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# SOME INEQUALITIES RELATED TO $M$–MATRICES

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(*communicated by B. Opic*)

*Abstract.* Several different forms of a Euler type inequality are investigated and their relation to $M$-matrices and doubly stochastic matrices is exhibited. The results are applied to the study of positive biquadratic forms.

## Introduction

In connection with positive biquadratic forms, the authors came across an interesting inequality (Theorem 2.). It turned out, however, that this inequality was essentially equivalent to a particular case of a result in [1], as well as to an inequality proved in [4]. The authors present a new simple proof thereof as well as its several consequences. Interesting connections with $M$-matrices and doubly stochastic matrices are also pointed out.

Let us recall the result posed as a problem by A. Berkes, proved by C. Bindscheidler [1]. In the book of D.S. Mitrinović [5] it appears under 2.41.

*Given $n+1$ positive numbers $x_1, \dots, x_{n+1}$ then the following implication holds:*

$$\text{If} \quad \sum_1^{n+1} \frac{1}{1+x_k} \geqslant n, \ \text{then} \quad \prod_1^{n+1} \frac{1}{x_k} \geqslant n^{n+1}.$$

For our purposes it will be convenient to restate this implication in the following form:

$(*)$ *Let $a_1, \dots, a_n$, $n \geqslant 3$, be positive numbers such that $\prod a_i \geqslant 1$. Then*

$$\sum \frac{1}{n-1+a_i} \leqslant 1,$$

*with equality if and only if $a_i = 1$ for $i = 1, \dots, n$.*

Let us recall that a real square matrix is called an $M$-matrix (in [3], a matrix of class $K$) if all its off-diagonal entries are nonpositive and all principal minors positive. It is called a *possibly singular* $M$-matrix (a matrix of class $K_0$) if all its off-diagonal

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entries are nonpositive and all principal minors nonnegative. A symmetric $M$-matrix is also called *Stieltjes matrix*.

A real nonnegative matrix is called *doubly stochastic* if all its row-sums as well as all its column-sums equal one.

In the sequel, we shall use the following notation.

If $x_1, \dots, x_n$ are real numbers, we denote by $A(x)$, resp. $A(x^2)$ the arithmetic mean of these numbers, resp. their squares. By $G(y)$ we denote the geometric mean of positive (or, nonnegative) numbers $y_i$.

## Results

We shall first give a simple proof of an inequality stated essentially in ([4], Lemma 5.1).

LEMMA 2.1. *Suppose that $u_1, \dots, u_n$ are real numbers. Then,*

$$A(u)^2 \leqslant \frac{n-1}{n} A(u^2) + \frac{1}{n} G(u^2). \tag{1}$$

*For $n \geqslant 3$, equality is attained if and only if $u_1 = \dots = u_n$. (For $n = 2$, equality is attained if and only if $u_1 u_2 \geqslant 0$, for $n = 1$ always.)*

*Proof.* It is immediate that it suffices to prove this for the case that $n \geqslant 3$, all numbers $u_i$ are positive and $G(u)$ equals one. Let then $\mathscr{S}$ denote the set of all $n$-tuples $y_i$ which satisfy $G(y) = 1$ and

$$\min(u_k) \leqslant y_i \leqslant \max(u_k) \quad \text{for all} \quad i.$$

Since $\mathscr{S}$ is compact, the function $\Phi(y) := A(y)^2 - \frac{n-1}{n} A(y^2)$ attains in $\mathscr{S}$ its maximum. To prove that this maximum is $\frac{1}{n}$, attained for $y_i = 1$ for all $i$, it suffices to show that for any other $(y) \in \mathscr{S}$ its value can be augmented for some $\tilde{y} \in \mathscr{S}$.

Let thus $y_1 \geqslant y_2 \geqslant \dots \geqslant y_n$ and suppose that $y_1 > y_n$. Denote by $f(x)$ the function

$$f(x) = (xy_1 + \frac{1}{x} y_n + b)^2 - (n-1)(x^2 y_1^2 + \frac{1}{x^2} y_n^2 + B)$$

defined for positive $x$, where $b = \sum_2^{n-1} y_j$ and $B = \sum_2^{n-1} y_j^2$. Its derivative can be written as

$$f'(x) = 2[-(n-2)(xy_1 + \frac{y_n}{x}) + b](y_1 - \frac{1}{x^2} y_n).$$

Since $y_n > 0$ and $(n-2)y_1 \geqslant b$, this derivative is negative for $x = 1$.

This implies that for $x < 1$ sufficiently close to 1, the point $\tilde{y} = (\tilde{y}_i)$, $\tilde{y}_1 = xy_1$, $\tilde{y}_n = \frac{1}{x} y_n$, $\tilde{y}_k = y_k$ for $2 \leqslant k \leqslant n-1$, belongs to $\mathscr{S}$ and $\Phi(\tilde{y}) > \Phi(y)$. $\square$

THEOREM 2.2. *Suppose* $a_1, \dots, a_n$, $n \geqslant 3$, *are positive numbers satisfying* $\prod a_i = 1$. *Then, for an arbitrary* $n$*-tuple of real numbers* $u_1, \dots, u_n$,

$$\left(\sum u_i\right)^2 \leqslant (n-1)\sum u_i^2 + \sum u_i^2 a_i.$$

*Equality is attained if and only if* $a_i = 1$ *for all i and* $u_1 = \dots = u_n$.

*Proof.* By Lemma 2.1,

$$\begin{aligned} \left(\sum u_i\right)^2 &\leqslant (n-1)\sum u_i^2 + nG(u^2) \\ &= (n-1)\sum u_i^2 + nG(a_1u_1^2, \dots, a_nu_n^2) \\ &\leqslant (n-1)\sum u_i^2 + nA(a_1u_1^2, \dots, a_nu_n^2) \\ &= (n-1)\sum u_i^2 + \sum a_iu_i^2. \end{aligned}$$

The rest is obvious. □

THEOREM 2.3. *Let* $a_1, \dots, a_n$, $n \geqslant 3$, *be positive numbers such that* $\prod_1^n a_i \geqslant 1$. *Consider the matrix*

$$M(a) = \begin{pmatrix} n-2+a_1 & -1 & \dots & -1 \\ -1 & n-2+a_2 & \dots & -1 \\ \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & n-2+a_n \end{pmatrix}.$$

*Then* $M(a)$ *is a possibly singular* $M$*-matrix.*

*If* $a_i = 1$ *for all* $i$, *then* $M(a)$ *is singular with all row-sums equal to zero. In all other cases,* $M(a)$ *is a (nonsingular)* $M$*-matrix.*

*Proof.* By Theorem 2.2, we have for all real $n$-tuples $u_1, \dots, u_n$

$$(n-1)\sum u_i^2 + \sum u_i^2 a_i - \left(\sum u_i\right)^2 \geqslant 0.$$

Since $M(a)$ is the matrix of this quadratic form, it is positive semidefinite. Thus all its principal minors are nonnegative and, at the same time, all its off-diagonal entries are negative, so it is a possibly singular $M$-matrix as asserted. The last assertion follows from the fact that unless all the $a_i$'s equal one, the last inequality is strict for all non-zero $n$-tuples which means that $M(a)$ is positive definite.

REMARK 2.4. It is easy to show that the determinant of $M(a)$ is equal to

$$\Pi(n-1+a_i)\left(1 - \sum \frac{1}{n-1+a_i}\right).$$

Therefore, the assertion (*) follows from Theorem 2.3.

COROLLARY 2.5. *Let* $y_1,\dots,y_n$ *be real numbers,* $P=(p(1),p(2),\dots,p(n))$ *a permutation of* $1,2,\dots,n$, $n\geqslant 2$. *Denote by* $Q(y,P)$ *the matrix*

$$\begin{pmatrix} (n-2)y_1^2+y_{p(1)}^2 & -y_1y_2 & -y_1y_3 & \dots & -y_1y_n \\ -y_2y_1 & (n-2)y_2^2+y_{p(2)}^2 & -y_2y_3 & \dots & -y_2y_n \\ \dots & \dots & \dots & \dots & \dots \\ -y_ny_1 & -y_ny_2 & y_ny_3 & \dots & (n-2)y_n^2+y_{p(n)}^2 \end{pmatrix}.$$

*Then,* $Q(y,P)$ *is positive semidefinite.*

*Proof.* It suffices to consider the case that all numbers $y_i$ are different from zero. It is then easy to see that

$$Q(y,P)=D(y)M(a)D(y),$$

where $D(y)$ is the diagonal matrix $\operatorname{diag}\{y_1,\dots,y_n\}$ and $a_i=(y_{P(i)}/y_i)^2$ so that $\prod a_i=1$. For $n\geqslant 3$, Theorem 2.4 applies. If $n=2$, the result is also true. □

COROLLARY 2.6. *Let* $P=(p(1),p(2),\dots,p(n))$, $n\geqslant 2$, *be a permutation of* $1,2,\dots,n$. *Given two n-tuples of real numbers* $x_1,\dots,x_n,y_1,\dots,y_n$, *the following inequality holds:*

$$(n-1)\sum x_i^2y_i^2+\sum x_i^2y_{p(i)}^2-\Big(\sum x_iy_i\Big)^2\geqslant 0. \tag{2}$$

*Proof.* The expression above equals, in the usual notation, $(Q(y,P)x,x)$. □

COROLLARY 2.7. *Suppose* $D=(d_{ik})$ *is an* $n$*-by-*$n$ *doubly stochastic matrix,* $n\geqslant 2$. *Then for any pair of* $n$*-tuples of real numbers* $x_1,\dots,x_n,y_1,\dots,y_n$, *the following inequality holds:*

$$\Big(\sum x_iy_i\Big)^2\leqslant (n-1)\sum x_i^2y_i^2+\sum_{i,k}d_{ik}x_i^2y_k^2.$$

*Proof.* By Birkhoff's theorem, the matrix $D$ may be written in the form of a convex combination of permutation matrices, $D=\sum\lambda_PP$. The inequality then follows from the fact that $\sum\lambda_PQ(y,P)$ is a convex combination of positive semidefinite matrices.

CORROLLARY 2.8. *Let* $B$, $D$ *be* $n$*-by-*$n$ *matrices,* $n\geqslant 2$, $D=(d_{ik})$ *doubly stochastic and* $B=(b_{ik})$ *positive semidefinite. Then, for every* $n$*-tuple of real numbers* $x_1,\dots,x_n$,

$$\sum_{i,k}b_{ik}x_ix_k\leqslant (n-1)\sum_k b_{kk}x_k^2+\sum_{i,k}d_{ik}b_{kk}x_i^2.$$

*Proof.* The matrix $B$ may be written in the form $B=\sum y^{(j)}y^{(j)T}$ where $y^{(j)}$ are some real vectors. We have then, using Corollary 2.7,

$$\begin{aligned}
\sum_{i,k} b_{ik}x_ix_k &= \sum_j \Big(\sum_i x_i y_i^{(j)}\Big)^2 \\
&\leqslant \sum_j (n-1)\sum_i x_i^2 (y_i^{(j)})^2 + \sum_j \sum_{i,k} d_{ik} x_i^2 (y_k^{(j)})^2 \\
&= (n-1)\sum_i b_{ii}x_i^2 + \sum_{i,k} d_{ik} b_{kk} x_i^2 .
\end{aligned}$$

COROLLARY 2.9. *Given $n$ real numbers $u_1, \dots, u_n$, the following estimate for the modified dispersion holds:*

$$\frac{1}{n}\sum_{i<k}(u_i - u_k)^2 \leqslant A(u^2) - G(u^2). \tag{3}$$

*Proof.* The inequality (1) may be rewritten in the form

$$n(n-1)A(u^2) + nG(u^2) - \Big(\sum u_i\Big)^2 \geqslant 0.$$

The expression on the left-hand-side of this inequality equals

$$n^2A(u^2) - \Big(\sum u_i\Big)^2 - n(A(u^2) - G(u^2)) = \sum_{i<k}(u_i - u_k)^2 - n(A(u^2) - G(u^2)).$$

This proves the equivalence of (1) and (3).

REMARK 2.10. The inequality (3) appears, in a slightly different form, also in [4]. Let us conclude by a slight strengthening of a result in [2]:

THEOREM 2.11. *The biquadratic form*

$$2\sum_1^3 (x_iy_i)^2 - \Big(\sum_1^3 x_iy_i\Big)^2 + x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2$$

*is positive semidefinite, equal to zero in all points $x_1 = x_2 = x_3$, $y_1 = y_2 = y_3$ and cannot be expressed as a sum of squares of bilinear forms.*

*Proof.* The first assertion follows from Corollary 2.6 for $n = 3$. The second is obvious. The last follows in the manner identical with that in [2], Theorem 1.

REMARK 2.12. An analogous result holds for the case of the biquadratic form in (2) if $P$ is the cyclic permutation ($p(k) \equiv k + 1 \bmod n$).

ADDED IN PROOF. The strengthening 2.11 is essentially contained in M. D. Choi, *Positive linear maps*, Proc. Symp. in Pure Math., Amer. Math. Soc. 38(2) (1982), 583–590.

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