

WILKER TYPE INEQUALITIES WITH TWO-PARAMETERIZED QUADRATIC POWER EXPONENTIAL FUNCTIONS

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Abstract. In this paper, we present new Wilker type inequality with power exponential functions as follows. The inequality

$$\left(\frac{\sin x}{x}\right)^{2+ax^2} + \left(\frac{\tan x}{x}\right)^{1-bx^2} > 2$$

holds for $0 < x < \frac{\pi}{2}$ and (a, b) in some closed subset $D \subset \mathbb{R}^2$. This is the first study to assign a function of x to the exponential part of $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ so that the Wilker inequality holds.

1. Introduction

Wilker [16] proposed two open problems as follows:

(a) Prove that if $0 < x < \frac{\pi}{2}$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2.$$

(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x$$

for $0 < x < \frac{\pi}{2}$. After, Sumner et al. [15] proved the problems and showed a stronger inequality. It is known that the inequality in (a) is called Wilker inequality. Many mathematicians have studied Wilker inequality and it has been extended in various ways, such as for inverse functions, hyperbolic functions, and trigonometric functions via integration. As results of recent researches, the following are known: [1]–[10], [13], [14], [17]–[23]. An interesting property of Wilker inequality is that even though $\frac{\sin x}{x}$ is monotonically decreasing, $\frac{\tan x}{x}$ is monotonically increasing, so $\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}$ is monotonically increasing. Also, $\left(\frac{\sin x}{x}\right)^c$ and $\left(\frac{\tan x}{x}\right)^c$ are monotonically decreasing and

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monotonically increasing with respect to c , respectively. Then, the question arises: if c of $\left(\frac{\sin x}{x}\right)^c$ is greater than 2 or c of $\left(\frac{\tan x}{x}\right)^c$ is less than 1, then does Wilker inequality hold? To answer this question, we consider the parameterized quadratic power exponential functions $\left(\frac{\sin x}{x}\right)^{2+ax^2}$ and $\left(\frac{\tan x}{x}\right)^{1-bx^2}$ and show that Wilker inequality holds for some closed subset $D \subset \mathbb{R}^2$. This work is the first to assign a function of x to the exponential part of $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ so that Wilker inequality holds. Our main results are as follows.

THEOREM 1.1. *For $0 < x < \frac{\pi}{2}$ and $(a, b) \in D$, we have*

$$\left(\frac{\sin x}{x}\right)^{2+ax^2} + \left(\frac{\tan x}{x}\right)^{1-bx^2} > 2,$$

where

$$D = \left\{ (a, b) \mid 0 \leq a \leq \frac{2}{5}, 0 \leq b \leq \frac{1}{5} - \frac{a}{2} \right\}.$$

The closed subset $D \subset \mathbb{R}^2$ is not the largest closed subset that satisfies the inequality in Theorem 1.1. As shown in the following theorems 1.2 and 1.3, the inequality in Theorem 1.1 also holds at points (a, b) outside D .

THEOREM 1.2. *For $0 < x < \frac{\pi}{2}$, we have*

$$\left(\frac{\sin x}{x}\right)^{2+ax^2} + \left(\frac{\tan x}{x}\right) > 2,$$

where the constant $a = \frac{16}{15}$ is the best possible, in the sense that $a = \frac{16}{15}$ can not be replaced by a larger number.

THEOREM 1.3. *For $0 < x < \frac{\pi}{2}$, we have*

$$\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-bx^2} > 2$$

for $0 \leq b \leq \frac{3}{10}$. Also, we have

$$\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-bx^2} < 2,$$

where the constant $b = \frac{8}{15}$ is the best possible, in the sense that $b = \frac{8}{15}$ can not be replaced by a smaller number.

2. Preliminaries

We used Mathematica software for the calculations in this paper. It is known that for $0 < x < \frac{\pi}{2}$ and non-negative integers m and n (see [12]), we have

$$u(x, 2n + 1) < \sin x < u(x, 2n) \quad \text{and} \quad v(x, 2m + 1) < \cos x < v(x, 2m),$$

where

$$u(x, p) = \sum_{k=0}^p \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{and} \quad v(x, q) = \sum_{k=0}^q \frac{(-1)^k x^{2k}}{(2k)!}.$$

Moreover, $u(x, p)$ and $v(x, q)$ are truncations of

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

We prove some lemmas to prove the main theorems.

LEMMA 2.1. For $0 < x < \frac{\pi}{2}$, we have

$$e^{-\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{1890}} < \frac{\sin x}{x} < e^{-\frac{x^2}{6} - \frac{x^4}{180}}.$$

Proof. We set

$$f(x) = \ln(\sin x) - \ln x - \left(-\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{1890} \right).$$

The derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= -\frac{1}{x} + \frac{x}{3} + \frac{x^3}{45} + \frac{x^5}{315} + \frac{1}{\tan x} \\ &> -\frac{1}{x} + \frac{x}{3} + \frac{x^3}{45} + \frac{x^5}{315} + \frac{v(x, 3)}{u(x, 2)} \\ &= -\frac{1}{x} + \frac{x}{3} + \frac{x^3}{45} + \frac{x^5}{315} + \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}}{x - \frac{x^3}{6} + \frac{x^5}{120}} \\ &= \frac{x^5(65 - 26x^2 + 2x^4)}{630(120 - 20x^2 + x^4)}. \end{aligned}$$

Since we have $65 - 26x^2 + 2x^4 > 0$ and $120 - 20x^2 + x^4 > 0$ for $0 < x < \frac{\pi}{2}$, $f(x)$ is monotonically increasing for $0 < x < \frac{\pi}{2}$. From $\lim_{x \rightarrow 0} f(x) = 0$, we obtain $f(x) > 0$ for $0 < x < \frac{\pi}{2}$. We set

$$g(x) = \ln(\sin x) - \ln x - \left(-\frac{x^2}{6} - \frac{x^4}{180} \right).$$

The derivative of $g(x)$ is

$$\begin{aligned} g'(x) &= -\frac{1}{x} + \frac{x}{3} + \frac{x^3}{45} + \frac{1}{\tan x} \\ &< -\frac{1}{x} + \frac{x}{3} + \frac{x^3}{45} + \frac{v(x,2)}{u(x,3)} \\ &= -\frac{1}{x} + \frac{x}{3} + \frac{x^3}{45} + \frac{1 - \frac{x^2}{2} + \frac{x^4}{24}}{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}} \\ &= -\frac{x^5(165 - 27x^2 + x^4)}{45(5040 - 840x^2 + 42x^4 - x^6)}. \end{aligned}$$

Since we have $165 - 27x^2 + x^4 > 0$ and $5040 - 840x^2 + 42x^4 - x^6 > 0$ for $0 < x < \frac{\pi}{2}$, $g(x)$ is monotonically decreasing for $0 < x < \frac{\pi}{2}$. From $\lim_{x \rightarrow 0} g(x) = 0$, we obtain $g(x) < 0$ for $0 < x < \frac{\pi}{2}$. \square

LEMMA 2.2. For $0 < x < \frac{\pi}{2}$, we have

$$\cos x < e^{-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520}}.$$

Proof. We set

$$f(x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \ln(\cos x).$$

The derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= -x - \frac{x^3}{3} - \frac{2x^5}{15} - \frac{17x^7}{315} + \tan x \\ &> -x - \frac{x^3}{3} - \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{u(x,3)}{v(x,4)} \\ &= -x - \frac{x^3}{3} - \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}} \\ &= \frac{x^9(277725 - 26313x^2 + 910x^4 - 17x^6)}{315(40320 - 20160x^2 + 1680x^4 - 56x^6 + x^8)} \\ &> \frac{x^9 \left(277725 - 26313 \left(\frac{158}{100}\right)^2 + 910x^4 - 17x^4 \left(\frac{158}{100}\right)^2 \right)}{315(40320 - 20160x^2 + 1680x^4 - 56x^6 + x^8)} \\ &= \frac{x^9 \left(\frac{530093067}{2500} + \frac{2168903x^4}{2500} \right)}{315(g(x) + x^4h(x))}, \end{aligned}$$

where $g(x) = 40320 - 20160x^2 + 1680x^4 \left(\frac{9213}{10000}\right)$ and $h(x) = 1680 \left(\frac{787}{10000}\right) - 56x^2 + x^4$.

Here, we set $t = x^2$, then for $0 < t < \frac{\pi^2}{4}$, we have

$$g(x) = 40320 - 20160t + 1680t^2 \left(\frac{9213}{10000}\right) = j(t)$$

and

$$h(x) = 1680 \left(\frac{787}{10000} \right) - 56t + t^2 = k(t).$$

From $\frac{\pi}{2} < \frac{31416}{20000}$ and $j(t)$ is a downward convex function that takes the minimum value at $t = \frac{20000}{3071} \cong 6.51254$, so we have

$$j(t) > j \left(\left(\frac{31416}{20000} \right)^2 \right) = \frac{311782948224993}{4882812500000000} > 0$$

for $0 < t < \frac{\pi^2}{4}$ and $k(t)$ is a downward convex function that takes the minimum value at $t = 28$, so we have

$$k(t) > k \left(\left(\frac{31416}{20000} \right)^2 \right) = \frac{5039738126241}{39062500000000} > 0$$

for $0 < t < \frac{\pi^2}{4}$. Therefore, we can get $g(x) > 0$ and $h(x) > 0$ for $0 < x < \frac{\pi}{2}$. From $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$, $f(x)$ is monotonically increasing for $0 < x < \frac{\pi}{2}$. By $\lim_{x \rightarrow 0} f(x) = 0$, we obtain $f(x) > 0$ for $0 < x < \frac{\pi}{2}$. \square

LEMMA 2.3. For $0 < x < \frac{\pi}{12}$, we have

$$\tan x < x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{18x^7}{315}.$$

Proof. We set

$$f(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{18x^7}{315} - \tan x.$$

Then we have

$$\begin{aligned} f(x) &> x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{18x^7}{315} - \frac{u(x, 2)}{v(x, 3)} \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{18x^7}{315} - \frac{x - \frac{x^3}{6} + \frac{x^5}{120}}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}} \\ &= \frac{x^7(225 - 1775x^2 + 166x^4 - 6x^6)}{105(720 - 360x^2 + 30x^4 - x^6)}. \end{aligned}$$

Since we have $225 - 1775x^2 + 166x^4 - 6x^6 > 0$ and $720 - 360x^2 + 30x^4 - x^6 > 0$ for $0 < x < \frac{\pi}{12}$, Therefore, $f(x) > 0$ for $0 < x < \frac{\pi}{12}$. \square

LEMMA 2.4. For $0 < x < \frac{\pi}{2}$, we have

$$\left(\frac{\sin x}{x} \right)^{2 + \frac{16}{15}x^2} + 1 + \frac{x^2}{3} + \frac{2x^4}{15} > 2.$$

Proof. First, we may show that

$$\left(\frac{\sin x}{x}\right)^{2+\frac{16}{15}x^2} > 1 - \frac{x^2}{3} - \frac{2x^4}{15}$$

for $0 < x < \frac{\sqrt{\sqrt{145}-5}}{2} \cong 1.3268$. We set

$$f(x) = \left(2 + \frac{16}{15}x^2\right) \ln\left(\frac{\sin x}{x}\right) - \ln\left(1 - \frac{x^2}{3} - \frac{2x^4}{15}\right)$$

and we have

$$f(x) > \left(2 + \frac{16}{15}x^2\right) \left(-\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{1890}\right) - \ln\left(1 - \frac{x^2}{3} - \frac{2x^4}{15}\right) = g(x)$$

from Lemma 2.1. The derivative of $g(x)$ is

$$g'(x) = \frac{2x^5(31770 + 11715x^2 + 754x^4 + 64x^6)}{14175(15 - 5x^2 - 2x^4)}.$$

Since we have

$$15 - 5x^2 - 2x^4 > 15 - 5\left(\frac{\sqrt{\sqrt{145}-5}}{2}\right)^2 - 2\left(\frac{\sqrt{\sqrt{145}-5}}{2}\right)^4 = 0,$$

$g'(x) > 0$ and $g(x)$ is monotonically increasing for $0 < x < \frac{\sqrt{\sqrt{145}-5}}{2}$. By $\lim_{x \rightarrow 0} g(x) = 0$, we obtain $g(x) > 0$ and $f(x) > 0$ for $0 < x < \frac{\sqrt{\sqrt{145}-5}}{2}$. Next, we consider the case of $\frac{\sqrt{\sqrt{145}-5}}{2} \leq x < \frac{\pi}{2}$. Then we have

$$1 + \frac{x^2}{3} + \frac{2x^4}{15} \geq 1 + \frac{1}{3}\left(\frac{\sqrt{\sqrt{145}-5}}{2}\right)^2 + \frac{2}{15}\left(\frac{\sqrt{\sqrt{145}-5}}{2}\right)^4 = 2$$

for $\frac{\sqrt{\sqrt{145}-5}}{2} \leq x < \frac{\pi}{2}$. Thus, we have

$$\left(\frac{\sin x}{x}\right)^{2+\frac{16}{15}x^2} + 1 + \frac{x^2}{3} + \frac{2x^4}{15} > 2$$

for $\frac{\sqrt{\sqrt{145}-5}}{2} \leq x < \frac{\pi}{2}$, immediately. \square

LEMMA 2.5. For $0 < x < \frac{\pi}{2}$, we have

$$\left(1 - \frac{x^2}{6}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-\frac{3}{10}x^2} > 2.$$

Proof. We set

$$f(x) = \left(1 - \frac{3}{10}x^2\right) \ln\left(\frac{\tan x}{x}\right) - \ln\left(2 - \left(1 - \frac{x^2}{6}\right)^2\right).$$

Then, we have

$$\begin{aligned} f(x) &> \left(1 - \frac{3}{10}x^2\right) \ln\left(\frac{u(x,1)}{xv(x,2)}\right) - \ln\left(2 - \left(1 - \frac{x^2}{6}\right)^2\right) \\ &= \left(1 - \frac{3}{10}x^2\right) \ln\left(\frac{1 - \frac{x^2}{6}}{1 - \frac{x^2}{2} + \frac{x^4}{24}}\right) - \ln\left(2 - \left(1 - \frac{x^2}{6}\right)^2\right) = g(x). \end{aligned}$$

The derivative of $g(x)$ is $g'(x) = xh(x)$, where

$$h(x) = \frac{3x^2(3552 - 2024x^2 + 396x^4 - 34x^6 + x^8)}{5(6 - x^2)(36 + 12x^2 - x^4)(24 - 12x^2 + x^4)} - \frac{3}{5} \ln\left(\frac{1 - \frac{x^2}{6}}{1 - \frac{x^2}{2} + \frac{x^4}{24}}\right).$$

The derivative of $h(x)$ is

$$h'(x) = \frac{6xj(x)}{5(6 - x^2)^2(36 + 12x^2 - x^4)^2(24 - 12x^2 + x^4)^2},$$

where $j(x) = 9455616 - 18745344x^2 + 14135040x^4 - 4811904x^6 + 781536x^8 - 39024x^{10} - 5472x^{12} + 912x^{14} - 50x^{16} + x^{18}$. We set $t = x^2$, then for $0 < t < \frac{\pi^2}{4} \cong 2.4674$, we have $j(x) = 9455616 - 18745344t + 14135040t^2 - 4811904t^3 + 781536t^4 - 39024t^5 - 5472t^6 + 912t^7 - 50t^8 + t^9 = k(t)$ and

$$\begin{aligned} k(t) &> 9455616 - 18745344t + 14135040t^2 - 4811904t^3 \\ &\quad + 781536t^4 - 39024t^5 - 5472t^5 \left(\frac{247}{100}\right) + 912t^7 - 50t^7 \left(\frac{247}{100}\right) \\ &= 9455616 - 18745344t + 14135040t^2 - 4811904t^3 \\ &\quad + 781536t^4 - \frac{1313496t^5}{25} + \frac{1577t^7}{2} \\ &> 9455616 - 18745344t + 14135040t^2 - 4811904t^3 \\ &\quad + 781536t^4 - \frac{1313496t^4}{25} \left(\frac{247}{100}\right) \\ &= 9455616 - 18745344t + 14135040t^2 - 4811904t^3 + \frac{407351622t^4}{625} = l(t). \end{aligned}$$

The derivatives of $l(t)$ are

$$l'(t) = -18745344 + 28270080t - 14435712t^2 + \frac{1629406488t^3}{625}$$

and

$$l''(t) = 28270080 - 28871424t + \frac{4888219464t^2}{625}.$$

Since $l''(t)$ is convex downward and has a minimum value at $t = \frac{125310000}{67891937} \cong 1.84573$,

$$l''(t) \geq l''\left(\frac{125310000}{67891937}\right) = \frac{110371419624960}{67891937} > 0$$

for $0 < t < \frac{\pi^2}{4}$. Therefore, $l'(t)$ is monotonically increasing for $0 < t < \frac{\pi^2}{4}$. From

$$l'\left(\frac{1508}{1000}\right) = -\frac{1792495709637}{1220703125} < 0$$

and

$$l'\left(\frac{1509}{1000}\right) = \frac{81774966025719}{78125000000} > 0,$$

there exists a unique $t = c$ with $\frac{1508}{1000} < c < \frac{1509}{1000}$ such that $l'(t) < 0$ for $0 < t < c$ and $l'(t) > 0$ for $c < t < \frac{\pi^2}{4}$. Thus, $l(t)$ is monotonically decreasing for $0 < t < \frac{1508}{1000}$ and monotonically increasing for $\frac{1509}{1000} < t < \frac{\pi^2}{4}$. From

$$l\left(\frac{1508}{1000}\right) = \frac{245018760694966851}{1220703125000} > 0$$

and

$$l\left(\frac{1509}{1000}\right) = \frac{62724736985762809971}{312500000000000} > 0,$$

$l(t) > 0$ for $0 < t < \frac{1508}{1000}$ or $\frac{1509}{1000} < t < \frac{\pi^2}{4}$. Also, we have

$$\begin{aligned} l(t) &> 9455616 - 18745344 \left(\frac{1509}{1000}\right) + 14135040 \left(\frac{1508}{1000}\right)^2 \\ &\quad - 4811904 \left(\frac{1509}{1000}\right)^3 + \frac{407351622}{625} \left(\frac{1508}{1000}\right)^4 \\ &= \frac{182036765041685601}{1220703125000} > 0 \end{aligned}$$

for $\frac{1508}{1000} < t < \frac{1509}{1000}$. Hence, we obtain $l(t) > 0$ for $0 < t < \frac{\pi^2}{4}$ and $h'(x) > 0$ for $0 < x < \frac{\pi}{2}$. Since $h(x)$ is monotonically increasing for $0 < x < \frac{\pi}{2}$ and $\lim_{x \rightarrow 0} h(x) = 0$, we obtain $h(x) > 0$ for $0 < x < \frac{\pi}{2}$ and $g(x)$ is monotonically increasing for $0 < x < \frac{\pi}{2}$. From $\lim_{x \rightarrow 0} g(x) = 0$, we have $g(x) > 0$ and $f(x) > 0$ for $0 < x < \frac{\pi}{2}$. \square

LEMMA 2.6. For $0 < x < \frac{\sqrt{30}}{4} \cong 1.36931$, we have

$$\left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)^2 + \left(\frac{\tan x}{x}\right)^{1 - \frac{8}{15}x^2} < 2.$$

Proof. We set

$$f(x) = \left(1 - \frac{8}{15}x^2\right) \ln\left(\frac{\tan x}{x}\right) - \ln\left(2 - \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)^2\right).$$

Then, we have

$$\begin{aligned} f(x) &< \left(1 - \frac{8}{15}x^2\right) \ln\left(\frac{u(x, 2)}{xv(x, 3)}\right) - \ln\left(2 - \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)^2\right) \\ &= \left(1 - \frac{8}{15}x^2\right) \ln\left(\frac{1 - \frac{x^2}{6} + \frac{x^4}{120}}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}}\right) - \ln\left(2 - \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)^2\right) = g(x). \end{aligned}$$

The derivative of $g(x)$ is

$$g'(x) = x \left(\frac{h(x)}{j(x)} - \frac{16}{15} \ln\left(\frac{1 - \frac{x^2}{6} + \frac{x^4}{120}}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}}\right) \right) = xk(x),$$

where $h(x) = 2x^2(3317760000 - 3094848000x^2 + 814464000x^4 - 117979200x^6 + 11136000x^8 - 697880x^{10} + 28720x^{12} - 715x^{14} + 8x^{16})$ and $j(x) = 15(120 - 20x^2 + x^4)(720 - 360x^2 + 30x^4 - x^6)(14400 + 4800x^2 - 640x^4 + 40x^6 - x^8)$. The derivative of $k(x)$ is

$$\begin{aligned} k'(x) &= \frac{4x^3}{15(120 - 20x^2 + x^4)^2(720 - 360x^2 + 30x^4 - x^6)^2} \\ &\times \frac{l(x)}{(14400 + 4800x^2 - 640x^4 + 40x^6 - x^8)^2}, \end{aligned}$$

where

$$\begin{aligned} l(x) &= -6875407319040000000 + 5980328755200000000x^2 \\ &\quad - 2181512134656000000x^4 + 464307572736000000x^6 \\ &\quad - 59979427430400000x^8 + 3111511449600000x^{10} \\ &\quad + 489129615360000x^{12} - 141331760640000x^{14} \\ &\quad + 18902087424000x^{16} - 1706401152000x^{18} + 113029081600x^{20} \\ &\quad - 5633473600x^{22} + 211092000x^{24} - 5816880x^{26} \\ &\quad + 112160x^{28} - 1365x^{30} + 8x^{32}. \end{aligned}$$

Here, we set $t = x^2$, then for $0 < t < \frac{15}{8}$, we have

$$\begin{aligned} l(x) = & -6875407319040000000 + 5980328755200000000t \\ & - 2181512134656000000t^2 + 464307572736000000t^3 - 59979427430400000t^4 \\ & + 3111511449600000t^5 + 489129615360000t^6 - 141331760640000t^7 \\ & + 18902087424000t^8 - 1706401152000t^9 + 113029081600t^{10} \\ & - 5633473600t^{11} + 211092000t^{12} - 5816880t^{13} \\ & + 112160t^{14} - 1365t^{15} + 8t^{16} = m(t). \end{aligned}$$

$$\begin{aligned} m(t) < & -6875407319040000000 + 5980328755200000000t \\ & - 2181512134656000000t^2 + 464307572736000000t^2 \left(\frac{15}{8}\right) \\ & - 59979427430400000t^4 + 3111511449600000t^4 \left(\frac{15}{8}\right) \\ & + 489129615360000t^4 \left(\frac{15}{8}\right)^2 - 141331760640000t^7 \\ & + 18902087424000t^7 \left(\frac{15}{8}\right) - 1706401152000t^9 + 113029081600t^9 \left(\frac{15}{8}\right) \\ & - 5633473600t^{11} + 211092000t^{11} \left(\frac{15}{8}\right) - 5816880t^{13} \\ & + 112160t^{13} \left(\frac{15}{8}\right) - 1365t^{15} + 8t^{15} \left(\frac{15}{8}\right) \\ = & -6875407319040000000 + 5980328755200000000t - 1310935435776000000t^2 \\ & - 52425747158400000t^4 - 105890346720000t^7 - 1494471624000t^9 \\ & - 5237676100t^{11} - 5606580t^{13} - 1350t^{15} \\ < & -6875407319040000000 + 5980328755200000000t - 1310935435776000000t^2 \\ = & n(t). \end{aligned}$$

Since $n(t)$ is upward convex and has a maximum value at $t = \frac{8011200}{3512237} \cong 2.28094$, so we have

$$n(t) < n\left(\frac{15}{8}\right) = -271048294440000000 < 0$$

for $0 < t < \frac{15}{8}$. Therefore, $k'(x) < 0$ for $0 < x < \frac{\sqrt{30}}{4}$ and $k(x)$ is monotonically decreasing for $0 < x < \frac{\sqrt{30}}{4}$. From $\lim_{x \rightarrow 0} k(x) = 0$, $g'(x) < 0$ for $0 < x < \frac{\sqrt{30}}{4}$ and $g(x)$ is monotonically decreasing for $0 < x < \frac{\sqrt{30}}{4}$. By $\lim_{x \rightarrow 0} g(x) = 0$, we have $f(x) < 0$ for $0 < x < \frac{\sqrt{30}}{4}$. \square

3. Proof of main Theorems

Proof of Theorem 1.1. From the arithmetic and geometric means inequality, we have

$$\left(\frac{\sin x}{x}\right)^{2+ax^2} + \left(\frac{\tan x}{x}\right)^{1-bx^2} > 2\sqrt{\left(\frac{\sin x}{x}\right)^{2+ax^2} \left(\frac{\tan x}{x}\right)^{1-bx^2}}$$

for $0 < x < \frac{\pi}{2}$ and $(a, b) \in D$. Thus, it is sufficient to show

$$\left(\frac{\sin x}{x}\right)^{2+ax^2} \left(\frac{\tan x}{x}\right)^{1-bx^2} > 1$$

for $0 < x < \frac{\pi}{2}$ and $(a, b) \in D$. We set

$$\begin{aligned} f(a, b, x) &= (2+ax^2) \ln\left(\frac{\sin x}{x}\right) + (1-bx^2) \ln\left(\frac{\tan x}{x}\right) \\ &= (3+(a-b)x^2) \ln\left(\frac{\sin x}{x}\right) - (1-bx^2) \ln(\cos x). \end{aligned}$$

The derivative of $f(a, b, x)$ for b is

$$\frac{\partial f}{\partial b}(a, b, x) = x^2 \ln\left(\frac{x}{\tan x}\right) < 0$$

for $0 < x < \frac{\pi}{2}$ and $(a, b) \in D$. Therefore, $f(a, b, x)$ is monotonically decreasing with respect to b and we have

$$\begin{aligned} f(a, b, x) &> f\left(a, \frac{1}{5} - \frac{a}{2}, x\right) \\ &= \left(3 - \frac{x^2}{5} + \frac{3ax^2}{2}\right) \ln\left(\frac{\sin x}{x}\right) - \left(1 - \frac{x^2}{5} + \frac{ax^2}{2}\right) \ln(\cos x) = g(a, x). \end{aligned}$$

The derivative of $g(a, x)$ for a is

$$\frac{\partial g}{\partial a}(a, x) = \frac{x^2}{2} \left(3 \ln\left(\frac{\sin x}{x}\right) - \ln(\cos x)\right).$$

Mitrinović [11] in p.236, presented that the inequality

$$1 \geq \cos \frac{x}{2} \geq \cos \alpha x \geq \frac{\sin x}{x} \geq \cos \beta x \geq \sqrt[3]{\cos x} \geq \frac{\cos x}{1 - \frac{x^2}{3}} \geq \cos x$$

holds for $0 < x < \frac{\pi}{2}$, where the constants $\alpha = \frac{2}{\pi} \arccos \frac{2}{\pi} \cong 0.560664$ and $\beta = \frac{1}{\sqrt{3}} \cong 0.57735$ are the best possible. Hence, we have

$$\left(\frac{\sin x}{x}\right)^3 \geq \cos x$$

for $0 < x < \frac{\pi}{2}$ and $\frac{\partial g}{\partial a}(a, x) > 0$ for $0 < x < \frac{\pi}{2}$ and $(a, b) \in D$. Since $g(a, x)$ is monotonically increasing with respect to a , we have

$$g(a, x) > g(0, x) = \left(3 - \frac{x^2}{5}\right) \ln\left(\frac{\sin x}{x}\right) - \left(1 - \frac{x^2}{5}\right) \ln(\cos x) = h(x).$$

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} h(x) &> \left(3 - \frac{x^2}{5}\right) \left(-\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{1890}\right) - \left(1 - \frac{x^2}{5}\right) \left(-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520}\right) \\ &= \frac{x^6(192 + 91x^2 - 51x^4)}{37800} > \frac{x^6\left(192 + 91x^2 - 51x^2\left(\frac{158}{100}\right)^2\right)}{37800} \\ &= \frac{x^6(480000 - 90791x^2)}{94500000} > \frac{x^6\left(480000 - 90791\left(\frac{158}{100}\right)^2\right)}{94500000} \\ &= \frac{633373369x^6}{236250000000} > 0. \end{aligned}$$

Hence, we obtain $f(a, b, x) > 0$ for $0 < x < \frac{\pi}{2}$ and $(a, b) \in D$. The proof of Theorem 1.1 is complete. \square

Proof of Theorem 1.2. We set

$$f(a, x) = \left(\frac{\sin x}{x}\right)^{2+ax^2} + \frac{\tan x}{x}.$$

The derivative of $f(a, x)$ for a is

$$\frac{\partial f}{\partial a}(a, x) = x^2 \left(\ln\left(\frac{\sin x}{x}\right)\right) \left(\frac{\sin x}{x}\right)^{2+ax^2} < 0$$

for $0 < x < \frac{\pi}{2}$ and $a > 0$. Therefore, $f(a, x)$ is monotonically decreasing with respect to a and we have

$$f(a, x) > f\left(\frac{16}{15}, x\right).$$

From Lemma 2.4 and $\tan x > x + \frac{x^3}{3} + \frac{2x^5}{15}$ for $0 < x < \frac{\pi}{2}$, we have

$$f\left(\frac{16}{15}, x\right) > \left(\frac{\sin x}{x}\right)^{2+\frac{16}{15}x^2} + 1 + \frac{x^2}{3} + \frac{2x^4}{15} > 2.$$

Therefore, we can get $f(a, x) > 2$ for $0 < x < \frac{\pi}{2}$ and $0 \leq a \leq \frac{16}{15}$. Next, we prove that the constant $\frac{16}{15}$ is the best possible constant. We consider the equation

$$\left(\frac{\sin x}{x}\right)^{2+g(x)x^2} + \frac{\tan x}{x} = 2$$

and we have

$$g(x) = -\frac{2}{x^2} + \frac{\ln\left(2 - \frac{\tan x}{x}\right)}{x^2 \ln\left(\frac{\sin x}{x}\right)} > \frac{16}{15}$$

for $0 < x < \frac{\pi}{2}$. By Lemmas 2.1 and 2.3, we have

$$\frac{1}{-\frac{x^2}{6} - \frac{x^4}{180}} < \frac{1}{\ln\left(\frac{\sin x}{x}\right)} < 0$$

and

$$\ln\left(1 - \frac{x^2}{3} - \frac{2x^4}{15} - \frac{18x^6}{315}\right) < \ln\left(2 - \frac{\tan x}{x}\right) < 0$$

for $0 < x < \frac{\pi}{12}$. Therefore, we have

$$\begin{aligned} g(x) &< -\frac{2}{x^2} + \frac{\ln\left(1 - \frac{x^2}{3} - \frac{2x^4}{15} - \frac{18x^6}{315}\right)}{x^2\left(-\frac{x^2}{6} - \frac{x^4}{180}\right)} \\ &= \frac{-2\left(-\frac{x^2}{6} - \frac{x^4}{180}\right) + \ln\left(1 - \frac{x^2}{3} - \frac{2x^4}{15} - \frac{18x^6}{315}\right)}{x^2\left(-\frac{x^2}{6} - \frac{x^4}{180}\right)} = \frac{h(x)}{j(x)} \end{aligned}$$

for $0 < x < \frac{\pi}{12}$. Since the derivative of $h(x)$ is

$$h'(x) = \frac{2x^3(1680 + 1055x^2 + 104x^4 + 6x^6)}{45(-105 + 35x^2 + 14x^4 + 6x^6)}$$

and the derivative of $j(x)$ is

$$j'(x) = -\frac{1}{30}x^3(20 + x^2),$$

We have

$$\frac{h'(x)}{j'(x)} = -\frac{4(1680 + 1055x^2 + 104x^4 + 6x^6)}{3(20 + x^2)(-105 + 35x^2 + 14x^4 + 6x^6)}.$$

From L'Hopital's rule, we obtain $\lim_{x \rightarrow 0} \frac{h(x)}{j(x)} = \lim_{x \rightarrow 0} \frac{h'(x)}{j'(x)} = \frac{4 \cdot 1680}{3 \cdot 20 \cdot 105} = \frac{16}{15}$ and $\lim_{x \rightarrow 0} g(x) \leq \frac{16}{15}$. By $g(x) > \frac{16}{15}$ for $0 < x < \frac{\pi}{2}$, we can get $\lim_{x \rightarrow 0} g(x) = \frac{16}{15}$. Therefore, the proof of Theorem 1.2 is complete. \square

Proof of Theorem 1.3. We set

$$f(b, x) = \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-bx^2}.$$

The derivative of $f(b, x)$ for b is

$$\frac{\partial f}{\partial b}(b, x) = -x^2 \left(\ln \left(\frac{\tan x}{x} \right) \right) \left(\frac{\tan x}{x} \right)^{1-bx^2} < 0$$

for $0 < x < \frac{\pi}{2}$ and $b > 0$. Therefore, $f(b, x)$ is monotonically decreasing with respect to b . Hence, for $0 < x < \frac{\pi}{2}$, we have $f(b, x) \geq f\left(\frac{3}{10}, x\right)$ for $0 \leq b \leq \frac{3}{10}$ and $f(b, x) \leq f\left(\frac{8}{15}, x\right)$ for $b \geq \frac{8}{15}$. From Lemma 2.5, we have

$$f\left(\frac{3}{10}, x\right) > \left(\frac{u(x, 1)}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-\frac{3}{10}x^2} = \left(1 - \frac{x^2}{6}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-\frac{3}{10}x^2} > 2.$$

Therefore, we can get $f(b, x) > 2$ for $0 < x < \frac{\pi}{2}$ and $0 \leq b \leq \frac{3}{10}$. Also, by Lemma 2.6, for $0 < x < \frac{\sqrt{30}}{4} \cong 1.36931$, we have

$$\begin{aligned} f\left(\frac{8}{15}, x\right) &< \left(\frac{u(x, 2)}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-\frac{8}{15}x^2} \\ &= \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-\frac{8}{15}x^2} < 2. \end{aligned}$$

From $0 < \left(\frac{\tan x}{x}\right)^{1-\frac{8}{15}x^2} < 1$ for $\frac{\sqrt{30}}{4} < x < \frac{\pi}{2}$, we have $f\left(\frac{8}{15}, x\right) < 2$ for $0 < x < \frac{\pi}{2}$. Thus, we can get $f(b, x) < 2$ for $0 < x < \frac{\pi}{2}$ and $b \geq \frac{8}{15}$. Next, we prove that the constant $\frac{8}{15}$ is the best possible constant. We consider the equation

$$\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^{1-g(x)x^2} = 2$$

and we have

$$g(x) = \frac{1}{x^2} - \frac{\ln\left(2 - \left(\frac{\sin x}{x}\right)^2\right)}{x^2 \ln\left(\frac{\tan x}{x}\right)} < \frac{8}{15}$$

for $0 < x < \frac{\pi}{2}$. By $\tan x > x + \frac{x^3}{3} + \frac{2x^5}{15}$, we have

$$1 < 2 - \left(\frac{\sin x}{x}\right)^2 < 2 - \left(\frac{u(x, 3)}{x}\right)^2$$

and

$$0 < \ln\left(1 + \frac{x^2}{3} + \frac{2x^4}{15}\right) < \ln\left(\frac{\tan x}{x}\right)$$

for $0 < x < \frac{\pi}{2}$. Therefore, we have

$$\begin{aligned}
 g(x) &> \frac{1}{x^2} - \frac{\ln\left(2 - \left(\frac{u(x,3)}{x}\right)^2\right)}{x^2 \ln\left(1 + \frac{x^2}{3} + \frac{2x^4}{15}\right)} \\
 &= \frac{1}{x^2} - \frac{\ln\left(2 - \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right)^2\right)}{x^2 \ln\left(1 + \frac{x^2}{3} + \frac{2x^4}{15}\right)} = h(x)
 \end{aligned}$$

for $0 < x < \frac{\pi}{2}$. Here, we set $t = x^2$, then for $0 < t < \frac{\pi^2}{4}$, we have

$$\begin{aligned}
 h(x) &= \frac{1}{t} - \frac{\ln\left(2 - \left(1 - \frac{t}{6} + \frac{t^2}{120} - \frac{t^3}{5040}\right)^2\right)}{t \ln\left(1 + \frac{t}{3} + \frac{2t^2}{15}\right)} \\
 &= \frac{\ln\left(1 + \frac{t}{3} + \frac{2t^2}{15}\right) - \ln\left(2 - \left(1 - \frac{t}{6} + \frac{t^2}{120} - \frac{t^3}{5040}\right)^2\right)}{t \ln\left(1 + \frac{t}{3} + \frac{2t^2}{15}\right)} = \frac{k(t)}{j(t)}.
 \end{aligned}$$

The derivatives of $k(t)$ are

$$\begin{aligned}
 k'(t) &= \frac{t}{15 + 5t + 2t^2} \\
 &\times \frac{135475200 + 18950400t - 599760t^2 - 115920t^3 + 12186t^4 - 479t^5 + 8t^6}{25401600 + 8467200t - 1128960t^2 + 80640t^3 - 3444t^4 + 84t^5 - t^6}
 \end{aligned}$$

and

$$\begin{aligned}
 k''(t) &= \frac{1}{(15 + 5t + 2t^2)^2} \\
 &\times \frac{l(t)}{(25401600 + 8467200t - 1128960t^2 + 80640t^3 - 3444t^4 + 84t^5 - t^6)^2},
 \end{aligned}$$

where $l(t) = 51619302604800000 + 14441114419200000t - 6195724577280000t^2 - 3868033720320000t^3 + 427111392960000t^4 - 12876325056000t^5 - 4263601406400t^6 + 704056147200t^7 - 46511466120t^8 + 1341716040t^9 + 14327718t^{10} - 2574876t^{11} + 99329t^{12} - 1916t^{13} + 16t^{14}$. The derivatives of $j(t)$ are

$$j'(t) = \frac{t(5 + 4t)}{15 + 5t + 2t^2} + \ln\left(1 + \frac{t}{3} + \frac{2t^2}{15}\right)$$

and

$$j''(t) = \frac{150 + 205t + 40t^2 + 8t^3}{(15 + 5t + 2t^2)^2}.$$

Thus, we have

$$\frac{k''(t)}{j''(t)} = \frac{1}{150 + 205t + 40t^2 + 8t^3} \times \frac{l(t)}{(25401600 + 8467200t - 1128960t^2 + 80640t^3 - 3444t^4 + 84t^5 - t^6)^2}$$

From L'Hopital's rule, we obtain

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{k(t)}{j(t)} &= \lim_{t \rightarrow 0} \frac{k'(t)}{j'(t)} = \lim_{t \rightarrow 0} \frac{k''(t)}{j''(t)} \\ &= \frac{51619302604800000}{150 \cdot 25401600^2} = \frac{8}{15} \end{aligned}$$

and $\lim_{x \rightarrow 0} g(x) \geq \frac{8}{15}$. By $g(x) < \frac{8}{15}$ for $0 < x < \frac{\pi}{2}$, we can get $\lim_{x \rightarrow 0} g(x) = \frac{8}{15}$. The proof of Theorem 1.3 is complete. \square

4. Open problems

Finally, we present the following two open problems related to Theorem 1.1.

OPEN PROBLEM 4.1. For $0 < x < \frac{1564}{1000}$, we have

$$\left(\frac{\sin x}{x}\right)^{2 + \frac{472}{1365}x^2} + \left(\frac{\tan x}{x}\right)^{1 - \frac{164}{455}x^2} < 2.$$

OPEN PROBLEM 4.2. For $0 < x < \frac{\pi}{2}$, we have

$$\begin{aligned} &\left(\frac{\sin x}{x}\right)^{2 + \frac{472}{1365}x^2} + \left(\frac{\tan x}{x}\right)^{1 - \frac{164}{455}x^2} \\ &> 2 - \frac{221404}{50307075}x^8 - \frac{7904408}{8300667375}x^{10} \\ &\quad - \frac{42997546}{871570074375}x^{12} + \frac{151849988}{33991232900625}x^{14}. \end{aligned}$$

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