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ON FURTHER REFINEMENTS OF YOUNG–TYPE INEQUALITIES WITH KANTOROVICH CONSTANT

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Abstract. In this paper, we further study the Young-type inequalities by dividing the interval $[0,1]$ into m equal parts. Particularly, we obtain some squared forms of refined Young-type inequalities with Kantorovich constant. As applications, the corresponding operator version, as well as inequalities involving Hilbert-Schmidt norm, unitarily invariant norms and traces are given.

1. Introduction

As is known to all, the classical Young inequality states that if $a,b \geqslant 0$ and $v \in [0,1]$, then

$$a^{1-v}b^{v} \leqslant (1-v)a+vb. \tag{1.1}$$

The equality holds if and only if $a=b$. This inequality is also known as the weighted arithmetic-geometric mean inequality. Although the form is simple, it has attracted the attention of many celebrated scholars.

In the past few years, several researchers have improved Young inequality by introducing intermediate terms or adding certain positive quantities. For examples, in 2010, Kittaneh and Manasrah [4] showed a refinement of Young inequality through the addition of a positive term as follows.

$$a^{1-v}b^{v} + r_0(\sqrt{a}-\sqrt{b})^2 \leqslant (1-v)a+vb, \tag{1.2}$$

where $r_0 = \min\{v, 1-v\}$.

One of the significant approaches is improved Young inequality with Kantorovich constant. Zuo et al. [11] presented the following ratio inequality:

$$K(h_0,2)^{r_0}a^{1-v}b^{v} \leqslant (1-v)a+vb, \tag{1.3}$$

where $r_0 = \min\{v, 1-v\}$ and $h_0 = \frac{b}{a}$, $a \neq 0$.

$$K(h,2) = \frac{(h+1)^2}{4h} \quad (h>0),$$

is called Kantorovich constant and satisfies the following properties:

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(i) $K(1,2)=1$;

(ii) $K(h,2)=K(\frac{1}{h},2)$ for $h>0$;

(iii) $K(h,2)$ is monotone increasing on $[1,\infty)$ and monotone decreasing on $(0,1]$.

In 2014, Wu and Zhao [8] deduced the inequality by using the iterative method:

$$K(h_1,2)^{r_1}a^{1-v}b^v + r_0\left(\sqrt{a}-\sqrt{b}\right)^2 \leqslant (1-v)a+vb, \tag{1.4}$$

where $r_0=\min\{v,1-v\}$, $r_1=\min\{2r_0,1-2r_0\}$ and $h_1=\sqrt{\frac{b}{a}}$.

To further study Young inequality, in 2012, Hu [2] established the following squared Young-type inequalities:

$$\left[(va)^v b^{1-v}\right]^2 + v^2(a-b)^2 \leqslant v^2a^2+(1-v)^2b^2, \quad 0\leqslant v\leqslant \frac{1}{2}, \tag{1.5}$$

$$\left[a^v\left((1-v)b\right)^{1-v}\right]^2 + (1-v)^2(a-b)^2 \leqslant v^2a^2+(1-v)^2b^2, \quad \frac{1}{2}\leqslant v\leqslant 1. \tag{1.6}$$

Afterwards, Nasiri et al. established the following refinements in [6]:

(i) If $0\leqslant v\leqslant \frac{1}{2}$, then

$$\left[(va)^v b^{1-v}\right]^2 + r_0^2(a-b)^2 + r_1 b\left(\sqrt{r_0 a}-\sqrt{b}\right)^2 \leqslant v^2a^2+(1-v)^2b^2. \tag{1.7}$$

(ii) If $\frac{1}{2}\leqslant v\leqslant 1$, then

$$\left[a^v\left((1-v)b\right)^{1-v}\right]^2 + r_0^2(a-b)^2 + r_1 a\left(\sqrt{a}-\sqrt{r_0 b}\right)^2 \leqslant v^2a^2+(1-v)^2b^2. \tag{1.8}$$

In 2020, C. Yang and Y. Li [9] showed the following multiple-term refinements.

THEOREM 1.1. *Let a and b be two positive numbers and $0<v<1$. We have*

(i) If $0<v<\frac{1}{4}$, then

$$\begin{aligned}(va)^{2v}b^{2-2v} &+ v^2(a-b)^2 + 2vb(\sqrt{va}-\sqrt{b})^2 \\ &+ \min\{4v,1-4v\}b(\sqrt[4]{vab}-\sqrt{b})^2 \leqslant v^2a^2+(1-v)^2b^2.\end{aligned}$$

(ii) If $\frac{1}{4}\leqslant v<\frac{1}{2}$, then

$$\begin{aligned}(va)^{2v}b^{2-2v} &+ v^2(a-b)^2 + (1-2v)b(\sqrt{va}-\sqrt{b})^2 \\ &+ \min\{2-4v,4v-1\}b(\sqrt[4]{vab}-\sqrt{va})^2 \leqslant v^2a^2+(1-v)^2b^2.\end{aligned}$$

(iii) If $\frac{1}{2}\leqslant v<\frac{3}{4}$, then

$$\begin{aligned}&a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 + (2v-1)a(\sqrt{a}-\sqrt{(1-v)b})^2 \\ &+ \min\{4v-2,3-4v\}a(\sqrt[4]{(1-v)ab}-\sqrt{(1-v)b})^2 \leqslant v^2a^2+(1-v)^2b^2.\end{aligned}$$

(iv) If $\frac{3}{4} \leqslant v < 1$, then

$$a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 + (2-2v)a(\sqrt{a} - \sqrt{(1-v)b})^2 + \min\{4-4v, 4v-3\}a(\sqrt[4]{(1-v)ab} - \sqrt{a})^2 \leqslant v^2a^2 + (1-v)^2b^2.$$

Beyond the two main research approaches mentioned above, which involve refining the classical Young inequality by introducing non-negative terms or studying its squared forms, scholars have also employed various mathematical tools to further deepen and extend Young-type inequalities from different perspectives. Notably, Furuichi et al. systematically utilized the Hermite–Hadamard inequality to achieve refinements and generalizations of existing inequalities. In light of these advances, we recommend that readers refer to [1, 5], which not only advance the relevant theoretical framework but also introduce new research perspectives to the field.

Another type of improvement to Young inequality involves dividing the interval $[0,1]$ of the weighting parameter v into m equal parts. Let $a,b \geqslant 0$ and m be a positive integer. Yang and Zhang [10] showed that, if $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ $(i = 0, 1, \cdots, m-1)$, then

$$a^{1-v}b^v + (mv - i)\left(\left(1 - \frac{i+1}{m}\right)a + \frac{i+1}{m}b - a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\right) + \left(i + 1 - mv\right)\left(\left(1 - \frac{i}{m}\right)a + \frac{i}{m}b - a^{1-\frac{i}{m}}b^{\frac{i}{m}}\right) \leqslant (1-v)a + vb. \tag{1.9}$$

In addition, the above Young-type inequality can be squared. Inspired by this, we will devote the efforts to certain applications of the result.

Throughout this paper, $M_n(\mathbb{C})$ denotes the space of $n \times n$ complex matrices. A norm $|||\cdot|||$ on $M_n(\mathbb{C})$ is defined unitarily invariant if it satisfies the condition $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and all unitary matrices $U, V \in M_n(\mathbb{C})$. For a matrix $A = (a_{i,j}) \in M_n(\mathbb{C})$, Hilbert-Schmidt norm and the trace norm are defined by

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{i,j}|^2\right)^{\frac{1}{2}}, \quad \|A\|_1 = \operatorname{tr}(|A|),$$

respectively, where the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$. It is well known that the trace norm $\|\cdot\|_1$ and Hilbert-Schmidt norm $\|\cdot\|_2$ are unitarily invariant norms.

Let $B(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$. An operator $A \in B(\mathbb{H})$ is positive (written $A \geqslant 0$) if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathbb{H}$. We define $B(\mathbb{H})^+$ as the collection of all positive operators. The set of all invertible operators in $B(\mathbb{H})^+$ is denoted by $B(\mathbb{H})^{++}$.

Let $A, B \in B(\mathbb{H})^+$ and $0 \leqslant v \leqslant 1$. The v-weighted arithmetic operator mean of A and B is defined as

$$A \nabla_v B = (1-v)A + vB.$$

And the ν-weighted geometric operator mean of A and B, denoted by $A\#_\nu B$, is defined as

$$A\#_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}}.$$

The operator Young inequality states that for positive operators A and B

$$A\sharp_\nu B \leqslant A\nabla_\nu B \quad \text{holds for} \quad 0\leqslant \nu \leqslant 1. \tag{1.10}$$

In recent years, a number of significant operator inequalities have been formulated. Yang and Zhang [10] presented the operator version corresponding to inequality (1.9).

Let $A,B\in B(\mathbb{H})$ be two positive operators, $0\leqslant \nu\leqslant 1$ and m be a positive integer. If $\nu\in\left[\frac{i}{m},\frac{i+1}{m}\right]$ $(i=0,1,\cdots,m-1)$, then

$$\begin{aligned} A\sharp_\nu B &+ (m\nu - i)\left(A\nabla_{\frac{i+1}{m}}B - A\sharp_{\frac{i+1}{m}}B\right)\\ &+(i+1-m\nu)\left(A\nabla_{\frac{i}{m}}B - A\sharp_{\frac{i}{m}}B\right)\leqslant A\nabla_\nu B. \end{aligned} \tag{1.11}$$

The organization of this paper is as follows. In section 2, we show some new generalizations and unification of two important results by Nasiri et al. [6] and Yang et al. [9]. While, we also present new refinements of Young-type inequalities involving Kantorovich constant. In section 3, we devote our efforts to certain applications of the main results from the second section and use iterative techniques to derive new and general operator inequalities. In section 4, some new refinements of Young-type inequalities for Hilbert-Schmidt norms, unitarily invariant norms, and traces are established.

2. Further refinements of the Young-type inequalities

In order to give a proof of our first main result in this section, we need the following lemma.

LEMMA 2.1. *Let $a,b\geqslant 0$, $0\leqslant\nu\leqslant 1$, and m be a positive integer. If $\nu\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=0,1,\cdots,2m-1)$, then the following inequalities hold*

$$(2m\nu-i)b^{2-\frac{i+1}{m}}(\nu a)^{\frac{i+1}{m}} + (i+1-2m\nu)b^{2-\frac{i}{m}}(\nu a)^{\frac{i}{m}} \geqslant (\nu a)^{2\nu}b^{2(1-\nu)}, \tag{2.1}$$

$$(2m\nu-i)\big((1-\nu)b\big)^{2-\frac{i+1}{m}}a^{\frac{i+1}{m}} + (i+1-2m\nu)\big((1-\nu)b\big)^{2-\frac{i}{m}}a^{\frac{i}{m}} \geqslant a^{2\nu}\big((1-\nu)b\big)^{2(1-\nu)}. \tag{2.2}$$

Proof. If $\nu\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=0,1,\cdots,2m-1)$, then it follows that $2m\nu-i\in[0,1]$. Applying inequality (1.1), we can obtain the inequalities (2.1) and (2.2). □

THEOREM 2.1. *Let* $a,b \geqslant 0,\ 0 \leqslant v \leqslant 1$, *and* m *be a positive integer.*

(i) *If* $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = 0,1,\cdots,m-1)$, *then*

$$(va)^{2v}b^{2(1-v)} + v^2(a-b)^2 + (2mv-i)b\left(\left(1-\frac{i+1}{m}\right)b + \frac{i+1}{m}(va) - b^{1-\frac{i+1}{m}}(va)^{\frac{i+1}{m}}\right)$$
$$+ (i+1-2mv)b\left(\left(1-\frac{i}{m}\right)b + \frac{i}{m}(va) - b^{1-\frac{i}{m}}(va)^{\frac{i}{m}}\right) \leqslant v^2a^2 + (1-v)^2b^2. \quad (2.3)$$

(ii) *If* $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = m, m+1,\cdots,2m-1)$, *then*

$$a^{2v}\left((1-v)b\right)^{2(1-v)} + (1-v)^2(a-b)^2$$
$$+ (2mv-i)a\left(\left(2-\frac{i+1}{m}\right)((1-v)b) + \left(\frac{i+1}{m}-1\right)a - \left((1-v)b\right)^{2-\frac{i+1}{m}}a^{\frac{i+1}{m}-1}\right)$$
$$+ (i+1-2mv)a\left(\left(2-\frac{i}{m}\right)((1-v)b) + \left(\frac{i}{m}-1\right)a - \left((1-v)b\right)^{2-\frac{i}{m}}a^{\frac{i}{m}-1}\right)$$
$$\leqslant v^2a^2 + (1-v)^2b^2. \quad (2.4)$$

Proof. Firstly, we suppose that $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = 0,1,\cdots,m-1$. Straightforward computations and inequality (2.1) lead to inequality (2.3).

When $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = m, m+1,\cdots,2m-1$. By some straightforward computations and applying inequality (2.2), we can obtain the desired inequality (2.4). ☐

REMARK 1. Note that inequalities (1.7) and (1.8) are special cases of inequalities (2.3) and (2.4), respectively. While, Theorem 2.1 retrieves Theorem 1.1 when $m = 4$. Therefore, we obtain new generalizations and unification of two important squared inequalities by Nasiri et al. [6] and Yang et al. [9].

THEOREM 2.2. *Let* $a,b \geqslant 0,\ 0 \leqslant v \leqslant 1$ *and* m *be a positive integer. If* $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ $(i = 0,1,\cdots,m-1)$, *then the following inequality holds*

$$K(h_2,2)^{r_2}a^{1-v}b^v + (mv-i)\left(\left(1-\frac{i+1}{m}\right)a + \frac{i+1}{m}b - a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\right)$$
$$+ (i+1-mv)\left(\left(1-\frac{i}{m}\right)a + \frac{i}{m}b - a^{1-\frac{i}{m}}b^{\frac{i}{m}}\right) \leqslant (1-v)a + vb, \quad (2.5)$$

where $r_2 = \min\{mv-i, 1-mv+i\}$ *and* $h_2 = \left(\frac{b}{a}\right)^{\frac{1}{m}}$. *In particular,*

(i) *If* $v \in \left[0, \frac{1}{2m}\right]$, *then*

$$K(h_2,2)^{mv}a^{1-v}b^v + mv\left(\left(1-\frac{2}{m}\right)a + \frac{2}{m}\sqrt{ab} - a^{1-\frac{1}{m}}b^{\frac{1}{m}}\right)$$
$$+ v\left(\sqrt{a}-\sqrt{b}\right)^2 \leqslant (1-v)a + vb. \quad (2.6)$$

(ii) *If* $v \in \left[1-\frac{1}{2m}, 1\right]$, *then*

$$K(h_2,2)^{m(1-v)}a^{1-v}b^{v} + m(1-v)\left[\left(1-\frac{2}{m}\right)b + \frac{2}{m}\sqrt{ab} - a^{\frac{1}{m}}b^{1-\frac{1}{m}}\right] + (1-v)\left(\sqrt{a}-\sqrt{b}\right)^2 \leqslant (1-v)a + vb. \tag{2.7}$$

Proof. It's suffice to prove inequality (2.5). Firstly, we suppose that $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ with $i = 0,1,\cdots,m-1$. It follows that $mv - i \in [0,1]$, then

$$\begin{aligned}
&(1-v)a + vb - (mv-i)\left(\left(1-\frac{i+1}{m}\right)a + \frac{i+1}{m}b - a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\right) \\
&\qquad - (i+1-mv)\left(\left(1-\frac{i}{m}\right)a + \frac{i}{m}b - a^{1-\frac{i}{m}}b^{\frac{i}{m}}\right) \\
&= (mv-i)a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}} + (i+1-mv)a^{1-\frac{i}{m}}b^{\frac{i}{m}} \\
&\geqslant K(h_2,2)^{r_2}\left(a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\right)^{mv-i}\left(a^{1-\frac{i}{m}}b^{\frac{i}{m}}\right)^{i+1-mv} \quad \text{by (1.3)} \\
&= K(h_2,2)^{r_2}a^{1-v}b^{v}.
\end{aligned}$$

The proof is complete. □

REMARK 2. Observe that, take $m = 1$ in Theorem 2.2, inequality (2.5) reduces to (1.3). If $m = 2$, then Theorem 2.2 retrieves (1.4). When $m = 3$ in Theorem 2.2, there is no ordering between the left sides of the resulting inequality and the inequality (1.4). For example, take $v = \frac{5}{12} \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and set $a = 2$, $b = 3$, we have

$$K(h_2,2)^{3v-1}a^{1-v}b^{v} + (2-3v)\left(\frac{2}{3}a + \frac{1}{3}b - a^{\frac{2}{3}}b^{\frac{1}{3}}\right) + (3v-1)\left(\frac{1}{3}a + \frac{2}{3}b - a^{\frac{1}{3}}b^{\frac{2}{3}}\right) \approx 2.4139.$$

$$K(h_1,2)^{1-2v}a^{1-v}b^{v} + v\left(\sqrt{a}-\sqrt{b}\right)^2 \leqslant (1-v)a + vb \approx 2.4131.$$

However, if we take $a = 0.01$ and $b = 10000$, then

$$K(h_2,2)^{3v-1}a^{1-v}b^{v} + (2-3v)\left(\frac{2}{3}a + \frac{1}{3}b - a^{\frac{2}{3}}b^{\frac{1}{3}}\right) + (3v-1)\left(\frac{1}{3}a + \frac{2}{3}b - a^{\frac{1}{3}}b^{\frac{2}{3}}\right) \approx 4148.0135.$$

$$K(h_1,2)^{1-2v}a^{1-v}b^{v} + v\left(\sqrt{a}-\sqrt{b}\right)^2 \leqslant (1-v)a + vb \approx 4166.2435.$$

In what follows, we present new refinements of Young-type inequalities due to the results of Nasiri et al. [6] and Yang et al. [9] involving Kantorovich constant.

THEOREM 2.3. *Let* $a,b \geqslant 0$, $0 \leqslant v \leqslant 1$, *and* m *be a positive integer. Then the following inequalities hold*

(i) *If* $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = 0,1,\cdots,m-1)$, *then*

$$\begin{aligned}
&K(h_3,2)^{r_3}(va)^{2v}b^{2-2v} + (2mv-i)b\left(\left(1-\frac{i+1}{m}\right)b + \frac{i+1}{m}va - b^{1-\frac{i+1}{m}}(va)^{\frac{i+1}{m}}\right) \\
&\quad + v^2(a-b)^2 + \left(i+1-2mv\right)b\left(\left(1-\frac{i}{m}\right)b + \frac{i}{m}va - b^{1-\frac{i}{m}}(va)^{\frac{i}{m}}\right) \\
&\leqslant v^2a^2 + (1-v)^2b^2. \qquad (2.8)
\end{aligned}$$

(ii) *If* $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = m,m+1,\cdots,2m-1)$, *then we deduce that*

$$\begin{aligned}
&K(h_4,2)^{r_3}a^{2v}\left((1-v)b\right)^{2(1-v)} + (1-v)^2(a-b)^2 \\
&\quad + (2mv-i)a\left(\left(2-\frac{i+1}{m}\right)(1-v)b + \left(\frac{i+1}{m}-1\right)a - ((1-v)b)^{2-\frac{i+1}{m}}a^{\frac{i+1}{m}-1}\right) \\
&\quad + (i+1-2mv)a\left(\left(2-\frac{i}{m}\right)(1-v)b + \left(\frac{i}{m}-1\right)a - ((1-v)b)^{2-\frac{i}{m}}a^{\frac{i}{m}-1}\right) \\
&\leqslant v^2a^2 + (1-v)^2b^2, \qquad (2.9)
\end{aligned}$$

where $r_3 = \min\{2mv-i, 1-2mv+i\}$, $h_3 = \left(\frac{va}{b}\right)^{\frac{1}{m}}$ *and* $h_4 = \left(\frac{(1-v)b}{a}\right)^{\frac{1}{m}}$.

Proof. Firstly, we suppose that $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = 0,1,\cdots,m-1)$, and it follows that $2v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$. Then

$$\begin{aligned}
&v^2a^2 + (1-v)^2b^2 - v^2(a-b)^2 = b\left[(1-2v)b + 2v(va)\right] \\
&\geqslant b\Bigg[K(h_3,2)^{r_3}(va)^{2v}b^{1-2v} + (2mv-i)\left(\left(1-\frac{i+1}{m}\right)b + \frac{i+1}{m}va - b^{1-\frac{i+1}{m}}(va)^{\frac{i+1}{m}}\right) \\
&\quad + ((i+1)-2mv)\left(\left(1-\frac{i}{m}\right)b + \frac{i}{m}va - b^{1-\frac{i}{m}}(va)^{\frac{i}{m}}\right)\Bigg]. \quad \text{by (2.5)}
\end{aligned}$$

Thus, the above inequality is equivalent with the desired inequality (2.8), i.e.

$$\begin{aligned}
&K(h_3,2)^{r_3}(va)^{2v}b^{2-2v} + (2mv-i)b\left(\left(1-\frac{i+1}{m}\right)b + \frac{i+1}{m}va - b^{1-\frac{i+1}{m}}(va)^{\frac{i+1}{m}}\right) \\
&\quad + v^2(a-b)^2 + \left(i+1-2mv\right)b\left(\left(1-\frac{i}{m}\right)b + \frac{i}{m}va - b^{1-\frac{i}{m}}(va)^{\frac{i}{m}}\right) \\
&\leqslant v^2a^2 + (1-v)^2b^2.
\end{aligned}$$

Next, when $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = m,m+1,\cdots,2m-1$, we put $j = 2m-1-i$ for $j = 0,1,\cdots,m-1$, then $1-v \in \left[\frac{j}{2m}, \frac{j+1}{2m}\right]$. By replacing a, b and v with b, a and $1-v$ in (2.8) , respectively, we obtain an inequality about j. In this inequality, taking j =$2m-1-i$, then we can get the desired inequality (2.9). □

3. Young-type inequalities for operators

In this section, we will establish some new operator Young-type inequalities through two distinct approaches. Firstly, we recall the following key lemma.

LEMMA 3.1. [7] *Let $T \in B(\mathbb{H})$ be a self-adjoint operator. If f and g are both continuous functions satisfying $f(t) \geqslant g(t)$ for all $t \in \mathrm{Sp}(T)$ (where $\mathrm{Sp}(T)$ denotes the spectrum of the operator T), then $f(T) \geqslant g(T)$.*

THEOREM 3.1. *Let $A,B \in B(\mathbb{H})^+$ and $v \in [0,1]$. If $I \leqslant hI \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant h'I$ or $0 < hI \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant h'I \leqslant I$, then the following inequality holds:*

$$K\left(h^{\frac{1}{m}},2\right)^{r_2} A\sharp_v B + (mv-i)\left(A\nabla_{\frac{i+1}{m}}B - A\sharp_{\frac{i+1}{m}}B\right) + (i+1-mv)\left(A\nabla_{\frac{i}{m}}B - A\sharp_{\frac{i}{m}}B\right) \leqslant A\nabla_v B, \tag{3.1}$$

where $r_2 = \min\{mv-i, 1-mv+i\}$.

Proof. For the case of $I \leqslant hI \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant h'I$, taking $a=1$, $b=t>0$ in Theorem 2.2, we have

$$K\left(t^{\frac{1}{m}},2\right)^{r_2} t^v + (mv-i)\left(\left(1-\frac{i+1}{m}\right)+\frac{i+1}{m}t - t^{\frac{i+1}{m}}\right) + (i+1-mv)\left(\left(1-\frac{i}{m}\right)+\frac{i}{m}t - t^{\frac{i}{m}}\right) \leqslant (1-v)+vt.$$

Let $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Under the above condition, we have $I \leqslant hI \leqslant X \leqslant h'I$, which implies $\mathrm{Sp}(X) \subseteq [h,h'] \subseteq [1,+\infty)$. By Lemma 3.1 we have

$$\min_{h\leqslant t\leqslant h'} K\left(t^{\frac{1}{m}},2\right)^{r_2} X^v + (mv-i)\left(\left(1-\frac{i+1}{m}\right)I+\frac{i+1}{m}X - X^{\frac{i+1}{m}}\right) + (i+1-mv)\left(\left(1-\frac{i}{m}\right)I+\frac{i}{m}X - X^{\frac{i}{m}}\right) \leqslant (1-v)I+vX.$$

Since Kantorovich constant $K(h,2)$ is increasing on $(1,+\infty)$, we further derive

$$K\left(h^{\frac{1}{m}},2\right)^{r_2} X^v + (mv-i)\left(\left(1-\frac{i+1}{m}\right)I+\frac{i+1}{m}X - X^{\frac{i+1}{m}}\right) + (i+1-mv)\left(\left(1-\frac{i}{m}\right)I+\frac{i}{m}X - X^{\frac{i}{m}}\right) \leqslant (1-v)I+vX.$$

Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, we deduce the desired result.

For the case of $0 < h \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant h'I < I$, since $K(h,2)$ is decreasing on $(0,1)$, we can obtain the desired inequality (3.1). □

Next, we obtain some new operator Young-type inequalities via operator iterative technique. The main lemma is presented as follows.

LEMMA 3.2. *Let $A,B \in B(\mathbb{H})^+$, $0 \leqslant u,v \leqslant 1$, and a be a positive number. Then*

$$A\sharp_v\big(a(A\sharp_u B)\big) = a^v(A\sharp_{uv}B).$$

Proof.

$$A\sharp_v\big(a(A\sharp_u B)\big) = A^{\frac{1}{2}}\left(aA^{-\frac{1}{2}}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^u A^{\frac{1}{2}}A^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}} = a^v(A\sharp_{uv}B). \quad \square$$

THEOREM 3.2. *Let $A,B \in B(\mathbb{H})^+$, and $0 \leqslant v \leqslant 1$. We have*

(i) *If $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = 0,1,\cdots,m-1$, then*

$$\begin{aligned} &v^{2v}A\sharp_v B + 2v^2(A\nabla B - A\sharp B) + (2mv-i)\left(A\nabla_{\frac{i+1}{m}} vA\sharp B - v^{\frac{i+1}{m}}A\sharp_{\frac{i+1}{2m}}B\right) \\ &\quad + (i+1-2mv)\left(A\nabla_{\frac{i}{m}} vA\sharp B - v^{\frac{i}{m}}A\sharp_{\frac{i}{2m}}B\right) \leqslant (1-v)^2A + v^2B. \end{aligned} \tag{3.2}$$

(ii) *If $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = m,m+1,\cdots,2m-1$, then*

$$\begin{aligned} &(1-v)^{2-2v}B\sharp_v A + 2(1-v)^2(A\nabla B - A\sharp B) \\ &\quad + (2mv-i)\left[A\nabla_{2-\frac{i+1}{m}}(1-v)A\sharp B - (1-v)^{2-\frac{i+1}{m}}B\sharp_{\frac{i+1}{2m}}A\right] \\ &\quad + (i+1-2mv)\left[A\nabla_{2-\frac{i}{m}}(1-v)A\sharp B - (1-v)^{2-\frac{i}{m}}B\sharp_{\frac{i}{2m}}A\right] \\ &\leqslant (1-v)^2B + v^2A. \end{aligned} \tag{3.3}$$

Proof. If $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = 0,1,\cdots,m-1$, then it follows that $2v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$. Substituting B with $vA\sharp B$ and v with $2v$ in inequality (1.11), we obtain

$$\begin{aligned} &A\sharp_{2v}vA\sharp B + (2mv-i)\left(A\nabla_{\frac{i+1}{m}}vA\sharp B - A\sharp_{\frac{i+1}{m}}vA\sharp B\right) \\ &\quad + (i+1-2mv)\left(A\nabla_{\frac{i}{m}}vA\sharp B - A\sharp_{\frac{i}{m}}vA\sharp B\right) \leqslant A\nabla_{2v}\big(v(A\sharp B)\big). \end{aligned}$$

By Lemma 3.2, we have

$$A\sharp_{2v}vA\sharp B = v^{2v}A\sharp_v B$$

and

$$\begin{aligned} A\nabla_{2v}vA\sharp B &= (1-2v)A + 2v\cdot\big(vA\sharp B\big) = A - 2vA + 2v^2A\sharp B \\ &= (1-v)^2A + v^2B - 2v^2\big(A\nabla B - A\sharp B\big). \end{aligned}$$

Thus

$$\begin{aligned} &v^{2v}A\sharp_v B + 2v^2\big(A\nabla B - A\sharp B\big) + (2mv-i)\left(A\nabla_{\frac{i+1}{m}}vA\sharp B - v^{\frac{i+1}{m}}A\sharp_{\frac{i+1}{2m}}B\right) \\ &\quad + (i+1-2mv)\left(A\nabla_{\frac{i}{m}}vA\sharp B - v^{\frac{i}{m}}A\sharp_{\frac{i}{2m}}B\right) \leqslant (1-v)^2A + v^2B. \end{aligned}$$

Now, consider the case of $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = m, m+1, \cdots, 2m-1$, we put $j = 2m-1-i$ for $j = 0, 1, \cdots, m-1$, then $1-v \in \left[\frac{j}{2m}, \frac{j+1}{2m}\right]$. By the above inequality (3.2), we have

$$\begin{aligned}
&(1-v)^{2-2v} A\sharp_{1-v} B + 2(1-v)^2 (A\nabla B - 2A\sharp B) \\
&\quad + (2m(1-v)-j)\left[A\nabla_{\frac{j+1}{m}}(1-v)A\sharp B - (1-v)^{\frac{j+1}{m}} A\sharp_{\frac{j+1}{2m}} B\right] \\
&\quad + (j+1-2m(1-v))\left[A\nabla_{\frac{j}{m}}(1-v)A\sharp B - (1-v)^{\frac{j}{m}} A\sharp_{\frac{j}{2m}} B\right] \\
&\leqslant (1-v)^2 B + v^2 A.
\end{aligned}$$

Replace j by $2m-1-i$ in the above operator inequality. Since $A\sharp_v B = B\sharp_{1-v} A$ holds for $v \in [0,1]$, we obtain inequality (3.3). □

THEOREM 3.3. *Let $A, B \in B(\mathbb{H})^+$. If $0 \leqslant v \leqslant 1$, $I \leqslant hI \leqslant A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leqslant h'I$ or $0 < hI \leqslant A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leqslant h'I \leqslant I$, we have*

(i) *If $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = 0, 1, \cdots, m-1)$, then*

$$\begin{aligned}
&K\left(v^{\frac{1}{m}} h^{\frac{1}{2m}}, 2\right)^{r_3} v^{2v} A\sharp_v B + 2v^2 \big(A\nabla B - A\sharp B\big) \\
&+ (2mv-i)\left(A\nabla_{\frac{i+1}{m}} vA\sharp B - v^{\frac{i+1}{m}} A\sharp_{\frac{i+1}{2m}} B\right) \\
&+ (i+1-2mv)\left(A\nabla_{\frac{i}{m}} vA\sharp B - v^{\frac{i}{m}} A\sharp_{\frac{i}{2m}} B\right) \leqslant (1-v)^2 A + v^2 B. \qquad (3.4)
\end{aligned}$$

(ii) *If $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = m, m+1, \cdots, 2m-1)$, then*

$$\begin{aligned}
&K\left((1-v)^{\frac{1}{m}} h^{\frac{1}{2m}}, 2\right)^{r_3} (1-v)^{2-2v} B\sharp_v A + 2(1-v)^2 (A\nabla B - A\sharp B) \\
&+ (2mv-i)\left[A\nabla_{2-\frac{i+1}{m}}(1-v)A\sharp B - (1-v)^{2-\frac{i+1}{m}} B\sharp_{\frac{i+1}{2m}} A\right] + (i+1-2mv) \\
&\left[A\nabla_{2-\frac{i}{m}}(1-v)A\sharp B - (1-v)^{2-\frac{i}{m}} B\sharp_{\frac{i}{2m}} A\right] \leqslant (1-v)^2 B + v^2 A. \qquad (3.5)
\end{aligned}$$

where $r_3 = \min\{2mv-i,\ i+1-2mv\}$.

Proof. If $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ for $i = 0, 1, \cdots, m-1$, then $2v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$. When $I \leqslant hI \leqslant A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leqslant h'I$, we have

$$I \leqslant v\sqrt{h}I \leqslant A^{-\frac{1}{2}}(vA\sharp B)A^{-\frac{1}{2}} \leqslant v\sqrt{h'}I.$$

Correspondingly, when $0 < hI \leqslant A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leqslant h'I \leqslant I$, we have

$$0 < v\sqrt{h}I \leqslant A^{-\frac{1}{2}}(vA\sharp B)A^{-\frac{1}{2}} \leqslant v\sqrt{h'}I \leqslant I.$$

By substituting B with $vA\sharp B$ and v with $2v$ in inequality (3.1), we have

$$K\left((v\sqrt{h})^{\frac{1}{m}},2\right)^{r_3}A\sharp_{2v}vA\sharp B+(2mv-i)\left(A\nabla_{\frac{i+1}{m}}vA\sharp B-A\sharp_{\frac{i+1}{m}}vA\sharp B\right)$$
$$+(i+1-2mv)\left(A\nabla_{\frac{i}{m}}vA\sharp B-A\sharp_{\frac{i}{m}}vA\sharp B\right)\leqslant A\nabla_{2v}vA\sharp B.$$

By Lemma 2.1, we have $A\sharp_{2v}\big(v(A\sharp B)\big)=v^{2v}(A\sharp_v B)$ and

$$A\nabla_{2v}vA\sharp B=(1-2v)A+2v\cdot\big(vA\sharp B\big)=A-2vA+2v^2A\sharp B$$
$$=(1-v)^2A+v^2B-2v^2\big(A\nabla B-2A\sharp B\big).$$

Thus, we can obtain the desired inequality (3.4).

When $v\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ for $i=m,m+1,\cdots,2m-1$, we put $j=2m-1-i$ for $j=0,1,\cdots,m-1$, then $1-v\in\left[\frac{j}{2m},\frac{j+1}{2m}\right]$. By the above inequality (3.4), we have

$$K\left((1-v)^{\frac{1}{m}}h^{\frac{1}{2m}},2\right)^{r_3}(1-v)^{2-2v}A\sharp_{1-v}B+2(1-v)^2(A\nabla B-A\sharp B)$$
$$+(2m(1-v)-j)\left(A\nabla_{\frac{j+1}{m}}(1-v)A\sharp B-(1-v)^{\frac{j+1}{m}}A\sharp_{\frac{j+1}{2m}}B\right)$$
$$+(j+1-2m(1-v))\left(A\nabla_{\frac{j}{m}}(1-v)A\sharp B-(1-v)^{\frac{j}{m}}A\sharp_{\frac{j}{2m}}B\right)\leqslant(1-v)^2B+v^2A.$$

Since $A\sharp_v B=B\sharp_{1-v}A$ holds for any $v\in[0,1]$, and $j=2m-1-i$, we can obtain the desired inequality (3.5). □

4. Young-type inequalities for matrices

In this section, we will investigate manifestations of matrix Young-type inequalities involving Hilbert-Schmidt norm, unitarily invariant norm and the traces.

Firstly, we establish a new refinement of Young-type inequality under the Hilbert-Schmidt norm. The matrix inequality corresponding to Theorem 2.3 is presented below.

THEOREM 4.1. *Let $A,B\in M_n(\mathbb{C})$ be positive semidefinite matrices and $X\in M_n(\mathbb{C})$ such that $0\leqslant m_0I\leqslant A,B\leqslant MI$. Then, for every positive integer m and $0\leqslant v\leqslant 1$, we have*

(i) *If $v\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=0,1,\cdots,m-1)$, then*

$$K\left((vh)^{\frac{1}{m}},2\right)^{r_3}v^{2v}\|A^vXB^{1-v}\|_2^2+2v(1-v)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2+v^2\|AX-XB\|_2^2$$
$$+(2mv-i)\left(\left(1-\frac{i+1}{m}\right)\|XB\|_2^2+\frac{i+1}{m}v\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2-v^{\frac{i+1}{m}}\|A^{\frac{i+1}{2m}}XB^{1-\frac{i+1}{m}}\|_2^2\right)$$
$$+(i+1-2mv)\left(\left(1-\frac{i}{m}\right)\|XB\|_2^2+\frac{i}{m}v\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2-v^{\frac{i}{m}}\|A^{\frac{i}{2m}}XB^{1-\frac{i}{m}}\|_2^2\right)$$
$$\leqslant\|vAX+(1-v)XB\|_2^2. \tag{4.1}$$

(ii) *If* $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = m, m+1, \cdots, 2m-1)$, *then*

$$K\left(((1-v)h)^{\frac{1}{m}}, 2\right)^{r_3} (1-v)^{2-2v} \|A^v XB^{1-v}\|_2^2 + 2v(1-v)\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\|_2^2$$
$$+ (1-v)^2 \|AX - XB\|_2^2 + (2mv - i)\left(\left(\frac{i+1}{m} - 1\right)\|AX\|_2^2\right.$$
$$+ \left(2 - \frac{i+1}{m}\right)(1-v)\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\|_2^2 - (1-v)^{2-\frac{i+1}{m}} \left.\|A^{\frac{i+1}{2m}} XB^{1-\frac{i+1}{2m}}\|_2^2\right)$$
$$+ (i+1-2mv)\left(\left(\frac{i}{m} - 1\right)\|AX\|_2^2 + \left(2 - \frac{i}{m}\right)(1-v)\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\|_2^2\right.$$
$$\left. - (1-v)^{2-\frac{i}{m}} \|A^{\frac{i}{2m}} XB^{1-\frac{i}{2m}}\|_2^2\right) \leqslant \|vAX + (1-v)XB\|_2^2, \tag{4.2}$$

where $r_3 = \min\{2mv - i,\ i+1-2mv\}$, $h = \frac{M}{m_0}$.

Proof. Since A and B are positive semidefinite matrices, the spectral decomposition theorem implies that there exist unitary matrices $U, V \in M_n(\mathbb{C})$ such that $A = UD_1U^*$ and $B = VD_2V^*$, where $D_1 = \operatorname{diag}(a_1, a_2, \cdots, a_n)$, $D_2 = \operatorname{diag}(b_1, b_2, \cdots, b_n)$ with $a_i \geqslant 0$, $b_i \geqslant 0$ for $i = 1, 2, \cdots, n$. Suppose that $Y = U^*XV = [y_{i,j}]$, we have

$$vAX + (1-v)XB = U(vD_1Y + (1-v)YD_2)V^* = U((va_i + (1-v)b_j)y_{i,j})V^*,$$

$$A^v XB^{1-v} = U\left(a_i^v b_j^{1-v} y_{i,j}\right)V^*, \quad AX - XB = U((a_i - b_j)y_{i,j})V^*,$$

$$A^{\frac{i+1}{2m}} XB^{1-\frac{i+1}{2m}} = U\left(a_i^{\frac{i+1}{2m}} b_j^{1-\frac{i+1}{2m}} y_{i,j}\right)V^*, \quad A^{\frac{i}{2m}} XB^{1-\frac{i}{2m}} = U\left(a_i^{\frac{i}{2m}} b_j^{1-\frac{i}{2m}} y_{i,j}\right)V^*.$$

By Theorem 2.3 and unitary invariance of Hilbert-Schmidt norm, we obtain that

$$K\left((vt_{i,j})^{\frac{1}{m}}, 2\right)^{r_3} v^{2v} \sum_{i,j=1}^{n} \left(a_i^v b_j^{1-v}\right)^2 |y_{i,j}|^2$$
$$+ 2v(1-v)\sum_{i,j=1}^{n} a_i b_j |y_{i,j}|^2 + v^2 \sum_{i,j=1}^{n} (a_i - b_j)^2 |y_{i,j}|^2$$
$$+(2mv-i)\left(\left(1-\frac{i+1}{m}\right)\sum_{i,j=1}^{n} b_j^2 |y_{i,j}|^2 + \frac{i+1}{m} v \sum_{i,j=1}^{n} a_i b_j\ |y_{i,j}|^2 - v^{\frac{i+1}{m}} \sum_{i,j=1}^{n} a_i^{\frac{i+1}{m}} b_j^{2-\frac{i+1}{m}} |y_{i,j}|^2\right)$$
$$+ (i+1-2mv)\left(\left(1 - \frac{i}{m}\right)\sum_{i,j=1}^{n} b_j^2 |y_{i,j}|^2 + \frac{i}{m} v \sum_{i,j=1}^{n} a_i b_j |y_{i,j}|^2 - v^{\frac{i}{m}} \sum_{i,j=1}^{n} a_i^{\frac{i}{m}} b_j^{2-\frac{i}{m}} |y_{i,j}|^2\right)$$
$$\leqslant \sum_{i,j=1}^{n} (va_i + (1-v)b_j)^2 |y_{i,j}|^2 = \|vAX + (1-v)XB\|_2^2,$$

where $t_{i,j} = \frac{a_i}{b_j}$. According to the conditions $0 < m_0 I \leqslant A, B < MI$ with the eigenvalues satisfy $\frac{1}{h} = \frac{m_0}{M} \leqslant t_{i,j} = \frac{a_i}{b_j} \leqslant h = \frac{M}{m_0}$, and property of Kantorovich constant, we have

$$
\begin{aligned}
&K\left((vh)^{\frac{1}{m}},2\right)^{r_3} v^{2v} \sum_{i,j=1}^{n} \left(a_i^v b_j^{1-v}\right)^2 |y_{i,j}|^2 + 2v(1-v) \sum_{i,j=1}^{n} a_i b_j |y_{i,j}|^2 + v^2 \sum_{i,j=1}^{n} (a_i - b_j)^2 |y_{i,j}|^2 \\
&+(2mv-i)\left(\left(1-\frac{i+1}{m}\right)\sum_{i,j=1}^{n} b_j^2 |y_{i,j}|^2 + \frac{i+1}{m} v \sum_{i,j=1}^{n} a_i b_j |y_{i,j}|^2 - v^{\frac{i+1}{m}} \sum_{i,j=1}^{n} a_i^{\frac{i+1}{m}} b_j^{2-\frac{i+1}{m}} |y_{i,j}|^2\right) \\
&+(i+1-2mv)\left(\left(1-\frac{i}{m}\right)\sum_{i,j=1}^{n} b_j^2 |y_{i,j}|^2 + \frac{i}{m} v \sum_{i,j=1}^{n} a_i b_j |y_{i,j}|^2 - v^{\frac{i}{m}} \sum_{i,j=1}^{n} a_i^{\frac{i}{m}} b_j^{2-\frac{i}{m}} |y_{i,j}|^2\right) \\
&\leqslant \sum_{i,j=1}^{n} (va_i + (1-v)b_j)^2 |y_{i,j}|^2 = \|vAX + (1-v)XB\|_2^2.
\end{aligned}
$$

The above inequality implies that

$$
\begin{aligned}
&K\left((vh)^{\frac{1}{m}},2\right)^{r_3} v^{2v} \|A^v X B^{1-v}\|_2^2 + 2v(1-v)\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2 + v^2 \|AX - XB\|_2^2 \\
&\quad + (2mv-i)\left(\left(1-\frac{i+1}{m}\right)\|XB\|_2^2 + \frac{i+1}{m} v \|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2 - v^{\frac{i+1}{m}} \|A^{\frac{i+1}{2m}} X B^{1-\frac{i+1}{2m}}\|_2^2\right) \\
&\quad + (i+1-2mv)\left(\left(1-\frac{i}{m}\right)\|XB\|_2^2 + \frac{i}{m} v \|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2 - v^{\frac{i}{m}} \|A^{\frac{i}{2m}} X B^{1-\frac{i}{2m}}\|_2^2\right). \\
&\leqslant \|vAX + (1-v)XB\|_2^2.
\end{aligned}
$$

By a similar way as that in the above section, inequality (4.2) can be obtained. □

Secondly, we will establish some new refinements of the Young-type inequalities under unitarily invariant norm and the trace. The unitarily invariant norm and trace inequalities corresponding to Theorem 2.2 and Theorem 2.3 are showed below. For this purpose, we need the following lemma.

LEMMA 4.1. [2] *Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite matrices and $X \in M_n(\mathbb{C})$. Then*

$$
|||A^v X B^{1-v}||| \leqslant |||AX|||^v |||XB|||^{1-v}. \tag{4.3}
$$

In particular,

$$
\operatorname{tr}\left|A^v B^{1-v}\right| \leqslant (\operatorname{tr} A)^v (\operatorname{tr} B)^{1-v}. \tag{4.4}
$$

THEOREM 4.2. *Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite matrices, and $X \in M_n(\mathbb{C})$. For all positive integers m and $0 \leqslant v \leqslant 1$, the following inequalities hold*

(i) *If* $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ *for* $i = 0, 1, \cdots, m-1$, *then*

$$K\left(\left(v\frac{|||AX|||}{|||XB|||}\right)^{\frac{1}{m}}, 2\right)^{r_3} v^{2v}|||A^vXB^{1-v}|||^2 + v^2\left(|||AX||| - |||XB|||\right)^2$$

$$+ 2v(1-v)|||AX||||||XB||| + (2mv - i)\left(\left(1 - \frac{i+1}{m}\right)|||XB|||^2\right.$$

$$\left. + \frac{i+1}{m}v|||AX||||||XB||| - v^{\frac{i+1}{m}}|||AX|||^{\frac{i+1}{m}}|||XB|||^{2-\frac{i+1}{m}}\right)$$

$$+ (i+1-2mv)\left(\left(1 - \frac{i}{m}\right)|||XB|||^2 + \frac{i}{m}v|||AX||||||XB|||\right.$$

$$\left. - v^{\frac{i}{m}}|||AX|||^{\frac{i}{m}}|||XB|||^{2-\frac{i}{m}}\right) \leqslant \left(v|||AX||| + (1-v)|||XB|||\right)^2. \qquad (4.5)$$

(ii) *If* $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ *for* $i = m, m+1, \cdots, 2m-1$, *then*

$$K\left(\left((1-v)\frac{|||XB|||}{|||AX|||}\right)^{\frac{1}{m}}, 2\right)^{r_3} (1-v)^{2-2v}|||A^vXB^{1-v}|||^2$$

$$+ (1-v)^2\left(|||AX||| - |||XB|||\right)^2 + 2v(1-v)|||AX||||||XB|||$$

$$+ (2mv - i)\left(\left(\frac{i+1}{m} - 1\right)|||AX|||^2 + \left(2 - \frac{i+1}{m}\right)(1-v)|||AX||||||XB|||\right.$$

$$\left. - (1-v)^{2-\frac{i+1}{m}}|||AX|||^{\frac{i+1}{m}}|||XB|||^{2-\frac{i+1}{m}}\right)$$

$$+ (i+1-2mv)\left(\left(\frac{i}{m} - 1\right)|||AX|||^2 + \left(2 - \frac{i}{m}\right)(1-v)|||AX||||||XB|||\right.$$

$$\left. - (1-v)^{2-\frac{i}{m}}|||AX|||^{\frac{i}{m}}|||XB|||^{2-\frac{i}{m}}\right) \leqslant \left(v|||AX||| + (1-v)|||XB|||\right)^2. \qquad (4.6)$$

Proof. If $v \in \left[\frac{i}{2m}, \frac{i+1}{2m}\right]$ $(i = 0, 1, \cdots, m-1)$, then, by Theorem 2.3 and Lemma 4.1, we have

$$K\left(\left(v\frac{|||AX|||}{|||XB|||}\right)^{\frac{1}{m}}, 2\right)^{r_3} v^{2v}|||A^vXB^{1-v}|||^2 + v^2\left(|||AX||| - |||XB|||\right)^2 + 2v(1-v)|||AX||||||XB|||$$

$$+ (2mv - i)\left(\left(1 - \frac{i+1}{m}\right)|||XB|||^2 + \frac{i+1}{m}v|||AX||||||XB||| - v^{\frac{i+1}{m}}|||AX|||^{\frac{i+1}{m}}|||XB|||^{2-\frac{i+1}{m}}\right)$$

$$+ (i+1-2mv)\left(\left(1 - \frac{i}{m}\right)|||XB|||^2 + \frac{i}{m}v|||AX||||||XB||| - v^{\frac{i}{m}}|||AX|||^{\frac{i}{m}}|||XB|||^{2-\frac{i}{m}}\right)$$

$$\leqslant K\left(\left(v\frac{|||AX|||}{|||XB|||}\right)^{\frac{1}{m}},2\right)^{r_3} v^{2v}|||AX|||^{2v}|||XB|||^{2-2v}$$
$$+v^2\left(|||AX|||-|||XB|||\right)^2+2v(1-v)|||AX||||||XB|||$$
$$+(2mv-i)\left(\left(1-\frac{i+1}{m}\right)|||XB|||^2+\frac{i+1}{m}v|||AX||||||XB|||-v^{\frac{i+1}{m}}|||AX|||^{\frac{i+1}{m}}|||XB|||^{2-\frac{i+1}{m}}\right)$$
$$+(i+1-2mv)\left(\left(1-\frac{i}{m}\right)|||XB|||^2+\frac{i}{m}v|||AX||||||XB|||-v^{\frac{i}{m}}|||AX|||^{\frac{i}{m}}|||XB|||^{2-\frac{i}{m}}\right)$$
$$\leqslant\left(v|||AX|||+(1-v)|||XB|||\right)^2.$$

If $v\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=m,m+1,\cdots,2m-1)$, by the same method as the above, we can obtain the desired inequality (4.6) . □

THEOREM 4.3. *Let $A,B\in M_n(\mathbb{C})$ be positive semidefinite matrices and $0\leqslant v\leqslant 1$. Then, for all positive integers m, we have*

(i) *If $v\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=0,1,\cdots,m-1)$, then*

$$\begin{aligned}&v^{2v}\operatorname{tr}\left(\left|A^vB^{1-v}\right|\right)^2+v^2\left(\operatorname{tr}(A)-\operatorname{tr}(B)\right)^2+2v(1-v)\operatorname{tr}(A)\operatorname{tr}(B)\\&\quad+(2mv-i)\operatorname{tr}(B)\left(\left(1-\frac{i+1}{m}\right)\operatorname{tr}(B)+\frac{i+1}{m}v\operatorname{tr}(A)-v^{\frac{i+1}{m}}\operatorname{tr}(A)^{\frac{i+1}{m}}\operatorname{tr}(B)^{1-\frac{i+1}{m}}\right)\\&\quad+(i+1-2mv)\operatorname{tr}(B)\left(\left(1-\frac{i}{m}\right)\operatorname{tr}(B)+\frac{i}{m}v\operatorname{tr}(A)-v^{\frac{i}{m}}\operatorname{tr}(A)^{\frac{i}{m}}\operatorname{tr}(B)^{1-\frac{i}{m}}\right)\\&\leqslant\left(\operatorname{tr}\left(vA+(1-v)B\right)\right)^2.\end{aligned}\tag{4.7}$$

(ii) *If $v\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=m,m+1,\cdots,2m-1)$, then*

$$\begin{aligned}&(1-v)^{2-2v}\operatorname{tr}\left(\left|A^vB^{1-v}\right|\right)^2+(1-v)^2\left(\operatorname{tr}(A)-\operatorname{tr}(B)\right)^2\\&+2v(1-v)\operatorname{tr}(A)\operatorname{tr}(B)+(2m-i)\operatorname{tr}(A)\left(\left(\frac{i+1}{m}-1\right)\operatorname{tr}(A)\right.\\&+\left(2-\frac{i+1}{m}\right)(1-v)\operatorname{tr}(B)-(1-v)^{2-\frac{i+1}{m}}\operatorname{tr}(B)^{2-\frac{i+1}{m}}\operatorname{tr}(A)^{\frac{i+1}{m}-1}\Big)\\&+(i+1-2mv)\operatorname{tr}(A)\left(\left(\frac{i}{m}-1\right)\operatorname{tr}(A)+\left(2-\frac{i}{m}\right)(1-v)\operatorname{tr}(B)\right.\\&-(1-v)^{2-\frac{i}{m}}\operatorname{tr}(B)^{2-\frac{i}{m}}\operatorname{tr}(A)^{\frac{i}{m}-1}\Big)\leqslant\left(\operatorname{tr}\left(vA+(1-v)B\right)\right)^2.\end{aligned}\tag{4.8}$$

Proof. If $v\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=0,1,\cdots,m-1)$, then by Theorem 2.3 and Lemma

4.1, we have

$$
\begin{aligned}
& v^{2v}\operatorname{tr}\left(\left|A^{v}B^{1-v}\right|\right)^{2}+v^{2}\left(\operatorname{tr}(A)-\operatorname{tr}(B)\right)^{2}+2v(1-v)\operatorname{tr}(A)\operatorname{tr}(B)\\
&\quad+(2mv-i)\operatorname{tr}(B)\left(\left(1-\frac{i+1}{m}\right)\operatorname{tr}(B)+\frac{i+1}{m}v\operatorname{tr}(A)-v^{\frac{i+1}{m}}\operatorname{tr}(A)^{\frac{i+1}{m}}\operatorname{tr}(B)^{1-\frac{i+1}{m}}\right)\\
&\quad+(i+1-2mv)\operatorname{tr}(B)\left(\left(1-\frac{i}{m}\right)\operatorname{tr}(B)+\frac{i}{m}v\operatorname{tr}(A)-v^{\frac{i}{m}}\operatorname{tr}(A)^{\frac{i}{m}}\operatorname{tr}(B)^{1-\frac{i}{m}}\right)\\
&\leqslant v^{2v}\operatorname{tr}(A)^{2v}\operatorname{tr}(B)^{2(1-v)}+v^{2}\left(\operatorname{tr}(A)-\operatorname{tr}(B)\right)^{2}+2v(1-v)\operatorname{tr}(A)\operatorname{tr}(B)\\
&\quad+(2mv-i)\operatorname{tr}(B)\left(\left(1-\frac{i+1}{m}\right)\operatorname{tr}(B)+\frac{i+1}{m}v\operatorname{tr}(A)-v^{\frac{i+1}{m}}\operatorname{tr}(A)^{\frac{i+1}{m}}\operatorname{tr}(B)^{1-\frac{i+1}{m}}\right)\\
&\quad+(i+1-2mv)\operatorname{tr}(B)\left(\left(1-\frac{i}{m}\right)\operatorname{tr}(B)+\frac{i}{m}v\operatorname{tr}(A)-v^{\frac{i}{m}}\operatorname{tr}(A)^{\frac{i}{m}}\operatorname{tr}(B)^{1-\frac{i}{m}}\right)\\
&\leqslant\left(\operatorname{tr}\left(vA+(1-v)B\right)\right)^{2}.
\end{aligned}
$$

If $v\in\left[\frac{i}{2m},\frac{i+1}{2m}\right]$ $(i=m,m+1,\cdots,2m-1)$, then the desired inequality (4.8) is obtained by the same method. □

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