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COMPLETE CONVERGENCE AND COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF MARTINGALE DIFFERENCE RANDOM VECTORS

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Abstract. Let $\{X_{ni}, \mathscr{F}_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of $d \times 1$ martingale difference random vectors weakly summable dominated by a random vector X concerning the array $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of $m \times d$ matrices of real numbers. Under almost optimal conditions, we proved that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} A_{ni}X_i\right\| > \varepsilon n^{1/\alpha}\log^{1/\gamma} n\right) < \infty,$$

and

$$\sum_{n=1}^{\infty} n^{-1} E\left(b_n^{-1/\alpha}\log^{-1/\gamma} n \max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} A_{ni}X_{ni}\right\| - \varepsilon\right)^{+} < \infty.$$

The main results provide a multi-dimensional extension of some corresponding ones in the literature, improving upon the existing one-dimensional theory. Moreover, by imposing a slightly stronger assumption on the weight matrices, we also obtain the desired results under a weaker moment condition and a more flexible range for γ.

1. Introduction

Weighted sums of the form $\sum_{i=1}^{n} a_{ni}X_i$, where $\{a_{ni}\}$ is an array of constants and $\{X_n\}$ a sequence of random variables, are fundamental to many linear statistical methods, motivating sustained research into their convergence properties. A key tool in this analysis is the concept of complete convergence, introduced by Hsu and Robbins (1947). This property, defined by the summability condition $\sum\limits_{n=1}^{\infty} P(|X_n - c| > \varepsilon) < \infty$ for all $\varepsilon > 0$, implies almost sure convergence via the Borel-Cantelli lemma. A significant body of research has since been devoted to establishing complete convergence and related the strong laws for weighted sums under varying assumptions. The research trajectory began with foundational work on independent and identically distributed (i.i.d.) sequences, such as strong law of large numbers (SLLN) by Chow (1966) and the Marcinkiewicz-Zygmund SLLN by Bai and Cheng (2000). Subsequent work has

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extended these findings to diverse dependence frameworks, including negative association (Chen, 2005), strong mixing (Thanh and Yin, 2015), extended negative dependence (Shen, 2016), negative superadditive-dependence (Wu et al., 2016), ψ-mixing (Hu et al., 2017), and widely orthant dependence (Wu and Wang, 2023), often linking these theoretical advances to applications in regression models.

Sung (2011) obtained the following complete convergence for weighted sums of negatively associated random variables.

THEOREM A. *Set $b_n = n^{1/\alpha}\log^{1/\gamma} n$, where $0<\alpha\leqslant 2$ and $\gamma>0$. Let $\{X, X_n, n\geqslant 1\}$ be a sequence of identically distributed negatively associated random variables and $\{a_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ be an array of real numbers with $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$. Assume further that $EX=0$ if $1<\alpha\leqslant 2$. Then for any $\varepsilon>0$,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant m\leqslant n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty$$

provided that

$$\begin{cases} E|X|^{\alpha}<\infty, & \text{if } \alpha>\gamma,\\ E|X|^{\alpha}\log(1+|X|)<\infty, & \text{if } \alpha=\gamma,\\ E|X|^{\gamma}<\infty, & \text{if } \alpha<\gamma. \end{cases}$$

The extension of Theorem A has been pursued under various dependence structures and for different ranges of the parameter α relative to γ. For ρ^*-mixing random variables, Zhou et al. (2011) handled the case $\alpha>\gamma$ using a Rosenthal-type maximal moment inequality, a path later followed for $\alpha=\gamma$ by Sung (2013) and for $\alpha<\gamma$ by Wu et al. (2014). Subsequently, Chen and Sung (2016) employed a Rosenthal-type moment inequality and a novel technique to extend the results for the case $\alpha<\gamma$ to negatively orthant dependent random variables, simultaneously proving the optimality of the moment condition $E|X|^{\gamma}<\infty$ in this specific scenario. However, the optimality for $\alpha>\gamma$ or $\alpha=\gamma$ remains an open question. For $\alpha>\gamma$, Chen and Sung (2014) managed to refine the moment condition in Theorem A to $E|X|^{\alpha}/\log^{\alpha/\gamma-1}(1+|X|)<\infty$, a result which Li et al. (2017) later generalized to ρ^*-mixing random variables via a markedly different methodology. Wu and Wang (2024) further extended the results of Chen and Sung (2014) as well as Li et al. (2017) from non-randomly weighted sums to randomly weighted sums of negatively superadditive-dependent random variables.

However, these aforementioned results are all confined to the case of one-dimensional random variables. The main purpose of our work is to extend the results of Chen and Sung (2014) and Li et al. (2017) to multi-dimensional martingale difference random vectors.

Let $\{\mathscr{F}_n, n\geqslant 1\}$ be an increasing sequence of σ-fields with $\mathscr{F}_n\subset\mathscr{F}$ for each $n\geqslant 1$. If d-dimensional ($d\geqslant 1$) random vector X_n is $\mathscr{F}_n$ measurable for each $n\geqslant 1$, then σ-fields $\{\mathscr{F}_n, n\geqslant 1\}$ are said to be adapted to the sequence $\{X_n, n\geqslant 1\}$ and $\{X_n, \mathscr{F}_n; n\geqslant 1\}$ is said to be an adapted stochastic sequence.

DEFINITION 1.1. (Wu and Wang, 2024) If $\{X_n, \mathscr{F}_n; n \geqslant 1\}$ is an adapted stochastic sequence with $E\|X_n\| < \infty$ for each $n \geqslant 1$ and

$$E(X_n|\mathscr{F}_{n-1}) = 0 \text{ a.s.},$$

then the sequence $\{X_n, \mathscr{F}_n; n \geqslant 1\}$ is called a martingale difference sequence.

Wu and Wang (2024) also introduced the concept of weakly summable domination for array $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of random vectors as follows.

DEFINITION 1.2. Let $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of $m \times d$ matrices of real numbers. An array $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of $d \times 1$ random vectors is said to be weakly summable dominated by a random vector X concerning the array $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ if there exists a positive constant C such that the following inequality holds for each $n \geqslant 1$ and all $x \geqslant 0$,

$$\sum_{i=1}^{n} P(\|A_{ni}X_{ni}\| > x) \leqslant C \sum_{i=1}^{n} P(\|A_{ni}X\| > x).$$

As noted by Wu and Wang (2024), the condition of weakly mean domination represents a special case of weakly summable domination. This occurs when the matrix A_{ni} is set to the $d \times d$ identity matrix for every $1 \leqslant i \leqslant n$ and $n \geqslant 1$. It is noteworthy that weakly mean domination remains a weaker assumption than both stochastic domination and identical distribution.

In this work, we consider the complete convergence and complete moment convergence for weighted sums of martingale difference random vectors under the almost optimal conditions. The results extend and improve the corresponding ones of Chen and Sung (2014) and Li et al. (2017) from weighted sums of one-dimensional random variables to multi-dimensional martingale difference random vectors. As a corollary, the strong law of large numbers for weighted sums of multi-dimensional martingale difference random vectors is established. Furthermore, by strengthening the assumptions on the weight matrices, we can relax the requirements on the moment condition and the parameter γ.

Throughout this paper, the symbol C represents some positive constant, the value of which can be different in different places. Let $I(A)$ be the indicator function of the event A and $x^+ = xI(x \geqslant 0)$. For positive numbers a_n and b_n, $a_n = O(b_n)$ shows that $\limsup_{n\to\infty} a_n/b_n < \infty$. Define the norm of a matrix $A = (a_{ij})_{m\times d}$ by $\|A\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{d} a_{ij}^2}$. Denote $\log x = \ln\max\{x, e\}$.

The rest of this paper is organized as follows: The main results are stated in Section 2. Some important lemmas for proving the main results are provided in Section 3. The proofs of the main results are presented in Section 4.

2. Main results

The first result concerning the complete convergence for weighted sums of martingale difference random vectors is presented as follows.

THEOREM 2.1. *Set* $b_n = n^{1/\alpha}\log^{1/\gamma} n$*, where* $0<\gamma<\alpha\leqslant 2$*. Let* $\{X_{ni},\mathscr{F}_{ni};1\leqslant i\leqslant n,n\geqslant 1\}$ *be an array of* $d\times 1$ *martingale difference random vectors weakly summable dominated by a random vector* X *concerning the array* $\{A_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ *of* $m\times d$ *matrices of real numbers. If* $\sum_{i=1}^{n}\|A_{ni}\|^{\alpha}=O(n)$ *and* $E\|X\|^{\alpha}/\log^{\alpha/\gamma-1}(1+\|X\|)<\infty$*, then for any* $\varepsilon>0$*,*

$$\sum_{n=1}^{\infty} n^{-1}P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}A_{ni}X_{ni}\right\|>\varepsilon b_n\right)<\infty. \tag{2.1}$$

By Theorem 2.1, we can obtain the following strong law of large numbers for weighted sums of martingale difference random vectors.

COROLLARY 2.1. *Set* $b_n = n^{1/\alpha}\log^{1/\gamma} n$*, where* $0<\gamma<\alpha\leqslant 2$*. Let* $\{X_n,\mathscr{F}_n;n\geqslant 1\}$ *be a sequence of* $d\times 1$ *martingale difference random vectors weakly summable dominated by a random vector* X *concerning the sequence* $\{A_n,n\geqslant 1\}$ *of* $m\times d$ *matrices of real numbers. If* $\sum_{i=1}^{n}\|A_i\|^{\alpha}=O(n)$ *and* $E\|X\|^{\alpha}/\log^{\alpha/\gamma-1}(1+\|X\|)<\infty$*, then*

$$b_n^{-1}\left\|\sum_{i=1}^{n}A_iX_i\right\|\to 0 \ \text{a.s.}, \ \text{as} \ n\to\infty.$$

We further obtain the following result on complete moment convergence, which is much stronger than complete convergence.

THEOREM 2.2. *Set* $b_n = n^{1/\alpha}\log^{1/\gamma} n$*, where* $1<\alpha\leqslant 2$ *and* $0<\gamma<\alpha$*. Let* $\{X_{ni},\mathscr{F}_{ni};1\leqslant i\leqslant n,n\geqslant 1\}$ *be an array of* $d\times 1$ *martingale difference random vectors weakly summable dominated by a random vector* X *concerning the array* $\{A_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ *of* $m\times d$ *matrices of real numbers. If* $\sum_{i=1}^{n}\|A_{ni}\|^{\alpha}=O(n)$ *and* $E\|X\|^{\alpha}/\log^{\alpha/\gamma-1}(1+\|X\|)<\infty$*, then for any* $\varepsilon>0$*,*

$$\sum_{n=1}^{\infty} n^{-1}E\left(b_n^{-1}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}A_{ni}X_{ni}\right\|-\varepsilon\right)^{+}<\infty. \tag{2.2}$$

REMARK 2.1. It can be easily checked that

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{-1}E\left(b_n^{-1}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}A_{ni}X_{ni}\right\|-\varepsilon\right)^{+}\\ &= \sum_{n=1}^{\infty} n^{-1}\int_0^{\infty}P\left(b_n^{-1}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}A_{ni}X_{ni}\right\|-\varepsilon>t\right)dt\end{aligned}$$

$$\geqslant \sum_{n=1}^{\infty} n^{-1} \int_0^{\varepsilon} P\left(b_n^{-1} \max_{1 \leqslant m \leqslant n} \left\| \sum_{i=1}^{m} A_{ni} X_{ni} \right\| - \varepsilon > t \right) dt$$

$$\geqslant \varepsilon \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant m \leqslant n} \left\| \sum_{i=1}^{m} A_{ni} X_{ni} \right\| > 2\varepsilon b_n \right).$$

Since $\varepsilon > 0$ is arbitrary, we can see that Theorem 2.2 is much more stronger than Theorem 2.1. Therefore, Theorem 2.2 also improves and extends the corresponding results of Chen and Sung (2014) and Li et al. (2017) from weighted sums of one-dimensional random variables to weighted sums of martingale difference random vectors.

For $0 < \alpha < 2$, we can relax the restrictions on γ and the moment condition by imposing a stronger assumption on the weight matrices.

THEOREM 2.3. *Set $b_n = n^{1/\alpha} \log^{1/\gamma} n$, where $0 < \alpha < 2$ and $0 < \gamma \leqslant \infty$. Let $\{X_{ni}, \mathscr{F}_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of $d \times 1$ martingale difference random vectors weakly summable dominated by a random vector X concerning the array $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of $m \times d$ matrices of real numbers. If $\sum_{i=1}^{n} \|A_{ni}\|^2 = O(n)$ and $E\|X\|^{\alpha} / \log^{\alpha/\gamma}(1 + \|X\|) < \infty$, then for any $\varepsilon > 0$, (2.1) holds true.*

By Theorem 2.3 and similar proof as Corollary 2.1, we can obtain the following strong law of large numbers immediately.

COROLLARY 2.2. *Set $b_n = n^{1/\alpha} \log^{1/\gamma} n$, where $0 < \alpha < 2$ and $0 < \gamma \leqslant \infty$. Let $\{X_n, \mathscr{F}_n; n \geqslant 1\}$ be a sequence of $d \times 1$ martingale difference random vectors weakly summable dominated by a random vector X concerning the sequence $\{A_n, n \geqslant 1\}$ of $m \times d$ matrices of real numbers. If $\sum_{i=1}^{n} \|A_i\|^2 = O(n)$ and $E\|X\|^{\alpha} / \log^{\alpha/\gamma}(1 + \|X\|) < \infty$, then*

$$b_n^{-1} \left\| \sum_{i=1}^{n} A_i X_i \right\| \to 0 \ \text{a.s.}, \ \text{as} \ n \to \infty.$$

Similar to Theorem 2.2, we can also obtain the following result on complete moment convergence.

THEOREM 2.4. *Set $b_n = n^{1/\alpha} \log^{1/\gamma} n$, where $1 < \alpha < 2$ and $0 < \gamma \leqslant \infty$. Let $\{X_{ni}, \mathscr{F}_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of $d \times 1$ martingale difference random vectors weakly summable dominated by a random vector X concerning the array $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of $m \times d$ matrices of real numbers. If $\sum_{i=1}^{n} \|A_{ni}\|^2 = O(n)$ and $E\|X\|^{\alpha} / \log^{\alpha/\gamma}(1 + \|X\|) < \infty$, then for any $\varepsilon > 0$, (2.2) also holds true.*

REMARK 2.2. Compared with Theorems 2.1 and 2.2, Theorems 2.3 and 2.4 are superior in the following aspects. First, they require slightly weaker moment conditions. Second, they generalize the results by allowing a more flexible range for γ, which is no longer restricted to $0 < \gamma < \alpha$. The cost for these improvements is that the assumption on the weight matrices, $\sum_{i=1}^{n} \|A_{ni}\|^{\alpha} = O(n)$, is strengthened to $\sum_{i=1}^{n} \|A_{ni}\|^2 = O(n)$. Moreover, our results also cover the case of $\gamma = \infty$, where we set $b_n = n^{1/\alpha}$ and the moment condition becomes $E\|X\|^{\alpha} < \infty$.

3. Some lemmas

In this section, we will present some lemmas which will be used in proving our main results.

LEMMA 3.1. *Set* $b_n = n^{1/\alpha}\log^{1/\gamma} n$*, where* $0<\gamma<\alpha$*. Let* X *be a* $d\times 1$ *random vector and* $\{A_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ *be an array of* $m\times d$ *matrices of real numbers such that* $\sum_{i=1}^n \|A_{ni}\|^\alpha = O(n)$*. If* $E\|X\|^\alpha/\log^{\alpha/\gamma-1}(1+\|X\|)<\infty$*, then for any* $0<q<\alpha\leqslant s$*, we have*

$$\sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E\|A_{ni}X\|^q I(\|A_{ni}X\|>b_n)<\infty, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} n^{-1} b_n^{-s} \sum_{i=1}^{n} E\|A_{ni}X\|^s I(\|A_{ni}X\|\leqslant b_n)<\infty. \tag{3.2}$$

Proof. Observe that

$$\begin{aligned}
&E\|A_{ni}X\|^q I(\|A_{ni}X\|>b_n)\\
&= E\|A_{ni}X\|^q I(\|A_{ni}X\|>b_n, \|X\|>b_n) + E\|A_{ni}X\|^q I(\|A_{ni}X\|>b_n, \|X\|\leqslant b_n)\\
&\leqslant E\|A_{ni}X\|^q I(\|X\|>b_n) + b_n^{q-\alpha} E\|A_{ni}X\|^\alpha I(\|X\|\leqslant b_n)\\
&= \|A_{ni}\|^q E\|X\|^q I(\|X\|>b_n) + b_n^{q-\alpha}\|A_{ni}\|^\alpha E\|X\|^\alpha I(\|X\|\leqslant b_n)
\end{aligned}$$

and analogously,

$$\begin{aligned}
&E\|A_{ni}X\|^s I(\|A_{ni}X\|\leqslant b_n)\\
&= E\|A_{ni}X\|^s I(\|A_{ni}X\|\leqslant b_n, \|X\|>b_n) + E\|A_{ni}X\|^s I(\|A_{ni}X\|\leqslant b_n, \|X\|\leqslant b_n)\\
&\leqslant b_n^{s-q}\|A_{ni}\|^q E\|X\|^q I(\|X\|>b_n) + b_n^{s-\alpha}\|A_{ni}\|^\alpha E\|X\|^\alpha I(\|X\|\leqslant b_n).
\end{aligned}$$

Hence, to prove (3.1) and (3.2), it suffices to show

$$\sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} \|A_{ni}\|^q E\|X\|^q I(\|X\|>b_n)<\infty, \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} \|A_{ni}\|^\alpha E\|X\|^\alpha I(\|X\|\leqslant b_n)<\infty. \tag{3.4}$$

By virtue of Hölder's inequality, we obtain that for any $\alpha'\in(0,\alpha)$,

$$\sum_{i=1}^{n} \|A_{ni}\|^{\alpha'} \leqslant \left(\sum_{i=1}^{n}\|A_{ni}\|^{\alpha}\right)^{\alpha'/\alpha}\left(\sum_{i=1}^{n} 1\right)^{1-\alpha'/\alpha} = O(n). \tag{3.5}$$

Hence, it is easy to check that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} \|A_{ni}\|^q E\|X\|^q I(\|X\| > b_n) \\
&\leqslant C \sum_{n=1}^{\infty} b_n^{-q} E\|X\|^q I(\|X\| > b_n) \\
&= C \sum_{j=1}^{\infty} E\|X\|^q I(b_j < \|X\| \leqslant b_{j+1}) \sum_{n=1}^{j} n^{-q/\alpha} \log^{-q/\gamma} n \\
&\leqslant C \sum_{j=1}^{\infty} j^{1-q/\alpha} \log^{-q/\gamma} j E\|X\|^q I(b_j < \|X\| \leqslant b_{j+1}) \\
&= C \sum_{j=1}^{\infty} j^{1-q/\alpha} \log^{-q/\gamma} j E\left(\frac{\|X\|^\alpha}{\log^{\alpha/\gamma}(1+\|X\|)} \cdot \|X\|^{q-\alpha} \log^{\alpha/\gamma}(1+\|X\|)\right) \\
&\quad \times I(b_j < \|X\| \leqslant b_{j+1}) \\
&\leqslant CE\|X\|^\alpha / \log^{\alpha/\gamma}(1+\|X\|) < \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} \|A_{ni}\|^\alpha E\|X\|^\alpha I(\|X\| \leqslant b_n) \\
&\leqslant C \sum_{n=1}^{\infty} b_n^{-\alpha} E\|X\|^\alpha I(\|X\| \leqslant b_n) \\
&= C \sum_{j=1}^{\infty} E\|X\|^\alpha I(b_{j-1} < \|X\| \leqslant b_j) \sum_{n=j}^{\infty} n^{-1} \log^{-\alpha/\gamma} n \\
&\leqslant C \sum_{j=1}^{\infty} \log^{1-\alpha/\gamma} j E\|X\|^\alpha I(b_j < \|X\| \leqslant b_{j+1}) \\
&= C \sum_{j=1}^{\infty} \log^{1-\alpha/\gamma} j E\left(\frac{\|X\|^\alpha}{\log^{\alpha/\gamma-1}(1+\|X\|)} \cdot \log^{\alpha/\gamma-1}(1+\|X\|)\right) \\
&\quad \times I(b_j < \|X\| \leqslant b_{j+1}) \\
&\leqslant CE\|X\|^\alpha / \log^{\alpha/\gamma-1}(1+\|X\|) < \infty.
\end{aligned}$$

The proof of the lemma is completed. □

The following lemma is proved by Wu and Wang (2024).

LEMMA 3.2. *Assume that* $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *is an array of* $d \times 1$ *random vectors weakly summable dominated by a random vector* X *concerning the array* $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *of* $m \times d$ *matrices of real numbers. Then for any* $a > 0$ *and* $b > 0$,

$$\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^a I(\|A_{ni}X_{ni}\| > b) \leqslant C \sum_{i=1}^{n} E\|A_{ni}X\|^a I(\|A_{ni}X\| > b),$$

$$\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^{a}I(\|A_{ni}X_{ni}\| \leqslant b) \leqslant C\sum_{i=1}^{n}[b^{a}P(\|A_{ni}X\| > b) + E\|A_{ni}X\|^{a}I(\|A_{ni}X\| \leqslant b)].$$

The following Marcinkiewicz-Zygmund type maximum inequality for martingale difference random vectors can also be seen in Wu and Wang (2024).

LEMMA 3.3. *Let $1 < p \leqslant 2$. If $\{X_i, \mathscr{F}_i; i \geqslant 1\}$ is a sequence of d-dimensional martingale difference random vectors, then there exists a positive constant $C_{p,d}$ depending only on p and d such that*

$$E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k} X_i\right\|^{p}\right) \leqslant C_{p,d}\sum_{i=1}^{n} E\|X_i\|^{p}.$$

The following moment inequality for general random variables has been proved in Wu et al. (2020).

LEMMA 3.4. *Let $\{\xi_i, i \geqslant 1\}$ and $\{\eta_i, i \geqslant 1\}$ be two sequences of random vectors. Then for any $q > r > 0$, $\varepsilon > 0$, and $a > 0$, the following inequality holds:*

$$E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k}(\xi_i+\eta_i)\right\| - \varepsilon a\right)_{+}^{r} \leqslant C_r\left(\varepsilon^{-q}+\frac{r}{q-r}\right)a^{r-q}E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k}\xi_i\right\|^{q}\right) + C_r E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k}\eta_i\right\|^{r}\right),$$

where $C_r = 1$ if $0 < r \leqslant 1$ or $C_r = 2^{r-1}$ if $r > 1$.

The last lemma is adopted to prove Theorems 2.3 and 2.4.

LEMMA 3.5. *Set $b_n = n^{1/\alpha}\log^{1/\gamma} n$, where $\alpha > 0$ and $0 < \gamma \leqslant \infty$. Let X be a $d\times 1$ random vector and $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of $m\times d$ matrices of real numbers such that $\sum_{i=1}^{n}\|A_{ni}\|^{r} = O(n)$ for some $r > \alpha$. If $E\|X\|^{\alpha}/\log^{\alpha/\gamma}(1+\|X\|) < \infty$, then for any $0 < q < \alpha < r \leqslant s$, (3.1) and (3.2) hold true.*

Proof. Similar to the proof of Lemma 3.1, we obtain

$$\begin{aligned}
&E\|A_{ni}X\|^{q}I(\|A_{ni}X\| > b_n)\\
&= E\|A_{ni}X\|^{q}I(\|A_{ni}X\| > b_n, \|X\| > b_n) + E\|A_{ni}X\|^{q}I(\|A_{ni}X\| > b_n, \|X\| \leqslant b_n)\\
&\leqslant E\|A_{ni}X\|^{q}I(\|X\| > b_n) + b_n^{q-r}E\|A_{ni}X\|^{r}I(\|X\| \leqslant b_n)\\
&= \|A_{ni}\|^{q}E\|X\|^{q}I(\|X\| > b_n) + b_n^{q-r}\|A_{ni}\|^{r}E\|X\|^{r}I(\|X\| \leqslant b_n)
\end{aligned}$$

and

$$\begin{aligned}
&E\|A_{ni}X\|^{s}I(\|A_{ni}X\| \leqslant b_n)\\
&= E\|A_{ni}X\|^{s}I(\|A_{ni}X\| \leqslant b_n, \|X\| > b_n) + E\|A_{ni}X\|^{s}I(\|A_{ni}X\| \leqslant b_n, \|X\| \leqslant b_n)\\
&\leqslant b_n^{s-q}\|A_{ni}\|^{q}E\|X\|^{q}I(\|X\| > b_n) + b_n^{s-r}\|A_{ni}\|^{r}E\|X\|^{r}I(\|X\| \leqslant b_n).
\end{aligned}$$

Hence, to prove (3.1) and (3.2), it suffices to show

$$\sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} \|A_{ni}\|^q E\|X\|^q I(\|X\| > b_n) < \infty, \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} n^{-1} b_n^{-r} \sum_{i=1}^{n} \|A_{ni}\|^r E\|X\|^r I(\|X\| \leqslant b_n) < \infty. \tag{3.7}$$

Analogous to the proof of (3.3), we get (3.6). Now we prove (3.7). It is easy to get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} b_n^{-r} \sum_{i=1}^{n} \|A_{ni}\|^r E\|X\|^r I(\|X\| \leqslant b_n) \\
&\leqslant C \sum_{n=1}^{\infty} b_n^{-r} E\|X\|^r I(\|X\| \leqslant b_n) \\
&= C \sum_{j=1}^{\infty} E\|X\|^r I(b_{j-1} < \|X\| \leqslant b_j) \sum_{n=j}^{\infty} n^{-r/\alpha} \log^{-r/\gamma} n \\
&\leqslant C \sum_{j=1}^{\infty} j^{1-r/\alpha} \log^{-r/\gamma} j E\|X\|^r I(b_j < \|X\| \leqslant b_{j+1}) \\
&= C \sum_{j=1}^{\infty} j^{1-r/\alpha} \log^{-r/\gamma} j E\left(\frac{\|X\|^\alpha}{\log^{\alpha/\gamma}(1+\|X\|)} \cdot \|X\|^{r-\alpha} \log^{\alpha/\gamma}(1+\|X\|) \right) \\
&\quad \times I(b_j < \|X\| \leqslant b_{j+1}) \\
&\leqslant CE\|X\|^\alpha / \log^{\alpha/\gamma}(1+\|X\|) < \infty.
\end{aligned}$$

The proof of the lemma is completed. □

4. Proofs of the main results

Proof of Theorem 2.1. The proof will be proceeded under the following two cases.

Case 1. $0 < \alpha \leqslant 1$. Denote for each $1 \leqslant i \leqslant n, n \geqslant 1$ that

$$\begin{aligned}
X_{ni}(1) &= A_{ni} X_{ni} I(\|A_{ni} X_{ni}\| \leqslant b_n), \\
X_{ni}(2) &= A_{ni} X_{ni} I(\|A_{ni} X_{ni}\| > b_n).
\end{aligned}$$

It is easy to see that

$$\left\{ \max_{1\leqslant m\leqslant n} \left\| \sum_{i=1}^{m} A_{ni} X_{ni} \right\| > \varepsilon b_n \right\} \subset \bigcup_{i=1}^{n} \{\|A_{ni} X_{ni}\| > b_n\} \bigcup \left\{ \max_{1\leqslant m\leqslant n} \left\| \sum_{i=1}^{m} X_{ni}(1) \right\| > \varepsilon b_n \right\}.$$

Hence, we obtain by Markov's inequality, C_r-inequality, Lemma 3.1 and Lemma 3.2 that for $0 < q < \alpha$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} A_{ni}X_{ni}\right\| > \varepsilon b_n\right)\\
&\leqslant \sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P(\|A_{ni}X_{ni}\| > b_n) + \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_{ni}(1)\right\| > \varepsilon b_n\right)\\
&\leqslant \sum_{n=1}^{\infty} n^{-1} b_n^{-q}\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^q I(\|A_{ni}X_{ni}\| > b_n) + \frac{1}{\varepsilon}\sum_{n=1}^{\infty} n^{-1} b_n^{-1} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_{ni}(1)\right\|\right)\\
&\leqslant \sum_{n=1}^{\infty} n^{-1} b_n^{-q}\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^q I(\|A_{ni}X_{ni}\| > b_n)\\
&\quad + \frac{1}{\varepsilon}\sum_{n=1}^{\infty} n^{-1} b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X_{ni}\| I(\|A_{ni}X_{ni}\| \leqslant b_n)\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-q}\sum_{i=1}^{n} E\|A_{ni}X\|^q I(\|A_{ni}X\| > b_n)\\
&\quad + \frac{C}{\varepsilon}\sum_{n=1}^{\infty} n^{-1} b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X\| I(\|A_{ni}X\| \leqslant b_n)\\
&\quad + \frac{C}{\varepsilon}\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P(\|A_{ni}X\| > b_n)\\
&\leqslant C\left(1+\frac{1}{\varepsilon}\right)\sum_{n=1}^{\infty} n^{-1} b_n^{-q}\sum_{i=1}^{n} E\|A_{ni}X\|^q I(\|A_{ni}X\| > b_n)\\
&\quad + \frac{C}{\varepsilon}\sum_{n=1}^{\infty} n^{-1} b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X\| I(\|A_{ni}X\| \leqslant b_n)\\
&< \infty.
\end{aligned}
$$

It is deserved to mention that the conclusion above holds for any random vectors under the case $0 < \alpha \leqslant 1$.

Case 2. $1 < \alpha \leqslant 2$.
Define for each $1 \leqslant i \leqslant n$ and $n \geqslant 1$ that

$$
\begin{aligned}
X_{ni}(3) &= A_{ni}X_{ni} I(\|A_{ni}X_{ni}\| \leqslant b_n) - E[A_{ni}X_{ni} I(\|A_{ni}X_{ni}\| \leqslant b_n)|\mathscr{F}_{n,i-1}],\\
X_{ni}(4) &= A_{ni}X_{ni} I(\|A_{ni}X_{ni}\| > b_n) - E[A_{ni}X_{ni} I(\|A_{ni}X_{ni}\| > b_n)|\mathscr{F}_{n,i-1}].
\end{aligned}
$$

It is evident that $A_{ni}X_{ni} = X_{ni}(3) + X_{ni}(4)$ a.s., and $\{X_{ni}(3), \mathscr{F}_{n,i}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ is still an array of martingale difference random vectors. Hence, in order to prove (2.1), it suffices to show that for any $\varepsilon > 0$,

$$
\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_{ni}(3)\right\| > \varepsilon b_n\right) < \infty, \tag{4.1}
$$

and

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant m \leqslant n}\left\|\sum_{i=1}^{m} X_{ni}(4)\right\| > \varepsilon b_n\right) < \infty. \tag{4.2}$$

Noting that $f(x) = \|x\|^t$ is a convex function for all $t \geqslant 1$, we get by conditional Jensen's inequality that

$$\begin{aligned}
&E\left[\|E[A_{ni}X_{ni}I(\|A_{ni}X_{ni}\| \leqslant b_n)|\mathscr{F}_{n,i-1}]\|^t\right] \\
&\leqslant E\left[E[\|A_{ni}X_{ni}\|^t I(\|A_{ni}X_{ni}\| \leqslant b_n)|\mathscr{F}_{n,i-1}]\right] \\
&= E\|A_{ni}X_{ni}\|^t I(\|A_{ni}X_{ni}\| \leqslant b_n).
\end{aligned} \tag{4.3}$$

Therefore, it follows from Markov's inequality, C_r-inequality, (4.3), Lemmas 3.1–3.3 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant m \leqslant n}\left\|\sum_{i=1}^{m} X_{ni}(3)\right\| > \varepsilon b_n\right) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-2} E\left(\max_{1 \leqslant m \leqslant n}\left\|\sum_{i=1}^{m} X_{ni}(3)\right\|^2\right) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} E\|X_{ni}(3)\|^2 \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^2 I(\|A_{ni}X_{ni}\| \leqslant b_n) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} E\|A_{ni}X\|^2 I(\|A_{ni}X\| \leqslant b_n) + C\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(\|A_{ni}X\| > b_n) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} E\|A_{ni}X\|^2 I(\|A_{ni}X\| \leqslant b_n) \\
&\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-1} \sum_{i=1}^{n} E\|A_{ni}X\| I(\|A_{ni}X\| > b_n) \\
&< \infty,
\end{aligned}$$

which implies (4.1). For (4.2), we obtain by Lemmas 3.1 and 3.2 again that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant m \leqslant n}\left\|\sum_{i=1}^{m} X_{ni}(4)\right\| > \varepsilon b_n\right) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} P\left(\bigcup_{i=1}^{n}\{\|A_{ni}X_{ni}\| > b_n\}\right) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(\|A_{ni}X_{ni}\| > b_n)
\end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|I(\|A_{ni}X_{ni}\|>b_n)$$
$$\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X\|I(\|A_{ni}X\|>b_n)$$
$$<\infty.$$

The proof is completed. □

Proof of Corollary 2.1. Let $A_{ni}=A_i$ and $X_{ni}=X_i$ for each $1\leqslant i\leqslant n, n\geqslant 1$ in Theorem 2.1, we have that for any $\varepsilon>0$,

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{-1}P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}A_iX_i\right\|>\varepsilon b_n\right)\\ &=\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} n^{-1}P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}A_iX_i\right\|>\varepsilon b_n\right)\\ &\geqslant \frac{1}{2}\sum_{k=0}^{\infty}P\left(\max_{1\leqslant m\leqslant 2^k}\left\|\sum_{i=1}^{m}A_iX_i\right\|>\varepsilon b_{2^{k+1}}\right),\end{aligned}$$

which together with Borel-Cantelli lemma yields that as $k\to\infty$,

$$b_{2^k}^{-1}\max_{1\leqslant m\leqslant 2^{k+1}}\left\|\sum_{i=1}^{m}A_iX_i\right\| = \frac{b_{2^{k+2}}}{b_{2^k}}\cdot b_{2^{k+2}}^{-1}\max_{1\leqslant m\leqslant 2^{k+1}}\left\|\sum_{i=1}^{m}A_iX_i\right\|\to 0 \text{ a.s.}.$$

Furthermore, for any fixed n, there always exists k such that $2^k\leqslant n<2^{k+1}$. Thus we derive that

$$b_n^{-1}\left\|\sum_{i=1}^{n}A_iX_i\right\|\leqslant b_{2^k}^{-1}\max_{1\leqslant m\leqslant 2^{k+1}}\left\|\sum_{i=1}^{m}A_iX_i\right\|\to 0 \text{ a.s., as } k\to\infty.$$

The proof is completed. □

Proof of Theorem 2.2. We use the same notations as those in the proof of Theorem 2.1. By virtue of Lemmas 3.1-3.4, (4.3), and C_r-inequality, we have that

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{-1}E\left(b_n^{-1}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}A_{ni}X_{ni}\right\|-\varepsilon\right)^{+}\\ &=\sum_{n=1}^{\infty} n^{-1}E\left(b_n^{-1}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}(X_{ni}(3)+X_{ni}(4))\right\|-\varepsilon\right)^{+}\\ &\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-2}E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}X_{ni}(3)\right\|^2\right)+C\sum_{n=1}^{\infty} n^{-1}b_n^{-1}E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}X_{ni}(4)\right\|\right)\end{aligned}$$

$$\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\sum_{i=1}^{n} E\|X_{ni}(3)\|^2 + C\sum_{n=1}^{\infty} n^{-1}b_n^{-1}\sum_{i=1}^{n} E\|X_{ni}(4)\| \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^2 I(\|A_{ni}X_{ni}\| \leqslant b_n) \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X_{ni}\| I(\|A_{ni}X_{ni}\| > b_n) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\sum_{i=1}^{n} E\|A_{ni}X\|^2 I(\|A_{ni}X\| \leqslant b_n) + C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P(\|A_{ni}X\| > b_n) \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X\| I(\|A_{ni}X\| > b_n) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\sum_{i=1}^{n} E\|A_{ni}X\|^2 I(\|A_{ni}X\| \leqslant b_n) \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}b_n^{-1}\sum_{i=1}^{n} E\|A_{ni}X\| I(\|A_{ni}X\| > b_n) \\
&< \infty.
\end{aligned}$$

The proof is completed. □

Proof of Theorem 2.3. The result under the assumptions of Theorem 2.3 follows similarly to the proof of Theorem 2.1, by replacing Lemma 3.1 with Lemma 3.5. Therefore, the details are omitted. □

Proof of Theorem 2.4. Following the proof of Theorem 2.2 and replacing Lemma 3.1 with Lemma 3.5, we can obtain the desired result under the assumptions of Theorem 2.4. The details are also omitted. □

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