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ON EQUIVALENCE OF COEFFICIENT CONDITIONS AND APPLICATION

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(*communicated by J. Pečarić*)

Abstract. We show that the simplest coefficient condition

$$\sum_{n=1}^{\infty} |c_n|^q \gamma_n \mu_n < \infty,$$

under specific assumptions on the sequence $\{\mu_n\}$, is equivalent to the conditions

$$\sum_{m=1}^{\infty} \beta_m \left(\sum_{n=1}^{m} \gamma_n |c_n|^q \right)^{p/q} < \infty,$$

and

$$\sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=m}^{\infty} \gamma_n |c_n|^q \right)^{p/q} < \infty,$$

respectively. Plainly the assumptions on $\{\mu_n\}$ depend on $\{\beta_m\}$, or $\{\lambda_m\}$, and $0 < p < q$.

An application to absolute $|C, \alpha|$-summability of general orthogonal series is also presented.

1. Introduction. In the theory of orthogonal series several families of coefficient conditions are being utilized. Among them the primarily used assumptions have the following structure:

$$\sum_{n=1}^{\infty} c_n^2 \rho_n < \infty, \tag{1.1}$$

$$\sum_{m=1}^{\infty} \alpha_m \left(\sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^2 \right)^{p/2} < \infty \tag{1.2}$$

and

$$\sum_{m=1}^{\infty} \kappa_m \left(\sum_{n=m}^{\infty} c_n^2 \right)^{p/2} < \infty, \tag{1.3}$$

where $p > 0$, $\rho := \{\rho_n\}$, $\alpha := \{\alpha_n\}$ and $\kappa := \{\kappa_n\}$ are certain monotone sequences of real numbers, $\nu := \{\nu_m\}$ is a subsequence of natural numbers and $c := \{c_n\}$ is

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a real coefficient sequence. We do believe that the reader knows plenty of results using one of the above mentioned conditions, but in any case in [5] are cited several different theorems incorporating conditions $(1.i)$. In the same paper we studied the relations between these conditions. Among others, we gave sufficient conditions for the equivalence of (1.2) and (1.3); moreover, we analyzed the relation between (1.1) and (1.2).

V. Totik and I. Vincze [9] continued our investigations replacing the exponent 2 by a positive number q in the conditions $(1.i)$, and gave necessary and sufficient conditions for the equivalences of the generalized conditions.

In [7] Y. Okuyama and T. Tsuchikura proved that for a specific sequence α and $p=1$ the condition (1.2) is equivalent to a condition of the type

$$\sum_{m=1}^{\infty}\beta_m\left(\sum_{n=1}^{m}\gamma_n c_n^2\right)^{1/2}<\infty \quad (\beta_n,\ \gamma_n>0). \tag{1.4}$$

As far we know, this is the first result verifying equivalence between conditions of type (1.2) and (1.4).

In [6] we proved a general equivalence theorem pertaining to the following conditions:

$$\sigma_1:=\sum_{m=1}^{\infty}\alpha_m\left(\sum_{n=\nu_m+1}^{\nu_{m+1}}|c_n|^q\right)^{p/q}<\infty \tag{1.5}$$

and

$$\sigma_2:=\sum_{m=1}^{\infty}\beta_m\left(\sum_{n=1}^{m}\gamma_n|c_n|^q\right)^{p/q}<\infty. \tag{1.6}$$

The equivalence of conditions (1.5) and (1.6) means that there exists a constant $K:=K(\alpha,\beta,\gamma,\nu,p,q)>0$ such that $K^{-1}\sigma_2\leqslant\sigma_1\leqslant K\sigma_2$ for any sequence $\{c_n\}$. We maintain this meaning of equivalence through the paper. In what follows K, K_i denote absolute constants or constants depending only those parameters which are irrelevant to the problem in question. The constants are not necessarily the same at any two occurrences.

Since the equivalence of (1.5) with the conditions

$$\sum_{n=1}^{\infty}|c_n|^q\rho_n<\infty \tag{1.7}$$

and

$$\sum_{m=1}^{\infty}\kappa_m\left(\sum_{n=m}^{\infty}|c_n|^q\right)^{p/q}<\infty \tag{1.8}$$

is settled by Totik and Vincze, thus all equivalences of conditions (1.5)–(1.8) are analyzed.

In an old paper [3], improving a celebrated theorem of W. Orlicz [8] related to the unconditional convergence of orthogonal series, we proved that the condition (1.3) with

$p = 1$ and $\kappa_m = 1/m$ is equivalent to the pair of the following conditions:

$$\sum_{n=1}^{\infty} c_n^2 \mu_n < \infty \tag{1.9}$$

and

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{\mu_{2^{2^n}}} < \infty,$$

where $\mu := \{\mu_n\}$ is a nondecreasing sequence of positive numbers.

Later we [4] generalized this equivalence statement as follows:

THEOREM A. *The condition*

$$\sum_{m=1}^{\infty} \frac{1}{\lambda_m} \left(\sum_{n=m}^{\infty} c_n^2 \right)^{1/2} < \infty$$

holds if and only if there exists a nondecreasing sequence $\mu := \{\mu_n\}$ of positive numbers satisfying (1.9) *and*

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{\lambda_n \mu_n} < \infty,$$

where $\Lambda_n := \sum_{k=1}^{n} \lambda_k^{-1}$ and $\lambda := \{\lambda_n\}$ is a monotone sequence of positive numbers.

This result was also utilized for problems in connection with orthogonal series. In the theory of orthogonal series an universal assertion on the equivalence of coefficient conditions with more general parameters would seem very strange. Maybe this was the reason that we hid the kernel of the proof of Theorem A as a lemma (see Hilfssatz). But now, treating the equivalence of very general coefficient conditions, our lemma, in my view, would be worth for recalling as a notable result. However, instead of doing this, we shall present a slightly more general version of it as Theorem 1, and we shall omit some assumptions from the original lemma, which were natural restrictions in connection with orthogonal series. We also remark that without the monotonicity assumption the proof requires some additional consideration.

2. Theorems. First we prove the following theorem.

THEOREM 1. *Let $0 < p < q$, $\lambda := \{\lambda_n\}$, $c := \{c_n\}$ and $\gamma := \{\gamma_n\}$ be sequences of nonnegative numbers, furthermore let $\Lambda_n := \sum_{k=1}^{n} \lambda_k$. The condition*

$$\sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=m}^{\infty} \gamma_n c_n^q \right)^{p/q} < \infty \tag{2.1}$$

holds if and only if there exists a nondecreasing sequence $\mu := \{\mu_n\}$ of positive numbers satisfying conditions

$$\sum_{n=1}^{\infty} c_n^q \gamma_n \mu_n < \infty \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{\Lambda_n}{\mu_n}\right)^{p/(q-p)} < \infty. \tag{2.3}$$

Next we establish the symmetrical analogue of Theorem 1 with condition (1.6) instead of (2.1).

THEOREM 2. *Let* $0 < p < q$, $\beta := \{\beta_n\}$, $c := \{c_n\}$ *and* $\gamma := \{\gamma_n\}$ *be sequences of nonnegative numbers,* $\sum_{n=1}^{\infty} \beta_n < \infty$, *furthermore let* $B_n := \sum_{k=n}^{\infty} \beta_k$. *Then the condition* (1.6) *holds if and only if there exists a nonincreasing sequence* $\mu := \{\mu_n\}$ *of positive numbers satisfying conditions* (2.2) *and*

$$\sum_{n=1}^{\infty} \beta_n \left(\frac{B_n}{\mu_n}\right)^{p/(q-p)} < \infty. \tag{2.4}$$

REMARK. We underline that if $p > q$ then Theorems 1 and 2 are not valid universally. To verify this we present only one counterexample. Let $0 < q < p$, $\alpha > 0$, $\lambda_n := n^{\frac{p\alpha}{q}-1}$, $\gamma_n := 1$, $c_n := n^{-(1+\alpha)/q}$ and $\mu_n := n^{\alpha-\varepsilon}$, $0 < \varepsilon < \alpha$. Then a simple calculation shows that the sum in (2.1) is infinite, but the sums in (2.2) and (2.3) are finite, i.e. (2.2) and (2.3) do not imply (2.1).

3. Proofs. The idea of the proofs are similar to that of the lemma used in the proof of Theorem A (see Hilfssatz in [4]).

Proof of Theorem 1. First we assume that all $\gamma_n = 1$. Then we set

$$t_n := \left(\sum_{k=n}^{\infty} c_k^q\right)^{1/q} \qquad \text{and} \qquad \mu_n := t_n^{p-q}\Lambda_n,$$

assuming that all t_n are positive, otherwise Theorem 1 is obvious. It is plain that this sequence $\mu := \{\mu_n\}$ is nondecreasing, and

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{\Lambda_n}{\mu_n}\right)^{p/(q-p)} = \sum_{n=1}^{\infty} \lambda_n t_n^p,$$

whence the implication (2.1) $\Rightarrow$ (2.3) clearly follows.

To prove (2.1) $\Rightarrow$ (2.2) we define an index-sequence $\{\ell_m\}$ as follows: Let $\ell_0 := 1$ and $\ell_1 := 2$. If $m \geqslant 2$ then let ℓ'_m be the smallest integer k with

$$4^{\left(1+\frac{1}{p}\right)} t_k < t_{\ell_{m-1}}, \tag{3.1}$$

and let

$$\ell_m := \max(\ell_{m-1}+1, \ell'_m - 1). \tag{3.1a}$$

By the definition of $\{\ell_m\}$

$$\sum_{k=m}^{\infty} t_{\ell_k}^p \leqslant K t_{\ell_m}^p \tag{3.2}$$

obviously holds.

An elementary consideration shows that

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^q \mu_n &= \sum_{n=1}^{\infty} c_n^q t_n^{p-q} \sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{\infty} \lambda_k \sum_{n=k}^{\infty} c_n^q t_n^{p-q} \\ &\leqslant \sum_{m=0}^{\infty} \sum_{k=\ell_m}^{\ell_{m+1}-1} \lambda_k \sum_{i=m}^{\infty} \sum_{n=\ell_i}^{\ell_{i+1}-1} c_n^q t_n^{p-q} \\ &\leqslant \sum_{m=0}^{\infty} \sum_{k=\ell_m}^{\ell_{m+1}-1} \lambda_k \sum_{i=m}^{\infty} t_{\ell_i}^p =: S_1. \end{aligned} \tag{3.3}$$

Now, using (3.1) and (3.2), we get that

$$S_1 \leqslant K \sum_{m=0}^{\infty} \sum_{k=\ell_m}^{\ell_{m+1}-1} \lambda_k t_{\ell_m}^p \leqslant K_1 \sum_{m=0}^{\infty} \sum_{k=\ell_m}^{\ell_{m+1}-1} \lambda_k t_k^p,$$

whence, by (3.3), the statement (2.1) $\Rightarrow$ (2.2) is obvious.

In order to verify that conditions (2.2) and (2.3) jointly imply (2.1) we can assume that $\sum_{k=1}^{\infty} \lambda_k = \infty$, otherwise the assertion is trivial. Then we define an index-sequence $\{p_m\}$ as follows: Let $p_0 := 0$ and $p_1 := 2$. If $m \geqslant 2$ then let p'_m be the smallest integer i with

$$\sum_{n=p_{m-1}+1}^{i} \lambda_n > 4 \sum_{n=p_{m-2}+1}^{p_{m-1}} \lambda_n, \tag{3.4}$$

and let

$$p_m := \max(p_{m-1}+1, p'_m - 1). \tag{3.5}$$

By (3.4) and (3.5), it is easy to see that

$$\left(\sum_{n=p_{m-1}+1}^{p_{m+1}} \lambda_n \right)^{q/(q-p)} \leqslant K \sum_{n=p_{m-1}+1}^{p_{m+1}} \lambda_n \Lambda_n^{p/(q-p)} \tag{3.6}$$

stays.

Now we estimate the sum in (2.1). By $p < q$ we can use the so-called power-sum inequality (see e.g. [1], p. 28), and thus, by (3.4) and (3.5), it is easy to see that

$$\begin{aligned}\sum_{n=3}^{\infty} \lambda_n t_n^p &\leqslant \sum_{m=1}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} \lambda_n \sum_{\nu=m}^{\infty} \left(\sum_{k=p_\nu+1}^{p_{\nu+1}} c_k^q \right)^{p/q} \\ &\leqslant 4 \sum_{\nu=1}^{\infty} \left(\sum_{k=p_\nu+1}^{p_{\nu+1}} c_k^q \right)^{p/q} \sum_{n=p_{\nu-1}+1}^{p_{\nu+1}} \lambda_n =: 4S_2. \end{aligned} \tag{3.7}$$

Hence, by (2.2), (2.3) and (3.6), using Hölder's inequality we get

$$
\begin{aligned}
S_2 &= \sum_{\nu=1}^{\infty}\left(\sum_{k=p_\nu+1}^{p_{\nu+1}} c_k^q\right)^{p/q} \mu_{p_\nu+1}^{p/q}\mu_{p_\nu+1}^{-p/q}\sum_{n=p_{\nu-1}+1}^{p_{\nu+1}}\lambda_n \\
&\leqslant \left(\sum_{k=p_1+1}^{\infty} c_k^q\mu_k\right)^{p/q}\left\{\sum_{\nu=1}^{\infty}\mu_{p_\nu+1}^{p/(p-q)}\left(\sum_{n=p_{\nu-1}+1}^{p_{\nu+1}}\lambda_n\right)^{q/(q-p)}\right\}^{(q-p)/q} \\
&\leqslant K\left\{\sum_{\nu=1}^{\infty}\left(\sum_{n=p_{\nu-1}+1}^{p_\nu}+\sum_{n=p_\nu+1}^{p_{\nu+1}}\right)\lambda_n\Lambda_n^{p/(q-p)}\mu_{p_\nu+1}^{p/(p-q)}\right\}^{(q-p)/q} \\
&\leqslant K\left\{K_1+\sum_{\nu=1}^{\infty}\sum_{n=p_\nu+1}^{p_{\nu+1}}\lambda_n\Lambda_n^{p/(q-p)}\mu_{p_\nu+1}^{p/(p-q)}\right\}^{(q-p)/q}. \qquad (3.8)
\end{aligned}
$$

If in the last summation we consider only terms where $p_{\nu+1} = p_\nu + 1,$ then this part of the sum, by (2.3), is plainly finite. If $p_{\nu+1} > p_\nu + 1,$ then by (3.4) and (3.5)

$$\sum_{n=p_\nu+1}^{p_{\nu+1}}\lambda_n \leqslant 4\sum_{n=p_{\nu-1}+1}^{p_\nu}\lambda_n,$$

and thus

$$\Lambda_{p_{\nu+1}} \leqslant 5\Lambda_{p_\nu} \leqslant 25\Lambda_{p_{\nu-1}+1}$$

also holds, and these inequalities clearly imply

$$\sum_{n=p_\nu+1}^{p_{\nu+1}}\lambda_n\Lambda_n^{p/(q-p)} \leqslant K_2\sum_{n=p_{\nu-1}+1}^{p_\nu}\lambda_n\Lambda_n^{p/(q-p)}. \qquad (3.9)$$

Taking into account these comments, by (3.9) we easily get that

$$
\begin{aligned}
&\sum_{\nu=1}^{\infty}\sum_{n=p_\nu+1}^{p_{\nu+1}}\lambda_n\Lambda_n^{p/(q-p)}\mu_{p_\nu+1}^{p/(p-q)} \\
&\qquad\leqslant K_1+K_2\sum_{\nu=1}^{\infty}\mu_{p_\nu+1}^{p/(p-q)}\sum_{n=p_{\nu-1}+1}^{p_\nu}\lambda_n\Lambda_n^{p/(q-p)} \\
&\qquad\leqslant K_3\sum_{n=1}^{\infty}\lambda_n\left(\frac{\Lambda_n}{\mu_n}\right)^{p/(q-p)}. \qquad (3.10)
\end{aligned}
$$

Combining of (3.7), (3.8) and (3.10) yields that (2.2) and (2.3) imply (2.1) with $\gamma_n = 1$.

Finally, if γ_n are arbitrary nonnegative numbers, we set

$$\overline{c}_n := c_n \gamma_n^{1/q},$$

and apply the previous results for these $\overline{c}_n$, whence the statements of Theorem 1 evidently follow, and the proof is complete.

Proof of Theorem 2. We begin again the proof with the assumption $\gamma_n = 1$ for all n. We also assume that $\sum_{n=1}^{\infty} c_n^q = \infty$, otherwise Theorem 2 is trivial.

The proof of the implication $(1.6) \Rightarrow (2.4)$ is very simple, namely we set

$$h_n := \left(\sum_{k=1}^{n} c_k^q \right)^{1/q} \quad \text{and} \quad \mu_n := B_n h_n^{p-q},$$

and thus

$$\sum_{n=1}^{\infty} \beta_n \left(\frac{B_n}{\mu_n} \right)^{p/(q-p)} = \sum_{n=1}^{\infty} \beta_n h_n^p,$$

and the sequence $\{\mu_n\}$ is obviously nonincreasing.

To prove the implication $(1.6) \Rightarrow (2.2)$ we first define an index-sequence $\{r_m\}$ as follows: $r_0 := 1$ and, if $m \geqslant 1$, let r_m be the smallest integer k with the property

$$h_k > 4h_{r_{m-1}}.$$

Furthermore, let $N(k)$ be defined by $r_{N(k)} \leqslant k < r_{N(k)+1}$.

Using these notations and properties we have

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n^q \mu_n = \sum_{n=1}^{\infty} c_n^q h_n^{p-q} \sum_{k=n}^{\infty} \beta_k &= \sum_{k=1}^{\infty} \beta_k \sum_{n=1}^{k} c_n^q h_n^{p-q} \\
&\leqslant \sum_{k=1}^{\infty} \beta_k \sum_{m=0}^{N(k)} \sum_{n=r_m}^{r_{m+1}-1} c_n^q h_n^{p-q} \\
&\leqslant 4^p \sum_{k=1}^{\infty} \beta_k \sum_{m=0}^{N(k)} h_{r_m}^p \leqslant K \sum_{k=1}^{\infty} \beta_k h_k^p,
\end{aligned}$$

and this proves the implication $(1.6) \Rightarrow (2.2)$.

In order to verify that conditions (2.2) and (2.4) jointly imply (1.6) we define another index-sequence $\{q_m\}$ as follows: $q_0 := 0$ and, if $m \geqslant 1$, let q_m be the smallest integer with the property

$$\sum_{n=q_{m-1}+1}^{q_m} \beta_n > \frac{1}{2} B_{q_{m-1}+1}.$$

Using this definition we get that

$$\sum_{n=q_m+1}^{q_{m+2}} \beta_n > \frac{3}{4} B_{q_m+1}$$

and

$$\sum_{n=q_m+1}^{q_{m+1}-1} \beta_n \leqslant \frac{1}{2} B_{q_m+1}, \tag{3.11}$$

and these inequalities imply

$$\sum_{n=q_{m+1}}^{q_{m+2}} \beta_n > \frac{1}{4} B_{q_m+1}. \tag{3.12}$$

Finally, (3.11) and (3.12) yield

$$2 \sum_{n=q_{m+1}}^{q_{m+2}} \beta_n > \sum_{n=q_m+1}^{q_{m+1}-1} \beta_n. \tag{3.13}$$

We also need the inequality

$$B_{q_m+1} \leqslant 2B_{q_{m+1}}. \tag{3.14}$$

This comes from the definition of q_{m+1}. Namely,

$$\sum_{n=q_m+1}^{q_{m+1}-1} \beta_n \leqslant \frac{1}{2} \sum_{n=q_m+1}^{\infty} \beta_n = \frac{1}{2} \left(\sum_{n=q_m+1}^{q_{m+1}-1} \beta_n + \sum_{n=q_{m+1}}^{\infty} \beta_n \right),$$

whence (3.14) clearly follows.

Now we turn to the estimation of the sum in (1.6). Since $p < q$ we can use the so-called power-sum inequality and get

$$\begin{aligned} \sum_{n=1}^{\infty} \beta_n h_n^p &= \sum_{m=0}^{\infty} \sum_{n=q_m+1}^{q_{m+1}} \beta_n \sum_{i=0}^{m} \left(\sum_{k=q_i+1}^{q_{i+1}} c_k^q \right)^{p/q} \\ &\leqslant \sum_{m=0}^{\infty} \sum_{n=q_m+1}^{q_{m+1}} \beta_n \sum_{i=0}^{m} \left(\sum_{k=q_i+1}^{q_{i+1}} \mu_k c_k^q \right)^{p/q} \mu_{q_{i+1}}^{-p/q} \\ &= \sum_{i=0}^{\infty} \left(\sum_{k=q_i+1}^{q_{i+1}} \mu_k c_k^q \right)^{p/q} \mu_{q_{i+1}}^{-p/q} \sum_{m=i}^{\infty} \sum_{n=q_m+1}^{q_{m+1}} \beta_n =: I_1. \end{aligned} \tag{3.15}$$

By the definition of q_m it is clear that

$$\sum_{m=i}^{\infty} \sum_{n=q_m+1}^{q_{m+1}} \beta_n \leqslant 2 \sum_{n=q_i+1}^{q_{i+1}} \beta_n.$$

Using this and Hölder's inequality, by (2.2), (2.4) and (3.13), we have

$$
\begin{aligned}
I_1 &\leqslant K\left\{\sum_{i=0}^{\infty}\sum_{k=q_i+1}^{q_{i+1}}\mu_k c_k^q\right\}^{p/q}\left\{\sum_{i=0}^{\infty}\mu_{q_{i+1}}^{\frac{p}{p-q}}\left(\sum_{n=q_i+1}^{q_{i+1}}\beta_n\right)^{q/(q-p)}\right\}^{1-\frac{p}{q}}\\
&\leqslant K_1\left\{\sum_{i=0}^{\infty}\mu_{q_{i+1}}^{\frac{p}{p-q}}\left[\left(\sum_{n=q_i+1}^{q_{i+1}-1}\beta_n\right)^{q/(q-p)}+\beta_{q_{i+1}}\beta_{q_{i+1}}^{\frac{p}{q-p}}\right]\right\}^{1-\frac{p}{q}}\\
&\leqslant K_2\left\{K+\sum_{i=0}^{\infty}\mu_{q_{i+1}}^{\frac{p}{p-q}}\left(\sum_{n=q_{i+1}}^{q_{i+2}}\beta_n\right)^{q/(q-p)}\right\}^{1-\frac{p}{q}}\\
&\leqslant K_3\left\{2K+\sum_{i=0}^{\infty}\mu_{q_{i+1}}^{\frac{p}{p-q}}\left(\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n\right)^{q/(q-p)}\right\}^{1-\frac{p}{q}}. \qquad (3.16)
\end{aligned}
$$

In the next step we utilize (3.14) as follows:

$$
\begin{aligned}
\left(\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n\right)^{q/(q-p)} &= \left(\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n\right)\left(\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n\right)^{p/(q-p)}\\
\leqslant K\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n B_{q_{i+2}}^{p/(q-p)} &\leqslant K\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n B_n^{p/(q-p)}. \qquad (3.17)
\end{aligned}
$$

Putting this estimation into the last sum in (3.16), furthermore using the monotonicity of $\{\mu_n\}$ and (2.4), we get

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\mu_{q_{i+1}}^{\frac{p}{p-q}}\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n B_n^{p/(q-p)}\\
&\qquad\leqslant\sum_{i=0}^{\infty}\sum_{n=q_{i+1}+1}^{q_{i+2}}\beta_n\left(\frac{B_n}{\mu_n}\right)^{p/(q-p)}<\infty. \qquad (3.18)
\end{aligned}
$$

Combining (3.15)–(3.18) we have (1.6) with $\gamma_n = 1$.

If γ_n are arbitrary nonnegative numbers we set

$$\overline{c}_n := c_n\gamma_n^{1/q},$$

and get the conclusion of Theorem 2 as in Theorem 1.

4. Application. Utilizing the result of Theorem 2 we present new conditions for the absolute $\left|C,\alpha>\frac{1}{2}\right|$-summability of general orthogonal series.

Since condition (1.6) and the conditions (2.2) and (2.4) jointly are equivalent, thus it is clear that if a condition of type (1.6) implies a certain property of a general orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x), \tag{4.1}$$

then the suited conditions (2.2) and (2.4) jointly yield the same property of (4.1). The same assertion is true regarding the conditions of Theorem 1. By this equivalence we could present several new sufficient conditions in pair utilizing Theorems 1 and 2. However, we shall establish only one sample result applying Theorem 2. Namely, in connection with Theorem 1, or more precisely with Theorem A which is a special case of Theorem 1, we have already the improvement of Orlicz's theorem as an applicat of Theorem A.

We establish the following result.

THEOREM 3. *If there exists a sequence* $\mu := \{\mu_n\}$ *of positive numbers such that* $\{\mu_n n^{-2}\}$ *is nonincreasing and*

$$\sum_{n=1}^{\infty} c_n^2 \mu_n < \infty \tag{4.2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n\mu_n} < \infty, \tag{4.3}$$

then series (4.1) *is* $\left|C, \alpha > \frac{1}{2}\right|$*-summable almost everywhere in* $(0,1)$.

Proof. Theorem 2 with $p = 1$, $q = 2$, $\gamma_n = n^2$ and $\beta_n = n^{-2}$ implies that conditions (4.2) and (4.3) are equivalent to

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \left(\sum_{n=1}^{m} n^2 c_n^2 \right)^{1/2} < \infty. \tag{4.4}$$

In [6] we proved that the condition (4.4) is equivalent to

$$\sum_{m=1}^{\infty} \left(\sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right)^{1/2} < \infty \tag{4.5}$$

(see Proposition 1 with $\nu_m = 2^m$, $\gamma(x) \equiv 1$, $\lambda(x) = x$ and $q = 2$).

Finally, in [2] (see Satz I) we proved that the condition (4.5) implies the $\left|C, \alpha > \frac{1}{2}\right|$-summability of series (4.1) almost every in $(0,1)$.

These results clearly convey the statement of Theorem 3, as desired.

REFERENCES

[1] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1959.
[2] L. LEINDLER, *Über die absolute Summierbarkeit der Orthogonalreihen*, Acta Sci. Math. (Szeged) **22** (1961), 243–268.
[3] L. LEINDLER, *Über unbedingte Konvergenz der Orthogonalreihen mit strukturellen Bedingungen*, Studia Math. **23** (1963), 113–117.
[4] L. LEINDLER, *Über einen Äquivalenzsatz*, Publ. Math. (Debrecen) **12** (1965), 213–218.
[5] L. LEINDLER, *On relations of coefficient-conditions*, Acta Math. Acad. Sci. Hungar. **39** (1982), 409–420.
[6] L. LEINDLER, *On equivalence of coefficient conditions with applications*, Acta Sci. Math. (Szeged) **60** (1995), 495–514.
[7] Y. OKUYAMA AND T. TSUCHIKURA, *On the absolute Riesz summability of orthogonal series*, Analysis Math. **7** (1981), 199–208.
[8] W. ORLICZ, *Zur Theorie der Orthogonalreihen*, Bull. Acad. Polonaise (1927), 81–115.
[9] V. TOTIK AND I. VINCZE, *On relations of coefficient conditions*, Acta Sci. Math. (Szeged) **50** (1986), 93–98.

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