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STABLE SEMINORMS REVISITED

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Abstract. A seminorm S on an algebra $\mathscr{A}$ is called *stable* if for some constant $\sigma > 0$,

$$S(x^k) \leqslant \sigma S(x)^k \quad \text{for all} \quad x \in \mathscr{A} \quad \text{and all} \quad k = 1, 2, 3, \ldots.$$

We call S *strongly stable* if the above holds with $\sigma = 1$. In this note we use several known and new results to shed light on the concepts of stability. In particular, the interrelation between stability and similar ideas is discussed.

Let $\mathscr{A}$ be an associative algebra over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$. As usual, a function

$$S : \mathscr{A} \to \mathbb{R}$$

is called a *seminorm* if for all $x, y \in \mathscr{A}$ and $\alpha \in \mathbb{F}$:

$$\begin{aligned} &S(x) \geqslant 0, \\ &S(\alpha x) = |\alpha| S(x), \\ &S(x+y) \leqslant S(x) + S(y). \end{aligned}$$

If in addition,

$$S(x) \neq 0 \quad \text{for all} \quad x \neq 0,$$

then S is a *norm*. Finally, we call a seminorm S *proper* if S does not vanish identically and $S(x) = 0$ for some $x \neq 0$.

In this note we deal mainly with *strongly stable* seminorms, namely those satisfying

$$S(x^k) \leqslant S(x)^k \quad \text{for all} \quad x \in \mathscr{A} \quad \text{and all} \quad k = 1, 2, 3, \ldots,$$

or in other words, the ones for which

$$S(x) \leqslant 1 \quad \text{implies} \quad S(x^k) \leqslant 1, \quad x \in \mathscr{A}, \quad k = 1, 2, 3, \ldots.$$

Since S being *submultiplicative* means

$$S(xy) \leqslant S(x)S(y) \quad \text{for all} \quad x, y \in \mathscr{A},$$

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we see that submultiplicativity implies strong stability. The converse is usually false. One of the first examples that come to mind is the well known *numerical radius*,

$$r(T) = \sup\{|(Tx,x)| : x \in \mathbf{H},\ (x,x) = 1\}, \tag{1}$$

defined on $\mathscr{B}(\mathbf{H})$, the algebra of bounded linear operators on a Hilbert space $\mathbf{H}$ over $\mathbb{C}$. Whereas r is a non-submultiplicative norm on $\mathscr{B}(\mathbf{H})$, it is strongly stable, since by Berger's celebrated inequality [P, H, GT]:

$$r(T^k) \leqslant r(T)^k, \quad T \in \mathscr{B}(\mathbf{H}), \quad k = 1,2,3,\ldots. \tag{2}$$

While often S will not be strongly stable, it may have weaker, yet related properties. For instance, we call S *quadrative* if

$$S(x^2) \leqslant S(x)^2 \quad \text{for all} \quad x \in \mathscr{A}.$$

As multiplicativity implies strong stability, which in turn implies quadrativity, we consult Theorems 1.4 in [AGL1] to obtain:

THEOREM 1. *Let $\mathscr{A}$ be an algebra of $\mathbb{F}$-valued functions*

$$f : \mathbf{T} \to \mathbb{F}$$

defined on a given nonempty set $\mathbf{T}$, with the usual pointwise multiplication

$$(fg)(t) = f(t)g(t), \quad f,g \in \mathscr{A},\ t \in \mathbf{T}. \tag{3}$$

Suppose $\mathscr{A}$ is closed under absolute values, that is,

$$f \in \mathscr{A} \quad \textit{implies} \quad |f| \in \mathscr{A};$$

and let S be monotonic, i.e.,

$$|f| \leqslant |g| \quad \textit{implies} \quad S(f) \leqslant S(g).$$

Then the following are equivalent:
(*a*) *S is submultiplicative.*
(*b*) *S is strongly stable.*
(*c*) *S is quadrative.*

Theorem 1 applies, of course, to algebras of bounded functions with seminorms of the form

$$S_c(f) = \sup_{t \in \mathbf{T}} |c(t)f(t)|, \tag{4}$$

$0 \neq c \in \mathscr{A}$ being a fixed element. Clearly, S_c is a norm if and only if c is not a zero-divisor in $\mathscr{A}$. Otherwise, S_c is a proper seminorm. Such monotonic seminorms were studied intensively in [AG1, AG2]. Appealing, for example, to Theorem 3.1 in [AG2], one can obtain:

THEOREM 2. *Let* $\mathscr{A}$ *be as in Theorem* 1, *with the seminorm in* (4). *Then the following are equivalent:*

(a) S_c *is submultiplicative.*
(b) S_c *is strongly stable.*
(c) S_c *is quadrative.*
(d) $\inf\{|c(t)| : t \in \mathbf{T},\ c(t) \neq 0\} \geqslant 1$.

Similar observations can be obtained at once from Theorems 3.2 and 3.3 in [AG2].

To illustrate Theorem 2, we consider l^∞, the algebra of bounded sequences $a = \{\alpha_i\}_{i=1}^\infty$ over $\mathbb{F}$, with the usual Hadamard multiplication,

$$ab = \{\alpha_i\beta_i\}, \quad a = \{\alpha_i\},\ b = \{\beta_i\} \in l^\infty. \tag{5}$$

Identifying l^∞ with the algebra of bounded functions on

$$\mathbf{T} = \mathbb{Z}^+ \equiv \{1, 2, 3, \ldots\}, \tag{6}$$

we fix an element $c = \{\gamma_i\}_{i=1}^\infty \in l^\infty$, $c \neq 0$, and define the seminorm

$$S_c(a) = \sup_i |\gamma_i\alpha_i|, \quad a \in l^\infty. \tag{7}$$

Obviously, S_c is a norm on l^∞ if and only if

$$\gamma_i \neq 0, \quad i = 1, 2, 3, \ldots.$$

Otherwise S_c is a proper seminorm.

By the theorem, S_c is strongly stable (in fact, multiplicative) if and only if

$$\inf_{\gamma_i \neq 0} |\gamma_i| \geqslant 1.$$

Hence, the four selections,

$$\begin{array}{lll} \gamma_i = 1, & i = 1, 2, 3, \ldots, & \\ \gamma_i = i^{-1}, & i = 1, 2, 3, \ldots, & \\ \gamma_1 = 0; & \gamma_i = 1, & i = 2, 3, 4, \ldots, \\ \gamma_1 = 0; & \gamma_i = i^{-1}, & i = 2, 3, 4, \ldots, \end{array}$$

show that norms and proper seminorms of the form (7) may or may not be strongly stable on l^∞.

A more complex situation arises when we consider a (monotonic) function norm η on $\mathscr{M} = \mathscr{M}(\mathbf{T}, \Omega, \mu)$, the algebra of measurable functions on a nonempty set $\mathbf{T}$, where Ω is a σ-algebra of subsets of $\mathbf{T}$, and μ is a countably additive, nonnegative measure. In [AGL1] we discussed $L_\eta = L_\eta(\mathbf{T}, \Omega, \mu)$, the space of all functions

$$\{f \in \mathscr{M} : \eta(f) < \infty\}$$

(modulo the null functions). η is surely a norm on this space, and without going into the definitions, we quote:

THEOREM 3. [AGL1, Theorem 2.5]. *If η is σ-subadditive, then L_η is an algebra (i.e., closed under multiplication) if and only if it is contained in $L^\infty = L^\infty(\mathbf{T},\Omega,\mu)$, the algebra of all μ-essentially bounded functions in $\mathscr{M}$.*

Another result can be readily deduced from Theorem 2.9 of [AGL1]:

THEOREM 4. *Let $(\mathbf{T},\Omega,\mu)$ be free of infinite atoms, and let η be a σ-subadditive, saturated function norm on $\mathscr{M}$. If L_η is an algebra, then the following are equivalent on L_η:*
(a) *ρ is submultiplicative.*
(b) *ρ is strongly stable.*
(c) *ρ is quadrative.*
(d) $\sup\{\|f\|_\infty : f \in L_\eta,\ \eta(f) \leqslant 1\} \leqslant 1$.

The results in Theorems 3 and 4 were applied in [AGL2] to Orlicz space function norms, a case that includes all L^p spaces. In particular, we showed that if μ is the Lebesgue or any other nonatomic measure, then for $1 \leqslant p < \infty$, the corresponding L^p space is not an algebra; so the question of strong stability is irrelevant. On the other hand, if $\mathbf{T} = \mathbb{Z}^+$ is the set in (6), and μ is the *counting measure* assigning to each subset of $\mathbb{Z}^+$ its cardinality, then l^p, the corresponding space of all bounded sequences $a = \{\alpha_i\}_{i=1}^\infty$ satisfying

$$\|a\|_p = \Big(\sum_{i=1}^\infty |\alpha_i|^p\Big)^{1/p} < \infty, \tag{8}$$

is an algebra with respect to the pointwise multiplication in (5), and the norm in (8) is multiplicative, hence strongly stable.

Theorem 2 does not apply to monotonic seminorms on $\mathbb{F}^{n\times n}$, the algebra of $n\times n$ matrices over $\mathbb{F}$ with respect to usual matrix multiplication. The reason is that while any $A \in \mathbb{F}^{n\times n}$ can be viewed as a function on the set of pairs of integers

$$\mathbf{T}_{n\times n} = \{(i,j) : i,j = 1,\ldots,n\},$$

the standard matrix multiplication

$$A,B \mapsto AB, \quad (AB)_{ij} = \sum_{s=1}^n \alpha_{is}\beta_{sj}, \quad A = (\alpha_{ij}),\ B = (\beta_{ij}) \in \mathbb{F}^{n\times n}, \tag{9}$$

does not agree with the pointwise manner in (3).

In order to accomodate this case, let $W = (\omega_{ij})$ be a fixed $n\times n$ matrix of positive entries, and consider the *W-weighted l^∞* norm on $\mathbb{C}^{n\times n}$,

$$\|A\|_{W,\infty} = \max_{i,j} \omega_{ij}|\alpha_{ij}|, \quad A = (\alpha_{ij}) \in \mathbb{C}^{n\times n}.$$

For this norm we can prove,

THEOREM 5. [AG3, Theorem 1]. *The following are equivalent on $\mathbb{C}^{n\times n}$:*
(a) *$\|\cdot\|_{W,\infty}$ is submultiplicative.*
(b) *$\|\cdot\|_{W,\infty}$ is strongly stable.*

(c) $\|\cdot\|_{W,\infty}$ *is quadrative.*
(d) $(W_{-1})^2 \leqslant W_{-1}$,

where W_{-1}, *the Hadamard inverse of* $W = (\omega_{ij})$, *is the matrix of reciprocals defined by*

$$W_{-1} = (\omega_{ij}^{-1}),$$

and the inequality is construed entrywise.

We remark that Theorem 5 does not hold for all weighted l^p norms, $1 \leqslant p \leqslant \infty$, on $\mathbb{C}^{n\times n}$. For example, it was shown in [AG5] that for certain weight matrices W, the weighted l^1 norms

$$\|A\|_{W,1} = \sum_{i,j}^{n} \omega_{ij}|\alpha_{ij}|, \quad A = (\alpha_{ij}) \in \mathbb{C}^{n\times n},$$

are quadrative but not submultiplicative. Whether or not the strong stability of $\|\cdot\|_{W,1}$ on $\mathbb{C}^{n\times n}$ is equivalent to either quadrativity or to submultiplicativity, is unknown to us.

Returning to an arbitrary algebra $\mathscr{A}$ of $\mathbb{F}$-valued functions defined on a set $\mathbf{T}$, we adapt (2.6) in [AG4], and say that $\mathscr{A}$ is *homotonic* if

$$|f_1| \leqslant g_1,\ |f_2| \leqslant g_2 \quad \text{implies} \quad |f_1 f_2| \leqslant g_1 g_2, \quad f_1, f_2, g_1, g_2 \in \mathscr{A},$$

where for $h_1, h_2 \in \mathscr{A}$,

$$h_1 \leqslant h_2$$

means, of course,

$$h_1(t) \leqslant h_2(t), \quad t \in \mathbf{T}.$$

Consider the weighted sup norm

$$\|f\|_{w,\infty} = \sup_{t\in\mathbf{T}} w(t)|f(t)|, \quad f \in \mathscr{A}, \tag{10}$$

where w, the *weight function*, is a fixed positive function on $\mathbf{T}$, bounded away from zero. Let w^{-1} be the (positive) reciprocal function of w,

$$w^{-1}(t) \equiv w(t)^{-1}, \quad t \in \mathbf{T},$$

and assume that $w^{-1} \in \mathscr{A}$. Then in analogy with Theorem 5, we can prove,

THEOREM 6. [AG4, Theorem 4.2]. *If* $\mathscr{A}$ *is a homotonic algebra of* $\mathbb{F}$*-valued functions, and* $\|\cdot\|_{w,\infty}$ *is the norm in* (10)*, then the following are equivalent:*
(a) $\|\cdot\|_{w,\infty}$ *is submultiplicative.*
(b) $\|\cdot\|_{w,\infty}$ *is strongly stable.*
(c) $\|\cdot\|_{w,\infty}$ *is quadrative.*
(d) $w^{-2} \leqslant w^{-1}$, *where* $w^{-2} = (w^{-1})^2 \equiv w^{-1}w^{-1}$.

As it is not hard to verify that $\mathbb{F}^{n\times n}$ with the standard matrix multiplication in (9) is a homotonic algebra, we see that Theorem 5 is but a special case of Theorem 6.

Given an integer $k \geqslant 2$ we shall follow [AG3] and call a seminorm S on an algebra $\mathscr{A}$ *k-bounded* if

$$S(x^k) \leqslant S(x)^k \quad \text{for all} \quad x \in \mathscr{A}. \tag{11}$$

Hence, S is quadrative if it is 2-bounded, and strongly stable if it is k-bounded for all $k = 2, 3, 4, \ldots$.

Boundedness for a particular k larger than 2 does not usually ensure strong stability, not even quadrativity. To substantiate this statement we quote, for example,

THEOREM 7. [AG3, Theorem 2]. *If $k \geqslant 3$, then there exists a weight matrix W for which $\|\cdot\|_{W,\infty}$ is k-bounded but not strongly stable, not even quadrative on $\mathbb{C}^{n\times n}$.*

Whether, in general, quadrativity implies strong stability, we do not know.

Relaxing our definition of strong stability, we shall say that a seminorm S is *stable* on an algebra $\mathscr{A}$ if for some positive constant σ,

$$S(x^k) \leqslant \sigma S(x)^k \quad \text{for all} \quad x \in \mathscr{A}, \quad k = 1, 2, 3, \ldots. \tag{12}$$

With this definition we can improve our observation in Theorem 7 as far as norms on $\mathbb{C}^{n\times n}$ are concerned:

THEOREM 8. *If $k \geqslant 2$ and N is a k-bounded norm on $\mathbb{C}^{n\times n}$, then N is stable.*

Proof. By hypothesis, for any $A \in \mathbb{C}^{n\times n}$,

$$N\left(A^{k^j}\right) \leqslant N(A)^{k^j}, \quad j = 1, 2, 3, \ldots.$$

Thus,

$$N(A) \geqslant N\left(A^{k^j}\right)^{1/k^j} @>> j \to \infty > \rho(A),$$

where $\rho(A)$ denotes the spectral radius of A. Consequently, N is *spectrally dominant*, i.e.,

$$N(A) \geqslant \rho(A), \quad A \in \mathbb{C}^{n\times n};$$

hence N is stable by the renowned Friedland-Zenger Theorem [FZ, Theorem 1]. □

Contrary to what one might expect, the converse of Theorem 8 is false; that is, *stability of matrix norms does not imply k-boundedness for any $k \geqslant 2$.* This was demonstrated in [AG3] by the action of $\|\cdot\|_{W,\infty}$ on $\mathbb{C}^{2\times 2}$, where W was the 2×2 weight matrix

$$W = \begin{pmatrix} \dfrac{2}{1+\theta} & \dfrac{2}{1-\theta} \\ \dfrac{2}{1-\theta} & \dfrac{2}{1+\theta} \end{pmatrix}, \quad \theta = \text{constant}, \quad 0 < \theta < 1. \tag{13}$$

The message conveyed by the above example is not limited to finite-dimensional algebras. In what follows we exhibit a norm on an arbitrary complex Hilbert space, which is stable but not strongly stable, not even quadrative.

Indeed, let $\mathbf{H}$ be any Hilbert space over $\mathbb{C}$, with $\dim \mathbf{H} \geqslant 2$, and consider the *generalized numerical radius*

$$r_g(T) = \sup\left\{|2(Tx, x) + (Ty, y)| : x, y \in \mathbf{H}; x, y \text{ orthonormal}\right\}, \quad T \in \mathscr{B}(\mathbf{H}).$$

Since

$$r_g(T) \geqslant 2\sup\left\{|(Tx, x) : (x, x) = 1\right\} - \sup\left\{|(Ty, y)| : (y, y) = 1\right\} = r(T)$$

and

$$r_g(T) \leqslant 2\sup\big\{|(Tx,x) : (x,x)=1\big\} + \sup\{|(Ty,y)| : (y,y)=1\big\} = 3r(T),$$

we get

$$r(T) \leqslant r_g(T) \leqslant 3r(T), \quad T \in \mathscr{B}(\mathbf{H}), \tag{14}$$

where r is the classical numerical radius in (1). Consequently, r_g is a well defined, positive definite function on $\mathscr{B}(\mathbf{H})$, and as it is moreover homogeneous and subadditive, it constitutes a norm on $\mathscr{B}(\mathbf{H})$.

Now, let S_1 and S_2 be seminorms on an arbitrary algebra $\mathscr{A}$, such that for constants $\nu_1 \geqslant \nu_2 > 0$, $\tau > 0$, and a fixed integer k,

$$\nu_1 S_1(x) \leqslant S_2(x) \leqslant \nu_2 S_1(x), \quad x \in \mathscr{A},$$

and

$$S_1(x^k) \leqslant \tau S_1(x)^k, \quad x \in \mathscr{A}.$$

It follows that

$$S_2(x^k) \leqslant \nu_2 S_1(x^k) \leqslant \nu_2 \tau S_1(x)^k \leqslant \frac{\nu_2\tau}{\nu_1^k} S_2(x)^k \quad \text{for all} \quad x \in \mathscr{A}; \tag{15}$$

so having (2) and (14), we apply (15) to $S_1 = r$ and $S_2 = r_g$ and get

$$r_g(T^k) \leqslant 3r_g(T)^k \quad \text{for all} \quad T \in \mathscr{B}(\mathbf{H}), \quad k = 1,2,3,\ldots,$$

showing that r_g is stable.

To prove that r_g is not strongly stable, fix linearly independent vectors a, $b \in \mathbf{H}$, and let $\mathbf{H}_2$ be the subspace of $\mathbf{H}$ spanned by these two vectors, so that

$$\mathbf{H} = \mathbf{H}_2 \oplus \mathbf{H}_2^{\perp}.$$

Let

$$Q \in \mathscr{B}(\mathbf{H})$$

be the linear operator defined on $\mathbf{H}$ by

$$Q(\alpha a + \beta b + x) = \alpha a - \beta b \quad \text{for all} \quad \alpha, \beta \in \mathbb{C}, \quad x \in \mathbf{H}_2^{\perp}.$$

Since

$$Q^2 = \begin{cases} I & \text{on } \mathbf{H}_2, \\ 0 & \text{on } \mathbf{H}_2^{\perp}, \end{cases}$$

we can write,

$$r_g(Q^2) = \sup\big\{|2(Ix,x) + (Iy,y)| : x,y \in \mathbf{H}_2; x,y \text{ orthonormal}\big\} = 3. \tag{16}$$

Similarly,

$$r_g(Q) = \sup\big\{|2(Qx,x) + (Qy,y)| : x,y \in \mathbf{H}_2; x,y \text{ orthonormal}\big\};$$

hence given the basis $\{a,b\}$, we may represent Q on $\mathbf{H}_2$ by the 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and write

$$r_g(Q) = \sup\{|2(Bx,x) + (By,y)| : x,y \in \mathbb{C}^2; x,y \text{ orthonormal}\},$$

where here, $(x,y) \equiv y^*x$ is the standard inner product on $\mathbb{C}^2$ (* denoting the conjugate transpose). We obtain,

$$\begin{aligned} r_g(Q) &= \max\{|2x^*Bx + y^*By| : x,y \in \mathbb{C}^2; x,y \text{ orthonormal}\} \\ &= \max\left\{\left|\operatorname{tr}\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} U^*BU\right]\right| : U \in \mathbb{C}^{2\times 2} \text{ unitary}\right\} \\ &= \max\left\{\left|\operatorname{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} U^*BU\right] + \operatorname{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^*BU\right]\right| : U \in \mathbb{C}^{2\times 2} \text{ unitary}\right\} \\ &= \max\{|\operatorname{tr} B + x^*Bx| : x \in \mathbb{C}^2, x^*x = 1\} \\ &= \max\{|x^*Bx| : x \in \mathbb{C}^2, x^*x = 1\} = r(B). \end{aligned}$$

As B is a normal matrix, we have $r(B) = \rho(B) = 1$ (e.g., [GT]). Thus,

$$r_g(Q) = 1, \tag{17}$$

so by (16) and (17),

$$r_g(Q^2) = 3r_g(Q)^2,$$

implying that r_g is not strongly stable on $\mathbf{H}$, not even quadrative.

Recalling the definition in (11), we follow Theorem 1.2 in [AG2] and prove,

THEOREM 9. *If S is a proper seminorm on an algebra $\mathscr{A}$, then it is k-bounded if and only if $\mathscr{K}$, the kernel of S, is closed under raising to the k-th power (i.e., $x \in \mathscr{K}$ implies $x^k \in \mathscr{K}$) and*

$$\lambda_k \equiv \sup\{S(x^k) : x \in \mathscr{A}, S(x) = 1\} \leqslant 1. \tag{18}$$

Proof. Let S be k-bounded. Then for any $x \in \mathscr{K}$,

$$S(x^k) \leqslant S(x^k) = 0;$$

hence $x^k \in \mathscr{K}$. So if $\mathscr{K}$ is not closed under the k-th power, S is not k-bounded.

Similarly, if $\lambda_k > 1$, then there exists an element $x_0 \in \mathscr{A}$ with $S(x_0) = 1$ such that

$$S(x_0^k) > S(x_0)^k;$$

and again, S is not k-bounded.

Conversely, suppose $\mathscr{K}$ is closed under raising to the k-th power and $\lambda_k \leqslant 1$. By the first assumption,

$$S(x^k) = 0 = S(x)^k, \ x \in \mathscr{K}. \tag{19}$$

By the second, since

$$\lambda_k = \sup\left\{\frac{S(x^k)}{S(x)^k} : x \in \mathscr{A}, x \notin \mathscr{K}\right\},$$

we get

$$S(x^k) \leqslant \lambda_k S(x)^k \leqslant S(x)^k, \quad x \notin \mathscr{K},$$

which together with (19) implies (11). $\square$

Note that *if* $\mathscr{K} \equiv \ker S$ *is closed under squaring, then* $\mathscr{K}$ *is closed under all natural powers,* $k = 2, 3, 4, \ldots$.

Indeed, $\mathscr{K}$ is clearly a subspace of $\mathscr{A}$. So if $x, y \in \mathscr{K}$ are commuting elements and $\mathscr{K}$ is closed under squaring, then

$$xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2] \in \mathscr{K}.$$

Thus

$$x \in \mathscr{K} \text{ implies } x^k \in \mathscr{K},\ k = 2, 3, 4, \ldots,$$

and the assertion follows.

Combining this assertion with Theorem 9, we get:

THEOREM 10. *Let* S *be a proper seminorm on* $\mathscr{A}$*, and let* $\mathscr{K} \equiv \ker S$ *be closed under squaring. Then:*

(*a*) S *is* k*-bounded if and only if* (18) *holds.*

(*b*) S *is stable if and only if the sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *is bounded, and strongly stable if and only if the bound is* 1.

If S is proper and λ_k is finite for some $k \geqslant 1$, then obviously, λ_k is the best k*-factor* for S, i.e., the least constant λ satisfying

$$S(x^k) \leqslant \lambda S(x)^k, \quad x \in \mathscr{A}.$$

In the same way, if S is proper and

$$\sigma_{\inf} \equiv \sup_{k \geqslant 1} \lambda_k < \infty,$$

then $\sigma_{\inf}$ is the best *stability factor* for S, that is, the least constant σ for which (12) holds.

For example, it was shown in [AG3] that the best k-factors for $\|\cdot\|_{W,\infty}$ on $\mathbb{C}^{2\times 2}$ with the weight matrix in (13) are

$$\lambda_k = \frac{1-\theta^k}{1-\theta}, \quad k = 1, 2, 3, \ldots;$$

hence the best stability factor in this case is

$$\sigma_{\inf} = \frac{1}{1-\theta}.$$

We conclude by noting that *if* S *is proper and* $\lambda_k < \infty$ *for some* $k \geqslant 2$*, then all sufficiently large multiples of* S *will always be* k*-bounded*. More precisely, *if* λ *is a positive constant such that*

$$\lambda \geqslant (\lambda_k)^{1/(k-1)},$$

then $S_\lambda \equiv \lambda S$ *will be a* k*-bounded seminorm on* $\mathscr{A}$. For then,

$$S_\lambda(x^k) = \lambda S(x^k) \leqslant \lambda \cdot \lambda_k S(x)^k \leqslant \lambda^k S(x)^k = S_\lambda(x)^k, \quad x \in \mathscr{A}.$$

Similarly, *if* S *is proper with* $\sigma_{\inf} < \infty$, *and* σ *is a constant satisfying* $\sigma \geqslant \sigma_{\inf}$, *then* $S_\sigma \equiv \sigma S$ *is strongly stable*. This is so because $\sigma_{\inf} \geqslant \lambda_1 = 1$; hence for all $x \in \mathscr{A}$ and $k = 2, 3, 4, \ldots$,

$$S_\sigma(x^k) = \sigma S(x^k) \leqslant \sigma \cdot \sigma_{\inf} S(x)^k \leqslant \sigma^2 S(x)^k \leqslant \sigma^k S(x)^k = S_\sigma(x)^k.$$

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