,* $n=1,2,\dots$ *are in* (8) *and* $K(h,2)$ *is the Kantorovich constant given by*

$$K(h,2) = \frac{(1+h)^2}{4h}. \tag{11}$$

In 2021, M. A. Ighachane and M. Akkouchi ([9]) proved that:

THEOREM 1.3. ([9]) *Assuming the hypotheses are as given in Theorem 1.2, we have:*

$$a\nabla_\nu b \geqslant \left(1+\frac{L\left(2^N\nu\right)}{2^{2N}}\ln^2\left(\frac{b}{a}\right)\right) a\sharp_\nu b + \sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n} f_{n,k}(a,b)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \tag{12}$$

and

$$a\nabla_\nu b \leqslant \left(1+\frac{L\left(2^N(1-\nu)\right)}{2^{2N}}\ln^2\left(\frac{b}{a}\right)\right)^{-1} a\sharp_\nu b + R_0(\nu)(\sqrt{a}-\sqrt{b})^2 - \sum_{n=1}^{N-1} r_n(\nu)\sum_{k=1}^{2^n} f_{n,k}(b,a)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu), \tag{13}$$

where the function L *is the 1-periodic function given by*

$$L(\nu) = \frac{\nu^2}{2}\left(\frac{1-\nu}{\nu}\right)^{2\nu} \quad \textit{for} \quad \nu\in(0,1) \quad \textit{and} \quad L(0)=L(1)=0. \tag{14}$$

Note that, L is symmetric about $\frac{1}{2}$, namely, $L(\nu)=L(1-\nu)$ for all $\nu\in[0,1]$. The quantity $1+\frac{L(2^N\nu)}{2^{2N}}\ln^2\left(\frac{b}{a}\right)$ is provisionally called the logarithmic coefficient. In a 2017 study ([17]), P. Kórus demonstrated that the Kantorovich constant and the logarithmic coefficient are not comparable. Consequently, results derived from these coefficients retain their individual significance and diverge from each other.

A novel approach that provided stronger estimates for Young's inequality was introduced in 2015 by Y. Manasrah and F. Kittaneh ([19]). This work was further refined by D. Choi in 2018 ([5]), where they presented power-type estimates with positive integer powers m for the term $a\nabla_\nu b$:

$$(a\sharp_\nu b)^m + (2r_0(\nu))^m\left[\left(\frac{a+b}{2}\right)^m - (\sqrt{ab})^m\right] \leqslant (a\nabla_\nu b)^m$$
$$\leqslant (a\sharp_\nu b)^m + (2R_0(\nu))^m\left[\left(\frac{a+b}{2}\right)^m - (\sqrt{ab})^m\right], \quad (15)$$

where $a,b>0$, $\nu\in[0,1]$, $r_0(\nu)=\min\{\nu,1-\nu\}$ and $R_0(\nu)=\max\{\nu,1-\nu\}$. For further related results, the reader is also referred to ([12], [13], [14], [24], [25]).

A natural question that arises is whether these results, having been established for positive integer powers m, can be extended to arbitrary real exponents. To this end, in 2023, D. Q. Huy et al. ([6]) leveraged the theory of weak submajorization to introduce the following remarkable results.

THEOREM 1.4. ([6]) *If $a,b>0$ and $\nu\in[0,1]$, then for all real number $p\geqslant 1$, we have*

$$(a\sharp_\nu b)^p + (2r_0(\nu))^p S_0 + (2r_1(\nu))^p\left(\sqrt{a}^p\chi_{(0,\frac{1}{2})}(\nu) + \sqrt{b}^p\chi_{(\frac{1}{2},1)}(\nu)\right)S_1 \leqslant (a\nabla_\nu b)^p$$
$$\leqslant (a\sharp_\nu b)^p + (2R_0(\nu))^p S_0 - (2r_1(\nu))^p\left(\sqrt{b}^p\chi_{(0,\frac{1}{2})}(\nu) + \sqrt{a}^p\chi_{(\frac{1}{2},1)}\right)(\nu)\Big)S_1, \quad (16)$$

where $r_0(\nu)=\min\{\nu,1-\nu\}$, $R_0(\nu)=\max\{\nu,1-\nu\}$, $r_1(\nu)=\min\{2r_0(\nu),1-2r_0(\nu)\}$, $S_0 := \left(\frac{a+b}{2}\right)^p - (\sqrt{ab})^p$ and $S_1 := \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^p - (\sqrt[4]{ab})^p$.

The key point in this work is that the authors utilized the following theory: For two vectors $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ in $\mathbb{R}^n, x$ is said to be weakly sub-majorized by y, written $x\prec_w y$, if

$$\sum_{i=1}^k x_i^* \leqslant \sum_{i=1}^k y_i^*, \quad (17)$$

for all $k=1,\ldots,n$, where $x^*=(x_1^*,\ldots,x_n^*)$ the vector obtained from the vector $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ by rearranging the components in non-increasing order. Theoretically, this method enables the extension of results to any vectors in $\mathbb{R}^n$. However, this approach faces substantial challenges in practice when relying solely on the fundamental definition of weak submajorization because of the number of sub-inequalities that need to be calculated is too large ([7]). To address this critical limitation, Huy et al. recently introduced, in 2025, new properties within the theory of weak submajorization, providing a novel perspective on the subject. This advancement has paved the way for the following promising results ([7]).

THEOREM 1.5. ([7]) *Let $a,b>0$, $0\leqslant\nu\leqslant 1$ and N be a positive integer. Then, for every positive real number $p\geqslant 1$, we have*

$$(a\nabla_\nu b)^p \geqslant (a\sharp_\nu b)^p + \sum_{n=0}^{N-1}(2r_n(\nu))^p\sum_{k=1}^{2^n}\Delta_{n,k}(a,b;p)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \quad (18)$$

and

$$(a\nabla_\nu b)^p \leqslant (a\sharp_\nu b)^p + (2R_0(\nu))^p\left[\left(\frac{a+b}{2}\right)^p - (\sqrt{ab})^p\right] - \sum_{n=1}^{N-1}(2r_n(\nu))^p\sum_{k=1}^{2^n}\Delta_{n,k}(b,a;p)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu), \tag{19}$$

where $\Delta_{n,k}(a,b;p)$ *are defined as follows*

$$\Delta_{n,k}(a,b;p) = \left(\frac{a\sharp_{(k-1)/2^n}b + a\sharp_{k/2^n}b}{2}\right)^p - \left(a\sharp_{(2k-1)/2^{n+1}}b\right)^p. \tag{20}$$

In 2015, H. Alzer, C. Fonseca and A. Kovačec ([1]) were the pioneers in introducing the two-weight refinement of Young's inequality. Their result can be expressed as.

THEOREM 1.6. ([1]) *If* λ,α,β *are real numbers satisfying* $\lambda \geqslant 1$ *and* $0<\alpha<\beta<1$ *then for all positive real numbers* a,b *we have*

$$\left(\frac{\alpha}{\beta}\right)^\lambda < \frac{(a\nabla_\alpha b)^\lambda - (a\sharp_\alpha b)^\lambda}{(a\nabla_\beta b)^\lambda - (a\sharp_\beta b)^\lambda} < \left(\frac{1-\alpha}{1-\beta}\right)^\lambda. \tag{21}$$

A remarkable aspect of the Alzer-Fonseca-Kovačec inequality is that the authors successfully proved it for the case of powers $\lambda \geqslant 1$. In Theorem 1.6, by choosing $\lambda = 1$ we immediately obtain the following refinement of Young's inequality.

$$\frac{\alpha}{\beta}\left(a\nabla_\beta b - a\sharp_\beta b\right) < a\nabla_\alpha b - a\sharp_\alpha b < \frac{1-\alpha}{1-\beta}\left(a\nabla_\beta b - a\sharp_\beta b\right).$$

These results were later extended by M. Sababheh ([23]), Y. Ren ([20]), Y. Ren and P. Li ([21]), J. Zhao ([27]), M. A. Ighachane ([10]), C. Yang and Z. Wang ([26]), M. A. Ighachane, M. Akkouchi and E. H. Benabdi ([8]) to various other contexts. For the power $\lambda = 1$, very recently, the Ighachane group of authors came up with a multiterm version as follows.

THEOREM 1.7. ([11]) *Let* $a,b>0$, $0<\alpha<\beta<1$ *and* $N\in\mathbb{N}^*$. *Then*

$$a\nabla_\alpha b \geqslant \left(1 + \frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{a\sharp_\beta b}{a}\right)\right)a\sharp_\alpha b + \frac{\alpha}{\beta}\left(a\nabla_\beta b - a\sharp_\beta b\right) + \sum_{n=0}^{N-1} r_n\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^n} f_{n,k}\left(a, a\sharp_\beta b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \tag{22}$$

and

$$a\nabla_\beta b \geqslant \left(1+\frac{L\left(2^N\frac{1-\beta}{1-\alpha}\right)}{2^{2N}}\ln^2\left(\frac{a\sharp_\alpha b}{b}\right)\right)a\sharp_\beta b+\frac{1-\beta}{1-\alpha}\left(a\nabla_\alpha b-a\sharp_\alpha b\right)$$
$$+\sum_{n=0}^{N-1} r_n\left(\frac{1-\beta}{1-\alpha}\right)\sum_{k=1}^{2^n} f_{n,k}\left(b,a\sharp_\alpha b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right), \quad (23)$$

where $f_{n,k}(a,b)$ *and* L *are defined as in* (7) *and* (14)*, respectively.*

However, further development of the Alzer-Fonseca-Kovačec inequality for real powers $\lambda \geqslant 1$ in the case of multiple terms remains an open question for mathematicians intrigued by this inequality. Motivated by the refinement of Young's inequality for real powers $\lambda \geqslant 1$ with an arbitrary number of terms, as presented in ([7]), which introduced new properties of the submajorization theory, we are optimistic about advancing the Alzer-Fonseca-Kovačec inequality for real powers $\lambda \geqslant 1$ in the multi-term case. Following that, based on these numerical results, we will introduce new corresponding achievements in operator theory, matrix determinants, and unitarily invariant norms. With this objective in mind, we will structure the article as follows: Section 1 introduces the motivation and objectives of the paper, Section 2 presents the scalar results, Section 3 discusses applications to operators, Section 4 addresses applications to unitarily invariant norms, and the final section presents applications to matrix determinants.

2. Some multiple-term refinements and reverses with real powers of Alzer-Fonseca-Kovačec inequality for convex functions and scalar versions

We begin this section with a key and important result for Alzer-Fonseca-Kovačec-type inequalities with logarithmic constants for convex functions.

THEOREM 2.1. *Let* $a,b>0,\ 0<\alpha<\beta<1,\ N\in\mathbb{N}^*$, *and* Φ *be a convex, increasing function defined on* $[0,\infty)$. *Then*

$$\Phi(a\nabla_\alpha b)\geqslant \Phi\left(\left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\frac{a\sharp_\beta b}{a}\right)a\sharp_\alpha b\right)+\Phi\left(\frac{\alpha}{\beta}a\nabla_\beta b\right)-\Phi\left(\frac{\alpha}{\beta}a\sharp_\beta b\right)$$
$$+\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Phi_{n,k}\left(a,a\sharp_\beta b;\frac{\alpha}{\beta},p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \quad (24)$$

and

$$\Phi(a\nabla_\beta b)\geqslant \Phi\left(\left(1+\frac{L\left(2^N\frac{1-\beta}{1-\alpha}\right)}{2^{2N}}\ln^2\frac{a\sharp_\alpha b}{b}\right)a\sharp_\beta b\right)+\Phi\left(\frac{1-\beta}{1-\alpha}a\nabla_\alpha b\right)$$
$$-\Phi\left(\frac{1-\beta}{1-\alpha}a\sharp_\alpha b\right)+\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Phi_{n,k}\left(b,a\sharp_\alpha b;\frac{1-\beta}{1-\alpha},p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right), \quad (25)$$

where

$$\Phi_{n,k}(a,b;\alpha,p) = \Phi\left(r_n(\alpha)\left(a\sharp_{\frac{k-1}{2^n}}b + a\sharp_{\frac{k}{2^n}}b\right)\right) - \Phi\left(2r_n(\alpha)\,a\sharp_{\frac{2k-1}{2^{n+1}}}b\right). \tag{26}$$

To prove Theorem 2.1, we need to use the following crucial lemma, which describes the relationship between the theory of weakly submajorization and the class of convex, increasing functions. This result is introduced by Huy et al. ([7]) in 2025.

LEMMA 2.2. ([7]) *Let $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$ be two vectors in $\mathbb{R}^n$ satisfying the conditions*

$$b_1 - a_1 \geqslant \sum_{i=2}^{n}(a_i - b_i) \geqslant a_j - b_j \geqslant 0, \quad j = 2,\dots,n \tag{27}$$

and ϕ is an increasing convex function defined on the interval I containing components of the vectors a and b. If $b_1 \geqslant a_j$ for every $j = 2,\dots,n$ and for every $k = 2,\dots,n-1$, the following condition

$$\sum_{i=1}^{k} b_i \geqslant \sum_{i=2}^{k+1} a_i \tag{28}$$

is satisfied, then the following inequality holds:

$$\sum_{i=1}^{n}\phi(a_i) \leqslant \sum_{i=1}^{n}\phi(b_i). \tag{29}$$

We also consider the following important lemma.

LEMMA 2.3. *Let the vectors $a = (a_1,a_2,\dots,a_{N+1})$ and $b = (b_1,b_2,\dots,b_{N+1})$ as two vectors in $\mathbb{R}^{N+1}$, with their components respectively determined as follows:* $a_1 = \left(1 + \frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{a\sharp_\beta b}{a}\right)\right)a\sharp_\alpha b$, $b_1 = a\nabla_\alpha b$, $a_2 = \frac{\alpha}{\beta}a\nabla_\beta b$, $b_2 = \frac{\alpha}{\beta}a\sharp_\beta b$ *and*

$$a_{n+3} = \sum_{k=1}^{2^n} r_n\left(\frac{\alpha}{\beta}\right)\left(a\sharp_{\beta\frac{k-1}{2^n}}b + a\sharp_{\beta\frac{k}{2^n}}b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),$$

$$b_{n+3} = \sum_{k=1}^{2^n} 2r_n\left(\frac{\alpha}{\beta}\right)a\sharp_{\beta\frac{2k-1}{2^{n+1}}}b\,\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),$$

where $a,b \in \mathbb{R}$, $a,b > 0$, $0 < \alpha < \beta < 1$ and $n = 0,\dots,N-2$, $N \in \mathbb{N}^*$ we have:

(i) $b_1 - a_1 \geqslant \sum_{i=2}^{N+1}(a_i - b_i) \geqslant a_i - b_i$, *for all* $i = 2,\dots,N+1$;

(ii) $b_1 \geqslant a_j$, *for all* $i = 2,\dots,N+1$;

(iii) $\sum_{i=1}^{k} b_i \geqslant \sum_{i=2}^{k+1} a_i$ *for all* $k = 2, \dots, N$.

Proof of Lemma 2.3.

(i) According to (22), we have

$$b_1 \geqslant a_1 + a_2 - b_2 + \sum_{i=3}^{N+1} (a_i - b_i),$$

which directly leads us to the issue that needs to be proven.

(ii) We consider the following two cases.

• In the case where $j = 2$, what we need to prove is specifically that $b_1 \geqslant a_2$. Simplifying further, we obtain $a(1 - \frac{\alpha}{\beta}) \geqslant 0$, which is always true. Therefore, we conclude that the inequality we set out to prove is satisfied.

• With $j > 2$, we need to prove

$$a\nabla_\alpha b \geqslant \sum_{k=1}^{2^n} r_n\left(\frac{\alpha}{\beta}\right)\left(a\sharp_{\beta\frac{k-1}{2^n}} b + a\sharp_{\beta\frac{k}{2^n}} b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),$$

where $n = 0, \dots, N-2$. We have

$$\begin{cases} \beta\frac{k-1}{2^n} & = \left(1 - \beta\frac{k-1}{2^n}\right)0 + \beta\frac{k-1}{2^n}1 \\ \beta\frac{k}{2^n} & = \left(1 - \beta\frac{k}{2^n}\right)0 + \beta\frac{k}{2^n}1 \end{cases}$$

therefore, due to the inequality (1), we obtain

$$\begin{cases} a^{1-\beta\frac{k-1}{2^n}} b^{\beta\frac{k-1}{2^n}} & \leqslant \left(1 - \beta\frac{k-1}{2^n}\right)a + \beta\frac{k-1}{2^n}b \\ a^{1-\beta\frac{k}{2^n}} b^{\beta\frac{k}{2^n}} & \leqslant \left(1 - \beta\frac{k}{2^n}\right)a + \beta\frac{k}{2^n}b \end{cases}$$

it follows that

$$\begin{aligned} & a\nabla_\alpha b - \left(k - 2^n\frac{\alpha}{\beta}\right)a^{1-\beta\frac{k-1}{2^n}} b^{\beta\frac{k-1}{2^n}} - \left(1 - k + 2^n\frac{\alpha}{\beta}\right)a^{1-\beta\frac{k}{2^n}} b^{\beta\frac{k}{2^n}} \\ & \geqslant a\nabla_\alpha b - \left(k - 2^n\frac{\alpha}{\beta}\right)\left[\left(1 - \beta\frac{k-1}{2^n}\right)a + \beta\frac{k-1}{2^n}b\right] \\ & \quad - \left(1 - k + 2^n\frac{\alpha}{\beta}\right)\left[\left(1 - \beta\frac{k}{2^n}\right)a + \beta\frac{k}{2^n}b\right] \\ & = 0. \end{aligned}$$

This leads to for all $\frac{\alpha}{\beta} \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$, $n = 0, \dots, N-2$, we have

$$\begin{aligned} a\nabla_\alpha b & \geqslant \left(k - 2^n\frac{\alpha}{\beta}\right)a^{1-\beta\frac{k-1}{2^n}} b^{\beta\frac{k-1}{2^n}} - \left(1 - k + 2^n\frac{\alpha}{\beta}\right)a^{1-\beta\frac{k}{2^n}} b^{\beta\frac{k}{2^n}} \\ & = \left(k - 2^n\frac{\alpha}{\beta}\right)a\sharp_{\beta\frac{k-1}{2^n}} b + \left(1 - k + 2^n\frac{\alpha}{\beta}\right)a\sharp_{\beta\frac{k}{2^n}} b \end{aligned}$$

$$\geqslant \min\left\{k-2^n\frac{\alpha}{\beta}, 1-k+2^n\frac{\alpha}{\beta}\right\}\left(a\sharp_{\beta\frac{k-1}{2^n}}b + a\sharp_{\beta\frac{k}{2^n}}b\right)$$
$$= r_n\left(\frac{\alpha}{\beta}\right)\left(a\sharp_{\beta\frac{k-1}{2^n}}b + a\sharp_{\beta\frac{k}{2^n}}b\right),$$

this means that $b_1 \geqslant a_j$, for all $j = 3,\dots,N+1$.

(iii) Following the proof of ([4], Lemma 2), we have

$$\begin{aligned}
& b_1 - \sum_{i=2}^{n+3}(a_i - b_i) \\
&= b_1 - (a_2 - b_2) - \sum_{i=3}^{n+3}(a_i - b_i) \\
&= \left(1-\frac{\alpha}{\beta}\right)a + \frac{\alpha}{\beta}a\sharp_\beta b \\
&\quad - \sum_{i=3}^{n+3}\sum_{k=1}^{2^n} r_n\left(\frac{\alpha}{\beta}\right)\left(a\sharp_{\beta\frac{k-1}{2^n}}b + a\sharp_{\beta\frac{k}{2^n}}b - 2a\sharp_{\beta\frac{2k-1}{2^{n+1}}}b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right) \\
&= \left(k-2^n\frac{\alpha}{\beta}\right)a\sharp_{\beta\frac{k-1}{2^n}}b + \left(2^n\frac{\alpha}{\beta}-k+1\right)a\sharp_{\beta\frac{k}{2^n}}b \\
&\geqslant \min\left\{k-2^n\frac{\alpha}{\beta}, 2^n\frac{\alpha}{\beta}-k+1\right\}\left(a\sharp_{\beta\frac{k-1}{2^n}}b + a\sharp_{\beta\frac{k}{2^n}}b\right) \\
&= r_n\left(\frac{\alpha}{\beta}\right)\left(a\sharp_{\beta\frac{k-1}{2^n}}b + a\sharp_{\beta\frac{k}{2^n}}b\right) \\
&= a_{n+3},
\end{aligned}$$

for all $n = 0,\dots,N-2$. At this point, we deduce that

$$b_1 - \sum_{i=2}^{n+3}(a_i - b_i) \geqslant a_{n+3}.$$

or

$$\sum_{i=1}^{n+2} b_i + b_{n+3} \geqslant \sum_{i=2}^{n+3} a_i + a_{n+3}$$

This is equivalent to

$$\sum_{i=1}^{n+2} b_i \geqslant \sum_{i=2}^{n+3} a_i,$$

the reason we arrive at the final inequality because $a_{n+3} \geqslant b_{n+3}$, for all $n = 0,\dots,N-2$. □

Proof of Theorem 2.1. Choosing a and b to be vectors in Lemma 2.3. At that time, vectors a and b satisfy the conditions of the Lemma 2.2, therefore, according to this lemma, we have:

$$\sum_{i=1}^{N+1}\Phi(a_i) \leqslant \sum_{i=1}^{N+1}\Phi(b_i),$$

where Φ be a convex, increasing function defined on $[0,\infty)$. This inequality is equivalent to (24). Therefore, we have what needs to be proven.

In inequality (24), by replacing a,b,α,β with $b,a,1-\beta,1-\alpha$ respectively, we obtain inequality (25). Thus, we complete the proof of Theorem 2.1. □

THEOREM 2.4. *Let $a,b>0$, $0<\alpha<\beta<1$, $\nu\in[0,1]$, $N\in\mathbb{N}^*$, and Φ be a convex, increasing function defined on $[0,\infty)$, we have*

$$\Phi(a\nabla_\alpha b) \geqslant \Phi\left(K\left(\sqrt[2^N]{\frac{a\sharp_\beta b}{a}},2\right)^{r_N\left(\frac{\alpha}{\beta}\right)} a\sharp_\alpha b\right) + \Phi\left(\frac{\alpha}{\beta}a\nabla_\beta b\right) - \Phi\left(\frac{\alpha}{\beta}a\sharp_\beta b\right) + \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Phi_{n,k}\left(a,a\sharp_\beta b;\frac{\alpha}{\beta},p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \tag{30}$$

and

$$\Phi(a\nabla_\beta b) \geqslant \Phi\left(K\left(\sqrt[2^N]{\frac{a\sharp_\alpha b}{b}},2\right)^{r_N\left(\frac{1-\beta}{1-\alpha}\right)} a\sharp_\beta b\right) + \Phi\left(\frac{1-\beta}{1-\alpha}a\nabla_\alpha b\right) - \Phi\left(\frac{1-\beta}{1-\alpha}a\sharp_\alpha b\right) + \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Phi_{n,k}\left(b,a\sharp_\alpha b;\frac{1-\beta}{1-\alpha},p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right), \tag{31}$$

where $\Phi_{n,k}(a,b;\cdot,p)$ are defined as in (26) and $K(\cdot,2)$ is the Kantorovich constant.

Proof. The proof of Theorem 2.4 follows a similar approach to that of Theorem 2.1. However, the term a_1 is replaced by $K\left(\sqrt[2^N]{\frac{a\sharp_\beta b}{a}},2\right)^{r_N\left(\frac{\alpha}{\beta}\right)} a\sharp_\alpha b$, which necessitates revisiting (i). This point is clarified in Lemma 2.5 presented below. □

LEMMA 2.5. *Let $a,b>0$, $0<\alpha<\beta<1$ and $N\in\mathbb{N}^*$, we have*

$$a\nabla_\alpha b \geqslant K\left(\sqrt[2^N]{\frac{a\sharp_\beta b}{a}},2\right)^{r_N\left(\frac{\alpha}{\beta}\right)} a\sharp_\alpha b + \frac{\alpha}{\beta}\left(a\nabla_\beta b - a\sharp_\beta b\right) + \sum_{n=0}^{N-1} r_n\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^n} f_{n,k}\left(a,a\sharp_\beta b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \tag{32}$$

where $f_{n,k}(a,b)$ is defined as in (7) and $K(\cdot,2)$ is the Kantorovich constant.

Proof of Lemma 2.5. According to (9), we have

$$\begin{aligned}
&a\nabla_{\alpha}b-\frac{\alpha}{\beta}\left(a\nabla_{\beta}b-a\sharp_{\beta}b\right)\\
&\quad=\left(1-\frac{\alpha}{\beta}\right)a+\frac{\alpha}{\beta}a^{1-\beta}b^{\beta}\\
&\quad=a\nabla_{\frac{\alpha}{\beta}}\left(a\sharp_{\beta}b\right)\\
&\quad\geqslant K\left(\sqrt[2^N]{\frac{a\sharp_{\beta}b}{a}},2\right)^{r_N\left(\frac{\alpha}{\beta}\right)}a\sharp_{\frac{\alpha}{\beta}}\left(a\sharp_{\beta}b\right)\\
&\qquad\quad+\sum_{n=0}^{N-1}r_n\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^n}f_{n,k}\left(a,a\sharp_{\beta}b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right)\\
&\quad=K\left(\sqrt[2^N]{\frac{a\sharp_{\beta}b}{a}},2\right)^{r_N\left(\frac{\alpha}{\beta}\right)}a\sharp_{\alpha}b\\
&\qquad\quad+\sum_{n=0}^{N-1}r_n\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^n}f_{n,k}\left(a,a\sharp_{\beta}b\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right).
\end{aligned}$$

From this, we obtain (32). □

In Theorems 2.1 and 2.4, choosing $\Phi(t)=t^p,\ p\geqslant 1,\ t\in(0,\infty)$, we have the following important theorems for scalar Alzer-Fonseca-Kovačec version.

THEOREM 2.6. *Let $a,b>0,\ \nu\in[0,1],\ 0<\alpha<\beta<1,\ N\in\mathbb{N}^*,\ p\geqslant 1$, we have*

$$\begin{aligned}
(a\nabla_{\alpha}b)^p\geqslant&\left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{a\sharp_{\beta}b}{a}\right)\right)^p(a\sharp_{\alpha}b)^p+\left(\frac{\alpha}{\beta}\right)^p\left[(a\nabla_{\beta}b)^p-(a\sharp_{\beta}b)^p\right]\\
&+\sum_{n=0}^{N-1}r_n^p\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^n}f_{n,k}\left(a,a\sharp_{\beta}b;p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),
\end{aligned}\tag{33}$$

and

$$\begin{aligned}
(a\nabla_{\beta}b)^p\geqslant&\left(1+\frac{L\left(2^N\frac{1-\beta}{1-\alpha}\right)}{2^{2N}}\ln^2\left(\frac{a\sharp_{\alpha}b}{b}\right)\right)^p(a\sharp_{\beta}b)^p\\
&+\left(\frac{1-\beta}{1-\alpha}\right)^p\left[(a\nabla_{\alpha}b)^p-(a\sharp_{\alpha}b)^p\right]\\
&+\sum_{n=0}^{N-1}r_n^p\left(\frac{1-\beta}{1-\alpha}\right)\sum_{k=1}^{2^n}f_{n,k}\left(b,a\sharp_{\alpha}b;p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),
\end{aligned}\tag{34}$$

where

$$f_{n,k}(a,b;p)=\left(a\sharp_{\frac{k-1}{2^n}}b+a\sharp_{\frac{k}{2^n}}b\right)^p-\left(2a\sharp_{\frac{2k-1}{2^{n+1}}}b\right)^p.\tag{35}$$

THEOREM 2.7. *Let $a,b>0,\ \nu\in[0,1],\ 0<\alpha<\beta<1,\ N\in\mathbb{N}^*,\ p\geqslant 1$, we have*

$$\begin{aligned}(a\nabla_\alpha b)^p \geqslant K\left(\sqrt[2^N]{\frac{a\sharp_\beta b}{a}},2\right)^{pr_N\left(\frac{\alpha}{\beta}\right)}(a\sharp_\alpha b)^p \\ +\left(\frac{\alpha}{\beta}\right)^p\left[(a\nabla_\beta b)^p-(a\sharp_\beta b)^p\right] \\ +\sum_{n=0}^{N-1} r_n^p\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^n} f_{n,k}\left(a,a\sharp_\beta b;p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),\end{aligned} \tag{36}$$

and

$$\begin{aligned}(a\nabla_\beta b)^p \geqslant K\left(\sqrt[2^N]{\frac{a\sharp_\alpha b}{b}},2\right)^{pr_N\left(\frac{1-\beta}{1-\alpha}\right)}(a\sharp_\beta b)^p \\ +\left(\frac{1-\beta}{1-\alpha}\right)^p\left[(a\nabla_\alpha b)^p-(a\sharp_\alpha b)^p\right] \\ +\sum_{n=0}^{N-1} r_n^p\left(\frac{1-\beta}{1-\alpha}\right)\sum_{k=1}^{2^n} f_{n,k}\left(b,a\sharp_\alpha b;p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),\end{aligned} \tag{37}$$

where $f_{n,k}(a,b;p)$ is defined as in (35) and $K(\cdot,2)$ is the Kantorovich constant.

From Theorem 2.6 or 2.7, using the property that the logarithmic and Kantorovich coefficients are always greater than or equal to 1, we directly derive the following important results.

THEOREM 2.8. *Let $a,b>0,\ \nu\in[0,1],\ 0<\alpha<\beta<1,\ N\in\mathbb{N}^*,\ p\geqslant 1$, we have*

$$\begin{aligned}(a\nabla_\alpha b)^p \geqslant (a\sharp_\alpha b)^p+\left(\frac{\alpha}{\beta}\right)^p\left[(a\nabla_\beta b)^p-(a\sharp_\beta b)^p\right] \\ +\sum_{n=0}^{N-1} r_n^p\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^n} f_{n,k}\left(a,a\sharp_\beta b;p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),\end{aligned} \tag{38}$$

and

$$\begin{aligned}(a\nabla_\beta b)^p \geqslant (a\sharp_\beta b)^p+\left(\frac{1-\beta}{1-\alpha}\right)^p\left[(a\nabla_\alpha b)^p-(a\sharp_\alpha b)^p\right] \\ +\sum_{n=0}^{N-1} r_n^p\left(\frac{1-\beta}{1-\alpha}\right)\sum_{k=1}^{2^n} f_{n,k}\left(b,a\sharp_\alpha b;p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),\end{aligned} \tag{39}$$

where $f_{n,k}(a,b;p)$ is defined as in (35).

3. Application for operator versions

Our main goal in this section is to use versions of multi-term two-weighted Young-type inequalities with real power $p \geqslant 1$ to establish their operator forms.

Let $\mathscr{B}(H)$ be an algebra of all bounded linear operators on a complex Hilbert space H. We denote invertible positive operators by capital letters and the identity operator by I. In addition, we also use the following notations

- $A \geqslant 0$ $(A > 0)$ if A is a positive (invertible positive) operator;
- $A \geqslant B$ $(A > B)$ if $A - B$ is a positive (invertible positive) operator.

For $A, B > 0$ and $\nu \in (0,1)$ the ν-weighted arithmetic and geometric means of A and B are defined respectively by

$$A \nabla_\nu B = (1-\nu)A + \nu B,$$

$$A \sharp_\nu B = A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^\nu A^{1/2}.$$

We also write $A \nabla B$ and $A \sharp B$ instead of $A \nabla_{\frac{1}{2}} B$ and $A \sharp_{\frac{1}{2}} B$, respectively. We also use the same symbol as geometric mean for $\nu \in \mathbb{R}$.

The main idea for showing operator inequalities corresponding to their scalar versions is to use the operator monotonicity of continuous functions in the following.

LEMMA 3.1. *Let $X \in \mathscr{B}(H)$ be an arbitrary self-adjoint operator. If f and g are continuous real functions satisfying $f(t) \geqslant g(t)$ for all t in the spectrum of X, we have $f(X) \geqslant g(X)$.*

The results we obtain are presented in the theorems below.

THEOREM 3.2. *Let $0 < \alpha < \beta < 1$, N be a positive integer and $A, B > 0$ satisfying one of the following conditions:*

(i) $0 < mI \leqslant A \leqslant \gamma I < \Gamma I \leqslant B \leqslant MI$,

(ii) $0 < mI \leqslant B \leqslant \gamma I < \Gamma I \leqslant A \leqslant MI$,

where $0 < m, M, \gamma, \Gamma < \infty$ are scalars. Then, for every positive real number $p \geqslant 1$, we have:

$$\begin{aligned} &A \sharp_p (A \nabla_\alpha B) \\ &\geqslant \left(1 + \frac{L\left(2^N \frac{\alpha}{\beta}\right)}{2^{2N}} \ln^2\left(\frac{\Gamma}{\gamma}\right)^\beta\right)^p A \sharp_{\alpha p} B + \left(\frac{\alpha}{\beta}\right)^p \left[A \sharp_p (A \nabla_\beta B) - A \sharp_{\beta p} B\right] \\ &\quad + \sum_{n=0}^{N-1}\left(2 r_n\left(\frac{\alpha}{\beta}\right)\right)^p \sum_{k=1}^{2^n}\left[A \sharp_p \left(\left(A \sharp_{\beta \frac{k-1}{2^n}} B\right) \nabla \left(A \sharp_{\beta \frac{k}{2^n}} B\right)\right)\right. \\ &\qquad\qquad \left. - A \sharp_{p\beta \frac{2k-1}{2^{n+1}}} B\right] \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \end{aligned} \tag{40}$$

and

$$\begin{aligned} &A\sharp_p(A\nabla_\beta B)\\ &\geqslant \left(1+\frac{L\left(2^N\frac{1-\beta}{1-\alpha}\right)}{2^{2N}}\ln^2\left(\frac{\Gamma}{\gamma}\right)^{1-\alpha}\right)^p A\sharp_{\beta p}B+\left(\frac{1-\beta}{1-\alpha}\right)^p[A\sharp_p(A\nabla_\alpha B)-A\sharp_{\alpha p}B]\\ &\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p\sum_{k=1}^{2^n}\left[A\sharp_p\left(\left(B\sharp_{(1-\alpha)\frac{k-1}{2^n}}A\right)\nabla\left(B\sharp_{(1-\alpha)\frac{k}{2^n}}A\right)\right)\right.\\ &\qquad\left.-B\sharp_{p(1-\alpha)\frac{2k-1}{2^{n+1}}}A\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right). \qquad (41)\end{aligned}$$

Proof of Theorem 3.2. We notice that $\frac{m}{M}\leqslant\frac{\gamma}{\Gamma}<1<\frac{\Gamma}{\gamma}\leqslant\frac{M}{m}$. Firstly, we suppose that the operators A,B satisfy the condition (i). Utilizing the inequality (33) and the increase of the function $\left(1+\frac{L(2^N\frac{\alpha}{\beta})}{2^{2N}}\ln^2(x^\beta)\right)$ on $[1;+\infty)$, we have, for all $x\in\left[\frac{\Gamma}{\gamma};\frac{M}{m}\right]\subset\left[\frac{m}{M},\frac{M}{m}\right]$:

$$\begin{aligned} &((1-\alpha)+\alpha x)^p\\ &\geqslant\left[1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2(x^\beta)\right]^p(x^\alpha)^p+\frac{\alpha}{\beta}\left[((1-\beta)+\beta x)^p-(x^\beta)^p\right]\\ &\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left(\left(\frac{x^{\beta\frac{k-1}{2^n}}+x^{\beta\frac{k}{2^n}}}{2}\right)^p-\left(x^{\beta\frac{2k-1}{2^{n+1}}}\right)^p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right)\\ &\geqslant\min_{\frac{\Gamma}{\gamma}\leqslant x\leqslant\frac{M}{m}}\left[1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2(x^\beta)\right]^p x^{\alpha p}+\frac{\alpha}{\beta}\left[((1-\beta)+\beta x)^p-x^{\beta p}\right]\\ &\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left(\left(\frac{x^{\beta\frac{k-1}{2^n}}+x^{\beta\frac{k}{2^n}}}{2}\right)^p-\left(x^{\beta\frac{2k-1}{2^{n+1}}}\right)^p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right)\\ &=\left[1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{\Gamma}{\gamma}\right)^\beta\right]^p x^{\alpha p}+\frac{\alpha}{\beta}\left[((1-\beta)+\beta x)^p-x^{\beta p}\right]\\ &\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left(\left(\frac{x^{\beta\frac{k-1}{2^n}}+x^{\beta\frac{k}{2^n}}}{2}\right)^p-\left(x^{\beta\frac{2k-1}{2^{n+1}}}\right)^p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

This, together with Lemma 3.1, implies that, for every positive operator X with its spectrum in $\left[\frac{\Gamma}{\gamma};\frac{M}{m}\right]$:

$$\begin{aligned}&((1-\alpha)I+\alpha X)^p\\&\geqslant\left[1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{\Gamma}{\gamma}\right)^{\beta}\right]^p X^{\alpha p}+\frac{\alpha}{\beta}\left[((1-\beta)I+\beta X)^p-X^{\beta p}\right]\\&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left(\left(\frac{X^{\beta\frac{k-1}{2^n}}+X^{\beta\frac{k}{2^n}}}{2}\right)^p-\left(X^{\beta\frac{2k-1}{2^{n+1}}}\right)^p\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

On the other hand, by the condition (i), the spectrum $\mathrm{Sp}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)$ of the operator $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ is in $\left[\frac{\Gamma}{\gamma};\frac{M}{m}\right]$. Thus replacing X in the above inequality with $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ and multiplying both sides of it by $A^{\frac{1}{2}}$, we get the inequality to be proved (40). Inequality (41) is tested in a similar way, so we omit the details. □

Using Theorems 2.7 and 2.8, along with the method applied in Theorem 3.2, we successively obtain the following operator versions of the inequalities.

THEOREM 3.3. *Under the assumptions of Theorem 3.2, we have*

$$\begin{aligned}A\sharp_p(A\nabla_\alpha B)\geqslant K\left(\sqrt[2^N]{\left(\frac{\Gamma}{\gamma}\right)^{\beta}},2\right)^{pr_N\left(\frac{\alpha}{\beta}\right)}A\sharp_{\alpha p}B+\left(\frac{\alpha}{\beta}\right)^p\left[A\sharp_p(A\nabla_\beta B)-A\sharp_{\beta p}B\right]\\+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\Bigg[A\sharp_p\left(\left(A\sharp_{\beta\frac{k-1}{2^n}}B\right)\nabla\left(A\sharp_{\beta\frac{k}{2^n}}B\right)\right)\\-A\sharp_{p\beta\frac{2k-1}{2^{n+1}}}B\Bigg]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),\end{aligned}\tag{42}$$

and

$$\begin{aligned}&A\sharp_p(A\nabla_\beta B)\\&\geqslant K\left(\sqrt[2^N]{\left(\frac{\Gamma}{\gamma}\right)^{1-\alpha}},2\right)^{pr_N\left(\frac{1-\beta}{1-\alpha}\right)}A\sharp_{\beta p}B+\left(\frac{1-\beta}{1-\alpha}\right)^p\left[A\sharp_p(A\nabla_\alpha B)-A\sharp_{\alpha p}B\right]\\&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p\sum_{k=1}^{2^n}\Bigg[A\sharp_p\left(\left(B\sharp_{(1-\alpha)\frac{k-1}{2^n}}A\right)\nabla\left(B\sharp_{(1-\alpha)\frac{k}{2^n}}A\right)\right)\\&\qquad-B\sharp_{p(1-\alpha)\frac{2k-1}{2^{n+1}}}A\Bigg]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),\end{aligned}\tag{43}$$

where $K(\cdot,2)$ is Kantorovich constant.

THEOREM 3.4. *Under the assumptions of Theorem 3.2, we have*

$$\begin{aligned}A\sharp_p(A\nabla_\alpha B) \geqslant A\sharp_{\alpha p}B + \left(\frac{\alpha}{\beta}\right)^p\left[A\sharp_p(A\nabla_\beta B) - A\sharp_{\beta p}B\right]\\ + \sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\Bigg[A\sharp_p\left(\left(A\sharp_{\beta\frac{k-1}{2^n}}B\right)\nabla\left(A\sharp_{\beta\frac{k}{2^n}}B\right)\right)\\ - A\sharp_{p\beta\frac{2k-1}{2^{n+1}}}B\Bigg]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),\end{aligned} \tag{44}$$

and

$$\begin{aligned}&A\sharp_p(A\nabla_\beta B)\\ &\geqslant A\sharp_{\beta p}B + \left(\frac{1-\beta}{1-\alpha}\right)^p\left[A\sharp_p(A\nabla_\alpha B) - A\sharp_{\alpha p}B\right]\\ &+ \sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p\sum_{k=1}^{2^n}\Bigg[A\sharp_p\left(\left(B\sharp_{(1-\alpha)\frac{k-1}{2^n}}A\right)\nabla\left(B\sharp_{(1-\alpha)\frac{k}{2^n}}A\right)\right)\\ &- B\sharp_{p(1-\alpha)\frac{2k-1}{2^{n+1}}}A\Bigg]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right).\end{aligned} \tag{45}$$

4. Applications for unitarily invariant norms

Let M_n be the algebra of all $n\times n$ complex matrices. A norm $|||\cdot|||$ on M_n is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A\in M_n$ and for all unitary matrices $U,V\in M_n$. Let $s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_n(A)$ be the singular values of $A\in M_n$. The Schatten p-norms, $p\in[1,\infty)$, written $\|\cdot\|_p$, and defined by

$$\|A\|_p = \left(\sum_{i=1}^n s_i^p(A)\right)^{1/p},$$

are typical examples of unitarily invariant norms. The trace norm of $A\in M_n$, as usually expressed as $\|A\|_1 = \operatorname{tr}|A|$, is defined as the Schatten 1-norm of A.

It is known ([2]) that if $X\in M_n$ and $A,B\in M_n$ are positive semidefinite, the inequality $|||A^{1-\nu}XB^\nu||| \leqslant |||(1-\nu)AX+\nu XB|||$ is not valid for $\nu\in[0,1]$. However, Kosaki ([18]) showed that

$$|||A^{1-\nu}XB^\nu||| \leqslant (1-\nu)\,|||AX|||+\nu\,|||XB|||.$$

This inequality can be regarded as a unitarily invariant norm inequality form of Young's inequality (1) for matrices.

Before stating our results in this subsection, we need to recall the following lemma, which is a Heinz-Kato type inequality for unitarily invariant norms (see ([19]) for details).

LEMMA 4.1. ([19]) *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant \nu \leqslant 1$, then*

$$|||\, A^{1-\nu}XB^{\nu} \,||| \leqslant |||AX\,|||^{1-\nu}\, |||XB\,|||^{\nu}. \tag{46}$$

In particular, we have

$$\operatorname{tr}\left|A^{1-\nu}B^{\nu}\right| \leqslant \operatorname{tr}(A)^{1-\nu}\operatorname{tr}(B)^{\nu}. \tag{47}$$

Now, we are ready to state the first main result of this subsection.

THEOREM 4.2. *Let $A,B>0$, $0<\alpha \leqslant \beta <1$ and N be a positive integer. Then for every positive real number $p \geqslant 1$, we have:*

$$\begin{aligned}
&(|||AX\,|||\,\nabla_{\alpha}\,|||\,XB|||)^{p}\\
&\geqslant \left(1+\frac{L\left(2^{N}\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^{2}\left(\frac{|||XB|||}{|||AX|||}\right)^{\beta}\right)^{p}\left(|||A^{1-\alpha}XB^{\alpha}|||\right)^{p}\\
&\quad+\left(\frac{\alpha}{\beta}\right)^{p}\left[(|||AX\,|||\,\nabla_{\beta}\,|||\,XB|||)^{p}-(|||AX\,|||\,\sharp_{\beta}\,|||\,XB|||)^{p}\right]\\
&\quad+\sum_{n=0}^{N-1}\left(2r_{n}\left(\frac{\alpha}{\beta}\right)\right)^{p}\sum_{k=1}^{2^{n}}\left[\left(\frac{|||A^{1-\beta\frac{k-1}{2^{n}}}XB^{\beta\frac{k-1}{2^{n}}}\,|||+|||\,A^{1-\beta\frac{k}{2^{n}}}XB^{\beta\frac{k}{2^{n}}}\,|||}{2}\right)^{p}\right.\\
&\qquad\left.-\left(|||AX\,|||\,\sharp_{\beta\frac{2k-1}{2^{n+1}}}\,|||\,XB|||\right)^{p}\right]\chi_{\left(\frac{k-1}{2^{n}},\frac{k}{2^{n}}\right)}\left(\frac{\alpha}{\beta}\right),
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
&(|||AX\,|||\,\nabla_{\beta}\,|||\,XB|||)^{p}\\
&\geqslant \left(1+\frac{L\left(2^{N}\frac{1-\beta}{1-\alpha}\right)}{2^{2N}}\ln^{2}\left(\frac{|||AX|||}{|||XB|||}\right)^{1-\alpha}\right)^{p}\left(|||A^{1-\beta}XB^{\beta}\,|||\right)^{p}\\
&\quad+\left(\frac{1-\beta}{1-\alpha}\right)^{p}\left[(|||AX\,|||\,\nabla_{\alpha}\,|||\,XB|||)^{p}-(|||AX\,|||\,\sharp_{\alpha}\,|||\,XB|||)^{p}\right]\\
&\quad+\sum_{n=0}^{N-1}\left(2r_{n}\left(\frac{1-\beta}{1-\alpha}\right)\right)^{p}\sum_{k=1}^{2^{n}}N_{\alpha,k,n}(A,B)\chi_{\left(\frac{k-1}{2^{n}},\frac{k}{2^{n}}\right)}\left(\frac{1-\beta}{1-\alpha}\right),
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
N_{\alpha,k,n}(A,B)=&\left(\frac{|||A^{(1-\alpha)\frac{k-1}{2^{n}}}XB^{1-(1-\alpha)\frac{k-1}{2^{n}}}\,|||+|||\,A^{(1-\alpha)\frac{k}{2^{n}}}XB^{1-(1-\alpha)\frac{k}{2^{n}}}\,|||}{2}\right)^{p}\\
&-\left(|||XB\,|||\,\sharp_{(1-\alpha)\frac{2k-1}{2^{n+1}}}\,|||\,AX\,|||\right)^{p}.
\end{aligned} \tag{50}$$

Proof of Theorem 4.2. We prove (48); to prove (49), we proceed similarly. Taking $a = |||AX|||$, $b = |||XB|||$ in the inequality (33) and using Lemma 4.1, we get:

$$
\begin{aligned}
&(|||AX|||\nabla_\alpha |||XB|||)^p \\
&\geqslant \left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{|||XB|||}{|||AX|||}\right)^\beta\right)^p (|||AX|||\sharp_\alpha |||XB|||)^p \\
&\quad+\left(\frac{\alpha}{\beta}\right)^p\left[(|||AX|||\nabla_\beta |||XB|||)^p-(|||AX|||\sharp_\beta |||XB|||)^p\right] \\
&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left[\left(\frac{|||AX|||\sharp_{\beta\frac{k-1}{2^n}}|||XB|||+|||AX|||\sharp_{\beta\frac{k}{2^n}}|||XB|||}{2}\right)^p\right. \\
&\qquad\left.-\left(|||AX|||\sharp_{\beta\frac{2k-1}{2^{n+1}}}|||XB|||\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right) \\
&\geqslant \left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{|||XB|||}{|||AX|||}\right)^\beta\right)^p \left(|||A^{1-\alpha}XB^\alpha|||\right)^p \\
&\quad+\left(\frac{\alpha}{\beta}\right)^p\left[(|||AX|||\nabla_\beta |||XB|||)^p-(|||AX|||\sharp_\beta |||XB|||)^p\right] \\
&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left[\left(\frac{|||A^{1-\beta\frac{k-1}{2^n}}XB^{\beta\frac{k-1}{2^n}}|||+|||A^{1-\beta\frac{k}{2^n}}XB^{\beta\frac{k}{2^n}}|||}{2}\right)^p\right. \\
&\qquad\left.-\left(|||AX|||\sharp_{\beta\frac{2k-1}{2^{n+1}}}|||XB|||\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right). \qquad \square
\end{aligned}
$$

Using Theorems 2.7 and 2.8, along with the method applied in Theorem 4.2, we successively obtain the following unitary invariant norm versions of the inequalities.

THEOREM 4.3. *Let $A,B>0$, $0<\alpha<\beta<1$ and N be a positive integer. Then for every positive real number $p \geqslant 1$, we have:*

$$
\begin{aligned}
&(|||AX|||\nabla_\alpha |||XB|||)^p \\
&\geqslant K\left(\sqrt[2^N]{\left(\frac{|||XB|||}{|||AX|||}\right)^\beta},2\right)^{pr_N\left(\frac{\alpha}{\beta}\right)} \left(|||A^{1-\alpha}XB^\alpha|||\right)^p \\
&\quad+\left(\frac{\alpha}{\beta}\right)^p\left[(|||AX|||\nabla_\beta |||XB|||)^p-(|||AX|||\sharp_\beta |||XB|||)^p\right] \\
&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left[\left(\frac{|||A^{1-\beta\frac{k-1}{2^n}}XB^{\beta\frac{k-1}{2^n}}|||+|||A^{1-\beta\frac{k}{2^n}}XB^{\beta\frac{k}{2^n}}|||}{2}\right)^p\right. \\
&\qquad\left.-\left(|||AX|||\sharp_{\beta\frac{2k-1}{2^{n+1}}}|||XB|||\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \qquad (51)
\end{aligned}
$$

and

$$
\begin{aligned}
&(|||AX|||\,\nabla_\beta\,|||XB|||)^p\\
&\geqslant K\left(\sqrt[2^N]{\left(\frac{|||XB|||}{|||AX|||}\right)}^{1-\alpha},2\right)^{pr_N\left(\frac{1-\beta}{1-\alpha}\right)}\left(|||A^{1-\beta}XB^{\beta}|||\right)^p\\
&\quad+\left(\frac{1-\beta}{1-\alpha}\right)^p[(|||AX|||\,\nabla_\alpha\,|||XB|||)^p-(|||AX|||\,\sharp_\alpha\,|||XB|||)^p]\\
&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p\sum_{k=1}^{2^n}N_{\alpha,k,n}(A,B)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),
\end{aligned}
\tag{52}
$$

where $K(\cdot,2)$ *is Kantorovich constant and* $N_{\alpha,k,n}(A,B)$ *is as in Theorem* 4.2.

THEOREM 4.4. *Let* $A,B>0$, $0<\alpha<\beta<1$ *and* N *be a positive integer. Then for every positive real number* $p\geqslant 1$, *we have:*

$$
\begin{aligned}
&(|||AX|||\,\nabla_\alpha\,|||XB|||)^p\\
&\geqslant(|||A^{1-\alpha}XB^{\alpha}|||)^p\\
&\quad+\left(\frac{\alpha}{\beta}\right)^p\left[(|||AX|||\,\nabla_\beta\,|||XB|||)^p-(|||AX|||\,\sharp_\beta\,|||XB|||)^p\right]\\
&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left[\left(\frac{|||A^{1-\beta\frac{k-1}{2^n}}XB^{\beta\frac{k-1}{2^n}}|||+|||A^{1-\beta\frac{k}{2^n}}XB^{\beta\frac{k}{2^n}}|||}{2}\right)^p\right.\\
&\qquad\left.-\left(|||AX|||\,\sharp_{\beta\frac{2k-1}{2^{n+1}}}\,|||XB|||\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
&(|||AX|||\,\nabla_\beta\,|||XB|||)^p\\
&\geqslant\left(|||A^{1-\beta}XB^{\beta}|||\right)^p\\
&\quad+\left(\frac{1-\beta}{1-\alpha}\right)^p[(|||AX|||\,\nabla_\alpha\,|||XB|||)^p-(|||AX|||\,\sharp_\alpha\,|||XB|||)^p]\\
&\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p\sum_{k=1}^{2^n}N_{\alpha,k,n}(A,B)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),
\end{aligned}
\tag{54}
$$

where $N_{\alpha,k,n}(A,B)$ *is as in Theorem* 4.2.

By taking the unitarily norm $|||\cdot|||$ is the trace norm and using the inequality (47) in Lemma 4.1 along with Theorem 4.2, 4.3 and 4.4, together with the remark that $\operatorname{tr}(A\nabla_\nu B)=(\operatorname{tr}A)\nabla_\nu(\operatorname{tr}B)$, we get.

COROLLARY 4.5. *Let $0<\alpha<\beta<1$, N be a positive integer. Let $A,B\in M_n$ be a positive semidefinite. Then, for every real number $p\geqslant 1$, we have:*

$$\begin{aligned} \operatorname{tr}(A\nabla_\alpha B)^p \geqslant & \left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{\operatorname{tr}B}{\operatorname{tr}A}\right)^{\beta}\right)^p \left(\operatorname{tr}|A^{1-\alpha}B^{\alpha}|\right)^p \\ & + \left(\frac{\alpha}{\beta}\right)^p \left[(\operatorname{tr}(A\nabla_\beta B))^p - (\operatorname{tr}A\sharp_\beta \operatorname{tr}B)^p\right] \\ & + \sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p \sum_{k=1}^{2^n}\left[\left(\frac{\operatorname{tr}|A^{1-\beta\frac{k-1}{2^n}}B^{\beta\frac{k-1}{2^n}}|+\operatorname{tr}|A^{1-\beta\frac{k}{2^n}}B^{\beta\frac{k}{2^n}}|}{2}\right)^p\right. \\ & \left. - \left(\operatorname{tr}A\sharp_{\beta\frac{2k-1}{2^{n+1}}}\operatorname{tr}B\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \end{aligned} \tag{55}$$

and

$$\begin{aligned} \operatorname{tr}(A\nabla_\beta B)^p \geqslant & \left(1+\frac{L\left(2^N\frac{1-\beta}{1-\alpha}\right)}{2^{2N}}\ln^2\left(\frac{\operatorname{tr}A}{\operatorname{tr}B}\right)^{1-\alpha}\right)^p (\operatorname{tr}|A^{1-\beta}B^{\beta}|)^p \\ & + \left(\frac{1-\beta}{1-\alpha}\right)^p \left[(\operatorname{tr}(A\nabla_\alpha B))^p - (\operatorname{tr}A\sharp_\alpha \operatorname{tr}B)^p\right] \\ & + \sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p \sum_{k=1}^{2^n} M_{\alpha,k,n}(A,B)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right), \end{aligned} \tag{56}$$

where L is defined as in (14) *and*

$$\begin{aligned} M_{\alpha,k,n}(A,B) = & \left(\frac{\operatorname{tr}|A^{(1-\alpha)\frac{k-1}{2^n}}B^{1-(1-\alpha)\frac{k-1}{2^n}}|+\operatorname{tr}|A^{(1-\alpha)\frac{k}{2^n}}B^{1-(1-\alpha)\frac{k}{2^n}}|}{2}\right)^p \\ & - \left(\operatorname{tr}B\sharp_{(1-\alpha)\frac{2k-1}{2^{n+1}}}\operatorname{tr}A\right)^p. \end{aligned} \tag{57}$$

COROLLARY 4.6. *Let $0<\alpha<\beta<1$, N be a positive integer. Let $A,B\in M_n$ be a positive semidefinite. Then, for every real number $p\geqslant 1$, we have:*

$$\begin{aligned} & \operatorname{tr}(A\nabla_\alpha B)^p \\ & \geqslant K\left(\sqrt[2^N]{\left(\frac{\operatorname{tr}B}{\operatorname{tr}A}\right)^{\beta}},2\right)^{pr_N\left(\frac{\alpha}{\beta}\right)} \left(\operatorname{tr}|A^{1-\alpha}B^{\alpha}|\right)^p + \left(\frac{\alpha}{\beta}\right)^p\left[(\operatorname{tr}(A\nabla_\beta B))^p - (\operatorname{tr}A\sharp_\beta\operatorname{tr}B)^p\right] \\ & \quad + \sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p \sum_{k=1}^{2^n}\left[\left(\frac{\operatorname{tr}|A^{1-\beta\frac{k-1}{2^n}}B^{\beta\frac{k-1}{2^n}}|+\operatorname{tr}|A^{1-\beta\frac{k}{2^n}}B^{\beta\frac{k}{2^n}}|}{2}\right)^p\right. \\ & \quad \left. - \left(\operatorname{tr}A\sharp_{\beta\frac{2k-1}{2^{n+1}}}\operatorname{tr}B\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right), \end{aligned} \tag{58}$$

and

$$\begin{aligned}\operatorname{tr}(A\nabla_\beta B)^p \geqslant{} & K\left(\sqrt[2^N]{\left(\frac{\operatorname{tr}B}{\operatorname{tr}A}\right)^{1-\alpha}},2\right)^{pr_N\left(\frac{1-\beta}{1-\alpha}\right)}(\operatorname{tr}|A^{1-\beta}B^\beta|)^p\\ &+\left(\frac{1-\beta}{1-\alpha}\right)^p\left[(\operatorname{tr}(A\nabla_\alpha B))^p-(\operatorname{tr}A\,\sharp_\alpha\operatorname{tr}B)^p\right]\\ &+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p\sum_{k=1}^{2^n}M_{\alpha,k,n}(A,B)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),\end{aligned}\tag{59}$$

where $K(\cdot,2)$ is Kantorovich constant and $M_{\alpha,k,n}(A,B)$ is defined as in (57).

COROLLARY 4.7. *Let $0<\alpha<\beta<1$, N be a positive integer. Let $A,B\in M_n$ be a positive semidefinite. Then, for every real number $p\geqslant 1$, we have:*

$$\begin{aligned}&\operatorname{tr}(A\nabla_\alpha B)^p\\ &\geqslant\left(\operatorname{tr}|A^{1-\alpha}B^\alpha|\right)^p+\left(\frac{\alpha}{\beta}\right)^p\left[(\operatorname{tr}(A\nabla_\beta B))^p-(\operatorname{tr}A\,\sharp_\beta\operatorname{tr}B)^p\right]\\ &\quad+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^p\sum_{k=1}^{2^n}\left[\left(\frac{\operatorname{tr}|A^{1-\beta\frac{k-1}{2^n}}B^{\beta\frac{k-1}{2^n}}|+\operatorname{tr}|A^{1-\beta\frac{k}{2^ln}}B^{\beta\frac{k}{2^n}}|}{2}\right)^p\right.\\ &\qquad\qquad\left.-\left(\operatorname{tr}A\,\sharp_{\beta\frac{2k-1}{2^{n+1}}}\operatorname{tr}B\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),\end{aligned}\tag{60}$$

and

$$\begin{aligned}\operatorname{tr}(A\nabla_\beta B)^p\geqslant{} & (\operatorname{tr}|A^{1-\beta}B^\beta|)^p\\ &+\left(\frac{1-\beta}{1-\alpha}\right)^p\left[(\operatorname{tr}(A\nabla_\alpha B))^p-(\operatorname{tr}A\,\sharp_\alpha\operatorname{tr}B)^p\right]\\ &+\sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^p\sum_{k=1}^{2^n}M_{\alpha,k,n}(A,B)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),\end{aligned}\tag{61}$$

where $M_{\alpha,k,n}(A,B)$ is as in (57).

5. Applications for determinants of matrices

Using the results obtained in Theorems 2.6, 2.7 and 2.8, in this section we will derive multi-term inequalities for determinants of matrices. Besides, in the proof process we also use the result stated in the following Lemma 5.1. This result is known as the Minkovski inequality for the determinant of matrices.

LEMMA 5.1. ([19]) *Let $A, B \in M_n$ be positive definite. Then*

$$[\det(A+B)]^{1/n} \geqslant (\det A)^{1/n} + (\det B)^{1/n}. \tag{62}$$

The main results of this subsection is as follows.

THEOREM 5.2. *Let $0 < \alpha < \beta < 1$, N be a positive integer and $A, B \in M_n$ be positive definite satisfying one of the following conditions:*

(i) $0 < mI \leqslant \det A \leqslant \gamma I < \Gamma I \leqslant \det B \leqslant MI$,

(ii) $0 < mI \leqslant \det B \leqslant \gamma I < \Gamma I \leqslant \det A \leqslant MI$,

where $0 < m, M, \gamma, \Gamma < \infty$ are scalars. Then, for every positive real number $p \geqslant 1$, we have:

$$\begin{aligned}
&[\det(A\nabla_\alpha B)]^p \\
&\geqslant \mathscr{M}^{np}[\det(A\sharp_\alpha B)]^p + \left(\frac{\alpha}{\beta}\right)^{np}\left\{\left[(\det A)^{\frac{1}{n}}\nabla_\beta(\det B)^{\frac{1}{n}}\right]^{np} - \left[\det(A\sharp_\beta B)\right]^p\right\} \\
&\quad + \sum_{n=0}^{N-1}\left(2r_n\left(\frac{\alpha}{\beta}\right)\right)^{np}\sum_{k=1}^{2^n}\left[\left(\frac{\det A^{\frac{1}{n}-\beta\frac{k-1}{n.2^n}}B^{\beta\frac{k-1}{n.2^n}} + \det A^{\frac{1}{n}-\beta\frac{k}{n.2^n}}B^{\beta\frac{k}{n.2^n}}}{2}\right)^{np}\right. \\
&\qquad\qquad \left. - \left(\det(A\sharp_{\beta\frac{2k-1}{2^{n+1}}}B)\right)^p\right]\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\alpha}{\beta}\right),
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
&[\det(A\nabla_\beta B)]^p \\
&\geqslant \mathscr{N}^{np}\,[\det(A\sharp_\beta B)]^p + \left(\frac{1-\beta}{1-\alpha}\right)^{np}\left\{\left[(\det A)^{\frac{1}{n}}\nabla_\alpha(\det B)^{\frac{1}{n}}\right]^{np} - [\det(A\sharp_\alpha B)]^p\right\} \\
&\quad + \sum_{n=0}^{N-1}\left(2r_n\left(\frac{1-\beta}{1-\alpha}\right)\right)^{np}\sum_{k=1}^{2^n}D_{\alpha,k,n}(A,B)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\beta}{1-\alpha}\right),
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
D_{\alpha,k,n}(A,B) = &\left(\frac{\det A^{(1-\alpha)\frac{k-1}{n.2^n}}B^{\frac{1}{n}-(1-\alpha)\frac{k-1}{n.2^n}} + \det A^{(1-\alpha)\frac{k}{n.2^n}}B^{\frac{1}{n}-(1-\alpha)\frac{k}{n.2^n}}}{2}\right)^{np} \\
&- \left(\det(B\sharp_{(1-\alpha)\frac{2k-1}{2^{n+1}}}A)\right)^p,
\end{aligned}$$

$$\mathscr{M} = \min\left\{1 + \frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\ln^2\left(\frac{\Gamma}{\gamma}\right)^\beta,\ K\left(\sqrt[2^N]{\left(\frac{\Gamma}{\gamma}\right)^\beta},2\right)^{r_N\left(\frac{\alpha}{\beta}\right)}\right\},$$

$$\mathscr{N} = \min\left\{1 + \frac{L\left(2^N \frac{1-\beta}{1-\alpha}\right)}{2^{2N}} \ln^2\left(\frac{\Gamma}{\gamma}\right)^{1-\alpha}, K\left(\sqrt[2^N]{\left(\frac{\Gamma}{\gamma}\right)^{1-\alpha}}, 2\right)^{r_N\left(\frac{1-\beta}{1-\alpha}\right)}\right\},$$

and $K(\cdot,2)$ *and* L *are defined as in (11) and (14), respectively.*

Proof of Theorem 5.2. Now, on the one hand, it follows from Lemma 5.1 that

$$\begin{aligned}[\det(A\nabla_\alpha B)]^p &= \left\{[\det((1-\alpha)A + \alpha B)]^{\frac{1}{n}}\right\}^{np} \\ &\geqslant \left\{[\det((1-\alpha)A)]^{\frac{1}{n}} + [\det(\alpha B)]^{\frac{1}{n}}\right\}^{np} \\ &= \left[(1-\alpha)(\det A)^{\frac{1}{n}} + \alpha(\det B)^{\frac{1}{n}}\right]^{np}.\end{aligned}$$

On the other hand, by using the substitution of variable $a = (\det A)^{\frac{1}{n}}$ and $b = (\det B)^{\frac{1}{n}}$ in the inequality (33) and (36), we obtain (63). By doing similarly, we can prove (64). $\square$

Acknowledgement. The author would like to express her sincere gratitude to the referee for his/her careful reading and helpful suggestions, which improved this paper. Doan Thi Thuy Van was funded by the Master, PhD Scholarship Programme of Vingroup Innovation Foundation (VINIF), code VINIF.2024.TS.005. This work is funded by the Research Project of Vietnam Ministry of Education and Training under grant number B2025-TTN-01.

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(Received October 14, 2025)

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