Mathematical
Inequalities
& Applications
Volume 1, Number 2 (1998), 177–191

NEW L^q INEQUALITIES FOR POLYNOMIALS

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(*communicated by J. Pečarić*)

Abstract. In this paper we establish some new L^q inequalities for polynomials which generalize and refine some results of Szegö, Zygmund, De Bruijn and others.

1. Introduction

For fixed μ, $1 \leqslant \mu \leqslant n$, let $\mathbb{P}_{n,\mu}$ denote the class of polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ of degree at most n. For $P \in \mathbb{P}_{n,\mu}$, we define

$$\|P\|_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \right\}^{1/q}, \quad 0 < q < \infty,$$

$$\|P\|_\infty = \max_{|z|=1} |P(z)| \quad \text{and} \quad m(P,k) = \min_{|z|=k} |P(z)|.$$

If $P \in \mathbb{P}_{n,1}$, then

$$\|P'\|_\infty \leqslant n\|P\|_\infty \tag{1}$$

and

$$\|P'\|_\infty \leqslant n\|\mathrm{Re}(P)\|_\infty. \tag{2}$$

Inequality (1) is well known result of S. Bernstein (for reference see [14] or [18]). Inequality (2), which is an interesting refinement of Bernstein's inequality (1), is due to Szegö [19] (for other proofs see [9, 13, 17]).

Inequalities (1) and (2) can be obtained by letting $q \to \infty$ in the inequalities

$$\|P'\|_q \leqslant n\|P\|_q, \quad q \geqslant 1 \tag{3}$$

and

$$\|P'\|_q \leqslant \frac{n}{\left\| \frac{1+z^n}{2} \right\|_q} \|\mathrm{Re}(P)\|_q, \quad q \geqslant 1 \tag{4}$$

Mathematics subject classification (1991): Primary 26D05, 30D15, Secondary 41A17.
Key words and phrases: Polynomials, Zygmund's inequality, L^q inequalities.

respectively. Inequality (3), which is an extension of the inequality (1) to L^q spaces, was found out by Zygmund [21] (see also [2]). Inequality (4), which is the corresponding extension of the inequality (2) to L^q spaces is also due to Zygmund [22].

It was shown by Frappier, Rahman and Ruscheweyh [9, inequality 7.18] that if $P \in \mathbb{P}_{n,1}$, then

$$\|P(Rz) - P(z)\|_\infty \leqslant (R^n - 1)\,\|Re(P)\|_\infty, \quad R > 1. \tag{5}$$

Recently the authors [5] have investigated the dependence of $\|P(Rz) - P(z)\|_q$ on $\|P\|_q$ and proved that if $P \in \mathbb{P}_{n,1}$, then for every $q \geqslant 1$ and $R \geqslant 1$,

$$\|P(Rz) - P(z)\|_q \leqslant (R^n - 1)\,\|P\|_q, \tag{6}$$

which contains the inequality (1) as a special case. In this paper we first prove the following result analogous to (6), which among other things includes inequality (5) as a special case.

THEOREM 1. *If $P \in \mathbb{P}_{n,1}$, then for every $q \geqslant 1$ and $R \geqslant 1$,*

$$\|P(Rz) - P(z)\|_q \leqslant \frac{(R^n - 1)}{\left\|\dfrac{1 + z^n}{2}\right\|_q}\,\|\mathrm{Re}(P)\|_q. \tag{7}$$

The result is best possible and equality holds for $P(z) = z^n$.

REMARK 1. If we let $q \to \infty$ in (7), we get (5). This inequality is an interesting generalization of the inequality (2).

If $P \in \mathbb{P}_{n,1}$ and $P(z)$ does not vanish in $|z| < k$, where $k \geqslant 1$, then

$$\|P'(z)\|_\infty \leqslant \frac{n}{1+k}\|P(z)\|_\infty. \tag{8}$$

Inequality (8) is due to Malik [13]. For $k = 1$, it was conjectured by P. Erdös and later verified by P. D. Lax [12]. Inequality (8) was further improved by Govil [10], who under the same hypothesis proved that

$$\|P'(z)\|_\infty \leqslant \frac{n}{1+k}\big\{\|P(z)\|_\infty - m(P,k)\big\}. \tag{9}$$

For $k = 1$ inequality (9) is due to A. Aziz and Q. M. Dawood [4]. As a generalization of (8), it was shown by Chan and Malik [6] (see also [8,16]) that if $P \in \mathbb{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$, then

$$\|P'(z)\|_\infty \leqslant \frac{n}{1+k^\mu}\|P(z)\|_\infty. \tag{10}$$

Next we present the following result which is a generalization of De Bruijn's Theorem [7, Theorem 13] and which includes the inequality (8) as a special case.

THEOREM 2. *If $p \in \mathbb{P}_{n,1}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$, then for every real or complex number β with $|\beta| \leqslant 1$, $q > 0$ and $R \geqslant 1$,*

$$\|P(Rz) - P(z) + \beta \frac{(R^n - 1)}{1+k} m(P,k)\|_q \leqslant \frac{(R^n - 1)}{\|k + z^n\|_q} \|P\|_q. \tag{11}$$

The result is best possible for $k = 1$ and equality holds for $P(z) = az^n + b$, $|a| = |b|$.

Instead of proving Theorem 2, we prove the following more general result which includes the inequality (10) as a special case and also have various other interesting consequences.

THEOREM 3. *If $P \in \mathbb{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$, then for every real or complex number β with $|\beta| \leqslant 1$, $q > 0$ and $R \geqslant 1$*

$$\|P(Rz) - P(z) + \beta \Big(\frac{R^n - 1}{1+k^\mu}\Big) m(P,k)\|_q \leqslant \frac{(R^n - 1)}{\|k^\mu + z^n\|_q} \|P\|_q. \tag{12}$$

REMARK 2. Making q tend to infinity in (12) and choosing argument of β, with $|\beta| = 1$, suitably we obtain that if $P \in \mathbb{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$, then

$$\|P(Rz) - P(z)\|_\infty + \Big(\frac{R^n - 1}{1+k^\mu}\Big) m(P,k) \leqslant \Big(\frac{R^n - 1}{1+k^\mu}\Big) \|P\|_\infty$$

or equivalently,

$$\|P(Rz) - P(z)\|_\infty \leqslant \Big(\frac{R^n - 1}{1+k^\mu}\Big) \big\{\|P(z)\|_\infty - m(P,k)\big\}. \tag{13}$$

Dividing two sides of this inequality by $R - 1$ and letting $R \to 1$, it follows that

$$\|P'(z)\|_\infty \leqslant \frac{n}{1+k^\mu} \big\{\|P(z)\|_\infty - m(P,k)\big\}, \tag{14}$$

which is an interesting refinement of the inequality (10). Inequality (9) is a special case of the inequality (14) for $\mu = 1$.

From inequality (13), it follows that

$$\|P(Rz)\|_\infty \leqslant \Big(\frac{R^n + k^\mu}{1+k^\mu}\Big) \|P(z)\|_\infty - \Big(\frac{R^n - 1}{1+k^\mu}\Big) m(P,k), \quad R \geqslant 1$$

which is a generalization of a result due to first author [3, Theorem 4].

If we divide two sides of the inequality (11) by $R - 1$ and let $R \to 1$, we get the following interesting result.

COROLLARY. *If $P \in \mathbb{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$, then for every real or complex number β with $|\beta| \leqslant 1$, $q > 0$ and $R \geqslant 1$,*

$$\Big\|zP'(z) + \frac{\beta mn}{1+k^\mu}\Big\|_q \leqslant \frac{n}{\|k^\mu + z^n\|} \|P\|_q.$$

For $\beta = 0$ and $\mu = 1$, above corollary reduces to a result due to Govil and Rahman [11, Theorem 9].

2. Lemmas

For the proofs of these theorems we need the following Lemmas.

LEMMA 1. *If $P \in \mathbb{P}_{n,1}$, then for every real α and $R \geqslant 1$,*

$$e^{i\alpha}P(Re^{i\theta}) = e^{i\alpha}P(e^{i\theta}) + \frac{1}{2\pi}\sum_{k=1}^{2n}(-1)^k A_k(R,\alpha)\,P\Big(e^{i(\theta+\frac{k\pi+\alpha}{n})}\Big)$$

where

$$A_k(R,\alpha) = (R^n-1) + 2\sum_{j=1}^{n}(R^{n-j}-1)\cos j\Big(\frac{k\pi+\alpha}{n}\Big).$$

The coefficients $A_k(R,\alpha)$ are all non-negative and

$$\frac{1}{2n}\sum_{k=1}^{2n}A_k(R,\alpha) = R^n - 1. \tag{15}$$

This result is due to Frappier, Rahman and Ruscheweyh [9].

LEMMA 2. *If $P \in \mathbb{P}_{n,1}$ and $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then*

$$|P(Rz)| > |P(z)| \quad \textit{for } |z| \geqslant 1 \textit{ and } R > 1. \tag{16}$$

Proof of Lemma 2. Since all the zeros of $P(z)$ lie in $|z| \leqslant k \leqslant 1$, we write

$$P(z) = C\prod_{j=1}^{n}(z - z_j e^{i\theta}j) \quad \text{where } r_j \leqslant k,\ j = 1,2,\ldots,n,$$

so that for each θ, $0 \leqslant \theta < 2\pi$ and $R > 1$, it can be easily seen that

$$\Big|\frac{P(Re^{i\theta})}{P(e^{i\theta})}\Big| = \prod_{j=1}^{n}\Big|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\Big| \geqslant \prod_{j=1}^{n}\Big(\frac{R+r_j}{1+r_j}\Big)$$
$$\geqslant \prod_{j=1}^{n}\Big(\frac{R+k}{1+k}\Big) = \Big(\frac{R+k}{1+k}\Big)^n.$$

This implies

$$|P(Re^{i\theta})| \geqslant \Big(\frac{R+k}{1+k}\Big)^n |P(e^{i\theta})| \quad \text{for } R > 1,\ 0 \leqslant \theta < 2\pi. \tag{17}$$

Clearly, $P(Re^{i\theta}) \neq 0$ for every $R > 1$ and $0 \leqslant \theta < 2\pi$, which implies

$$|P(Rz)| > 0 \quad \text{for } |z| = 1 \text{ and } R > 1. \tag{18}$$

Now, for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are not the zeros of $P(z)$, we get from (17)

$$|P(Re^{i\theta})| > |P(e^{i\theta})| \quad \text{for every } R > 1. \tag{19}$$

Since by (18), the inequality (19) is trivially true for those points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are the zeros of $P(z)$, it follows that

$$|P(z)| < |P(Rz)| \quad \text{for } |z| = 1 \text{ and } R > 1.$$

Since all the zeros of $P(Rz)$ lie in $|z| \leqslant (k/R) < 1$, it follows by the Maximum Modulus principle that

$$|P(z)| < |P(Rz)| \quad \text{for } |z| \geqslant 1 \text{ and } R > 1.$$

This completes the proof of Lemma 2.

LEMMA 3. *If* $P \in \mathbb{P}_{n,\mu}$ *and* $P(z)$ *does not vanish in* $|z| < k$ *where* $k \geqslant 1$ *and* $Q(z) = z^n\overline{P(1/\overline{z})}$, *then*

$$k^{\mu}|P(Rz) - P(z)| \leqslant |Q(Rz) - Q(z)| \quad \textit{for } |z| = 1 \textit{ and } R \geqslant 1. \tag{20}$$

Proof of Lemma 3. For $R = 1$ there is nothing to prove. Henceforth we assume that $R > 1$. Since the polynomial $P(z)$ has all its zeros in $|z| \geqslant k$ where $k \geqslant 1$, therefore, the polynomial $F(z) = P(kz)$ has all its zeros in $|z| \geqslant 1$. Hence for every real or complex number β with $|\beta| > 1$, the polynomial $f(z) = F(z) - \beta G(z)$, where $G(z) = z^n\overline{F(1/\overline{z})}$, has all its zeros in $|z| \leqslant 1$. Applying Lemma 1 to the polynomial $f(z)$ with $k = 1$, we get

$$|f(z)| < |f(Rz)| \quad \text{for } |z| = 1 \text{ and } R > 1.$$

Using Rouche's theorem and noting that all the zeros of $f(Rz)$ lie in $|z| \leqslant (1/R) < 1$, we conclude that the polynomial

$$g(z) = f(Rz) - f(z) = (F(Rz) - F(z)) - \beta(G(Rz) - G(z)), \tag{21}$$

where $G(z) = z^n\overline{F(1/\overline{z})}$, has all its zeros in $|z| < 1$ for every complex number β with $|\beta| > 1$ and $R > 1$. This implies

$$|F(Rz) - F(z)| \leqslant |G(Rz) - G(z)| \quad \text{for } |z| \geqslant 1 \text{ and } R > 1. \tag{22}$$

If inequality (22) is not true, then there is a point $z = z_0$ with $|z_0| \geqslant 1$ such that

$$|F(Rz_0) - F(z_0)| > |G(Rz_0) - G(z_0)|.$$

Since all the zeros of $G(z)$ lie in $|z| \leqslant 1$, it follows (as in the case of $f(z)$) that all the zeros of $G(Rz) - G(z)$ lie in $|z| < 1$ for every $R > 1$. Hence $G(Rz_0) - G(z_0) \neq 0$ with $|z_0| \geqslant 1$, we take

$$\beta = \frac{F(Rz_0) - F(z_0)}{G(Rz_0) - G(z_0)}$$

so that β is a well defined real or complex number with $|\beta| > 1$ and with this choice of β, from (21), we get

$$g(z_0) = 0 \quad \text{where } |z_0| \geqslant 1.$$

This is clearly contradiction to the fact that all the zeros of $g(z)$ lie in $|z| < 1$. Thus

$$|F(Rz) - F(z)| \leqslant |G(Rz) - G(z)| \quad \text{for } |z| \geqslant 1 \text{ and } R > 1.$$

Replacing $F(z)$ by $P(kz)$ and $G(z)$ by $z^n\overline{P(k/\overline{z})}$, we get

$$\begin{aligned}|P(Rkz) - P(kz)| &\leqslant \left|R^n z^n \overline{P(k/R\overline{z})} - z^n \overline{P(k/\overline{z})}\right| \\ &= |R^n P(kz/R) - P(kz)| \quad \text{for } |z| = 1 \text{ and } R > 1. \end{aligned} \tag{23}$$

Now $P \in \mathbb{P}_{n,\mu}$ implies that

$$\begin{aligned}P(Rkz) - P(kz) &= a_n(R^n - 1)k^n z^n + a_{n-1}(R^{n-1} - 1)k^{n-1}z^{n-1} + \\ &\qquad \ldots + a_{\mu+1}(R^{\mu+1} - 1)k^{\mu+1}z^{\mu+1} + a_\mu(R^\mu - 1)k^\mu z^\mu \\ &= k^\mu z^\mu \left\{a_n(R^n-1)k^{n-\mu}z^{n-\mu} + \ldots + a_{\mu+1}(R^{\mu+1}-1)kz + a_\mu(R^\mu-1)\right\}.\end{aligned}$$

Using this in (23), we obtain

$$\begin{aligned}k^\mu |a_n(R^n - 1)k^{n-\mu}z^{n-\mu} + \ldots + a_{\mu+1}(R^{\mu+1} - 1)kz + a_\mu(R^\mu - 1)| \\ \leqslant |R^n P(kz/R) - P(kz)| \quad \text{for} \quad |z| = 1 \text{ and } R > 1.\end{aligned} \tag{24}$$

Since the polynomial $R^nP(kz/R) - P(kz)$ does not vanish in $|z| \leqslant 1$, by the Maximum Modulus Principle, the inequality (24) holds for $|z| \leqslant 1$ also. That is

$$\begin{aligned}k^\mu |a_n(R^n - 1)k^{n-\mu}z^{n-\mu} + \ldots + a_{\mu+1}(R^{\mu+1} - 1)kz + a_\mu(R^\mu - 1)| \\ \leqslant |R^n P(kz/R) - P(kz)| \quad \text{for} \quad |z| \leqslant 1 \text{ and } R > 1.\end{aligned} \tag{25}$$

Replacing z by $e^{i\theta}/k$, $k \geqslant 1$, $0 \leqslant \theta < 2\pi$, in (25), it follows that

$$k^\mu |P(Re^{i\theta}) - P(e^{i\theta})| \leqslant |R^n P(e^{i\theta}/R) - P(e^{i\theta})|, \quad R > 1. \tag{26}$$

By hypothesis, $Q(z) = z^n\overline{P(1/\overline{z})}$, therefore for every $R > 1$ and $0 \leqslant \theta < 2\pi$, we have

$$|Q(Re^{i\theta}) - Q(e^{i\theta})| = |R^n e^{in\theta}\overline{P(e^{i\theta}/R)} - e^{in\theta}\overline{P(e^{i\theta})}| = |R^n P(e^{i\theta}/R) - P(e^{i\theta})|.$$

This in conjuction with (26) yields

$$k^\mu |P(Rz) - P(z)| \leqslant |Q(Rz) - Q(z)| \quad \text{for} \quad |z| = 1 \text{ and } R > 1,$$

which is inequality (20) and this completes the proof of Lemma 3.

LEMMA 4. *If $P \in \mathbb{P}_{n,1}$ and $P(z)$ has all its zeros in $|z| \leqslant t$ where $t \leqslant 1$, then*

$$|P(Rz) - P(z)| \geqslant \frac{(R^n - 1)}{t^n} m(P,t) \quad \textit{for} \quad |z| = 1 \textit{ and } R \geqslant 1.$$

Proof of Lemma 4. By hypothesis, all the zeros of $P(z)$ lie in $|z| \leqslant t$ where $|t| \leqslant 1$, therefore, the polynomial $G(z) = P(tz)$ has all its zeros in $|z| \leqslant 1$. Now

$$m(P,t) = \min_{|z|=t} |P(z)| = \min_{|z|=1} |G(z)| = m(G,1)$$

so that

$$m(G,1)\,|z^n| \leqslant |G(z)| \quad \text{for} \quad |z| = 1.$$

We first show that the polynomial $H(z) = G(z) - \alpha m(G,1)z^n$ has all its zeros in $|z| \leqslant 1$ for every real and complex number α with $|\alpha| < 1$. This is clear if $m(G,1) = 0$. Henceforth we assume $G(z)$ has all its zeros in $|z| < 1$, then $m(G,1) > 0$ and it follows by Rouche's theorem that the polynomial $H(z) = G(z) - \alpha m(G,1)z^n$ of degree n has all its zeros in $|z| < 1$ for every real and complex number α with $|\alpha| < 1$. Applying Lemma 2 to the polynomial $H(z)$, we get

$$|H(Rz)| > |H(z)| \quad \text{for } |z| \geqslant 1 \text{ and } R > 1.$$

This implies

$$H(Rz) - H(z) \neq 0 \quad \text{for } |z| \geqslant 1 \text{ and } R > 1,$$

or equivalently, for every real and complex number α with $|\alpha| < 1$,

$$\big(G(Rz) - \alpha m(G,1)R^n z^n\big) - \big(G(z) - \alpha m(G,1)z^n\big) \neq 0 \quad \text{for } |z| \geqslant 1 \text{ and } R > 1.$$

This gives for every α with $|\alpha| < 1$, the polynomial

$$T(z) = \big(G(Rz) - G(z)\big) + \alpha(R^n - 1)m(G,1)z^n \neq 0 \quad \text{for } |z| \geqslant 1 \text{ and } R > 1, \tag{28}$$

from which we conclude that

$$|G(Rz) - G(z)| \geqslant (R^n - 1)m(G,1)\,|z^n| \quad \text{for } |z| \geqslant 1 \text{ and } R > 1.$$

Because if this inequality is not true, then there is a point $z = z_0$ with $|z_0| \geqslant 1$ such that

$$|G(Rz_0) - G(z_0)| < (R^n - 1)\,m(G,1)\,|z_0|^n, \quad R > 1.$$

We choose

$$\alpha = \frac{G(Rz_0) - G(z_0)}{(R^n - 1)\,m(G,1)z_0^n},$$

so that $|\alpha| < 1$ and with this choice of α, $T(z_0) = 0$, $|z_0| \geqslant 1$, which is contradiction to (28). Hence

$$|G(Rz) - G(z)| \geqslant (R^n - 1)m(G,1)\,|z^n| \quad \text{for } |z| \geqslant 1 \text{ and } R > 1.$$

Replacing $G(z)$ by $P(tz)$, we get

$$|P(Rtz) - P(tz)| \geqslant (R^n - 1)m(P,t)\,|z^n| \quad \text{for } |z| \geqslant 1 \text{ and } R > 1.$$

Taking $z = e^{i\theta}/t$, $t \leqslant 1$, $0 \leqslant \theta < 2\pi$, we obtain for every $R > 1$,

$$|P(Re^{i\theta}) - P(e^{i\theta})| \geqslant \frac{(R^n - 1)}{t^n}\,m(P,t),$$

which is equivalent to desired result and this completes the proof of Lemma 4.

LEMMA 5. *If $P \in \mathbb{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for $|z| = 1$ and $R \geqslant 1$,*

$$k^{\mu}|P(Rz) - P(z)| \leqslant |Q(Rz) - Q(z)| - (R^n - 1)m(P,k). \tag{29}$$

Proof of Lemma 5. The result is trivial for $R = 1$, so we assume $R > 1$. By hypothesis, $P(z)$ has all its zeros in $|z| \geqslant k \geqslant 1$ and $m(P,k) = \min_{|z|=k} |P(z)|$, therefore

$$m(P,k) \leqslant |P(z)| \quad \text{for} \quad |z| = k. \tag{30}$$

We first show that for every real or complex number δ with $|\delta| \leqslant 1$, the polynomial $h(z) = P(z) - \delta m(P,k)$ has all its zeros in $|z| \geqslant k \geqslant 1$. This is clear if $P(z)$ has a zero on $|z| = k$, for then $m(P,k) = 0$ and $h(z) = P(z)$. In case $P(z)$ has no zero on $|z| = k$, then clearly $m(P,k) > 0$. Since $m/P(z)$ is not a constant, by the Maximum Modulus Principle, it follows from (30) that

$$m(P,k) < |P(z)| \quad \text{for} \quad |z| < k. \tag{31}$$

Now if $h(z)$ has a zero in $|z| < k$, say at $z = z_0$ with $|z_0| < k$, then $P(z_0) - \delta m(P,k) = h(z_0) = 0$. This implies

$$|P(z_0)| = |\delta m(P,k)| \leqslant m(P,k) \quad \text{where } |z_0| < k,$$

which contradicts (31) and hence in any case $h(z) = P(z) - \delta m(P,k)$ has all its zeros in $|z| \geqslant k \geqslant 1$ for every real or complex number δ with $|\delta| \leqslant 1$. Applying Lemma 3 to the polynomial $h(z) = P(z) - \delta m(P,k)$ we get for every real or complex number δ with $|\delta| \leqslant 1$,

$$k^{\mu}|P(Rz) - P(z)| \leqslant |Q(Rz) - Q(z) - \delta(R^n - 1)m(P,k)| \qquad \text{for} \quad |z| = 1 \quad \text{and} \quad R > 1. \tag{32}$$

Since all the zeros of $Q(z)$ lie in $|z| \leqslant (1/k) \leqslant 1$ it follows by Lemma 4 (with $P(z)$ replaced by $Q(z)$ and t replaced by $1/k$) that

$$|Q(Rz) - Q(z)| \geqslant (R^n - 1)k^n \min_{|z|=1/k} |Q(z)|.$$

But

$$\begin{aligned} m(Q,1/k) &= \min_{|z|=1/k} |Q(z)| = \min_{|z|=1/k} |z^n\overline{P(1/\overline{z})}| \\ &= \min_{|z|=1}\left|\frac{z^n}{k^n}\overline{P(k/\overline{z})}\right| = \frac{1}{k^n}\min_{|z|=1} |P(kz)| \\ &= \frac{1}{k^n}\min_{|z|=k} |P(z)| = \frac{1}{k^n}m(P,k), \end{aligned}$$

therefore, we have

$$|Q(Rz) - Q(z)| \geqslant (R^n - 1)m(P,k) \quad \text{for } |z| = 1 \text{ and } R > 1. \tag{33}$$

Choosing now argument of δ with $|\delta| = 1$ on the right hand side of (32) such that for $|z| = 1$ and $R > 1$,

$$|Q(Rz) - Q(z) - \delta(R^n - 1)m(P,k)| = |Q(Rz) - Q(z)| - (R^n - 1)m(P,k)$$

(which is possible by (33)), we conclude that

$$k^{\mu}|P(Rz) - P(z)| \leqslant |Q(Rz) - Q(z)| - (R^n - 1)m(P,k)| \quad \text{for } |z| = 1 \text{ and } R > 1,$$

which is equivalent to (29) and this proves Lemma 5.

Next we describe a result of Arestov.

For $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, we define

$$\Lambda_\gamma P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leqslant 1\}$,

(ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geqslant 1\}$.

The result of Arestov may now be stated as follows.

LEMMA 6. [1, Theorem 4] *Let $\phi(x) = \psi(\log x)$ where ψ is a convex nondecreasing function on $\mathbb{R}$. Then for all polynomials $P(z)$ of degree at most n and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \phi\big(|\Lambda_\gamma P(e^{i\theta})|\big)\, d\theta \leqslant \int_0^{2\pi} \phi\big(C(\gamma,n)\,|P(e^{i\theta})|\big)\, d\theta$$

where $C(\gamma,n) = \max(|\gamma_0|, |\gamma_n|)$.

In particular, Lemma 6 applies with $\phi : x \to x^q$ for every $q \in (0,\infty)$. Therefore, we have

$$\|\Lambda_\gamma P\|_q \leqslant C(\gamma,n)\|P\|_q, \quad 0 < q < \infty. \tag{34}$$

From Lemma 6, we deduce the following result which is also of independent interest.

LEMMA 7. *If $P \in \mathbb{P}_{n,1}$ and $P(z)$ does not vanish in $|z| < 1$, then for each $q > 0$, $R \geqslant 1$ and α real,*

$$\|(P(Rz) - P(z)) + e^{i\alpha}(R^n P(z/R) - P(z))\|_q \leqslant (R^n - 1)\|P\|_q. \tag{35}$$

The result is best possible and equality holds for $P(z) = \lambda z^n + \mu$, $|\lambda| = |\mu|$.

Proof of Lemma 7. The result is trivial for $R = 1$. Henceforth we assume $R > 1$. First we show that for every $R > 1$ and α real, all the zeros of polynomial

$$R(z) = \sum_{j=0}^{n} \binom{n}{j}\{(R^j - 1) + e^{i\alpha}(R^{n-j} - 1)\}z^j$$

lie on the unit circle. Let

$$H(z) = \sum_{j=0}^{n} \binom{n}{j} (R^j - 1) z^j = (Rz+1)^n - (z+1)^n.$$

The zeros z_ν, $\nu = 1, 2, \dots, n$ of $H(z)$ are given by

$$z_\nu = \frac{1 - e^{(2\nu\pi i/n)}}{e^{(2\nu\pi i/n)} - R}.$$

Since $R > 1$, it can be easily seen that $|z_\nu| < 1$, $\nu = 1, 2, \dots, n$. Hence all the zeros of $H(z)$ lie in $|z| < 1$ for every $R > 1$. If now

$$G(z) = z^n \overline{H(1/\overline{z})} = z^n H(1/z) = \sum_{j=0}^{n} \binom{n}{j} (R^j - 1) z^{n-j} = \sum_{j=0}^{n} \binom{n}{j} (R^{n-j} - 1) z^j,$$

then all the zeros of $G(z)$ lie in $|z| > 1$ and it follows that (see [15, Prob. 26, P.108]) the polynomial

$$R(z) = H(z) + e^{i\alpha} z^n \overline{H(1/\overline{z})} = \sum_{j=0}^{n} \binom{n}{j} \{(R^j - 1) + e^{i\alpha}(R^{n-j} - 1)\} z^j$$

has all its zeros on the circle $|z| = 1$ for every $R > 1$ and α real. Now by hypothesis $P(z)$ has all its zeros in $|z| \geqslant 1$ therefore, by Szegö's convolution theorem [20],

$$\begin{aligned}\Lambda P(z) &= (P(Rz) - P(z)) + e^{i\alpha}(R^n P(z/R) - P(z)) \\ &= (R^n - 1) a_n z^n + \{(R^{n-1} - 1) + e^{i\alpha}(R - 1)\} a_{n-1} z^{n-1} + \dots \\ &\quad + \{(R - 1) + e^{i\alpha}(R^{n-1} - 1)\} a_1 z + (R^n - 1) a_0,\end{aligned}$$

does not vanish in $|z| < 1$ for every $R > 1$ and α real. Therefore, Λ is an admissible operator. Hence by (34), we obtain for each $q > 0$, $R > 1$ and α real,

$$\int_0^{2\pi} \left| \left(P(Re^{i\theta}) - P(e^{i\theta})\right) + e^{i\alpha}\left(R^n P(e^{i\theta}/R) - P(e^{i\theta})\right) \right|^q d\theta \leqslant (R^n - 1)^q \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta,$$

which is equivalent to the desired result and this completes the proof of Lemma 7.

LEMMA 8. *If A, B, C are non-negative real numbers such that $B + C \leqslant A$ then for every real number α,*

$$|(A - C)e^{i\alpha} + (B + C)| \leqslant |Ae^{i\alpha} + B|.$$

Proof of Lemma 8. If $C = 0$, then Lemma 8 is obvious, so we suppose that $C > 0$. Since $\cos\alpha \leqslant 1$ for every real α and by hypothesis $(A - B - C) \geqslant 0$, it follows that

$$(A - B - C)\cos\alpha \leqslant (A - B - C). \tag{36}$$

Multiplying both sides of (36) by $2C$ and noting that $C > 0$, we get

$$\{2C(A-B)-2C^2\}\cos\alpha \leqslant 2C(A-B-C),$$

or equivalently,

$$2\{C(A-B)-C^2\}\cos\alpha + 2C^2 - 2C(A-B) \leqslant 0. \tag{37}$$

Adding $A^2+B^2+2AB\cos\alpha$ to the both sides of (37) and rearranging the terms, we obtain

$$(A^2-2AC+C^2)+(B^2+2BC+C^2)+2(A-C)(B+C)\cos\alpha \leqslant A^2+B^2+2AB\cos\alpha,$$

which implies

$$|(A-C)e^{i\alpha}+(B+C)|^2 \leqslant |Ae^{i\alpha}+B|^2$$

and hence

$$|(A-C)e^{i\alpha}+(B+C)| \leqslant |Ae^{i\alpha}+B|$$

for every real α. This computes the proof of Lemma 8.

Proof of Theorem 1. Using Lemma 1, we have for every $q \geqslant 1$, $R \geqslant 1$ and α real

$$\left|\operatorname{Re}\left\{e^{i\alpha}\left(P(Re^{i\theta})-P(e^{i\theta})\right)\right\}\right|^q = \left|\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,\alpha)\operatorname{Re}\left\{P\left(e^{i(\theta+\frac{k\pi+\alpha}{n})}\right)\right\}\right|^q. \tag{38}$$

Integrating both sides of (38) with respect to θ from 0 to 2π, we get

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\alpha}\left(P(Re^{i\theta})-P(e^{i\theta})\right)\right\}\right|^q d\theta = \int_0^{2\pi}\left|\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,\alpha)\operatorname{Re}\left\{P\left(e^{i(\theta+\frac{k\pi+\alpha}{n})}\right)\right\}\right|^q d\theta. \tag{39}$$

Let $\gamma(\theta)$ be the argument of $P(Re^{i\theta})-P(e^{i\theta})$, then from (39), we have by Minkowski's inequality for every $q \geqslant 1$, $R \geqslant 1$ and α real

$$\begin{aligned}\left\{\int_0^{2\pi}\left|\left(P(Re^{i\theta})-P(e^{i\theta})\right)\right|^q\left|\operatorname{Re}\left(e^{i(\alpha+\gamma(\theta))}\right)\right|^q d\theta\right\}^{1/q} &\leqslant \frac{1}{2n}\sum_{k=1}^{2n}A_k(R,\alpha)\left\{\int_0^{2\pi}\left|\operatorname{Re}\left\{P\left(e^{i(\theta+\frac{k\pi+\alpha}{n})}\right)\right\}\right|^q d\theta\right\}^{1/q}\\ &= \left\{\frac{1}{2n}\sum_{k=1}^{2n}A_k(R,\alpha)\right\}\left\{\int_0^{2\pi}\left|\operatorname{Re}\left(P(e^{i\theta})\right)\right|^q d\theta\right\}^{1/q}.\end{aligned}$$

This, with the help of (15), gives

$$\int_0^{2\pi} \left|\big(P(Re^{i\theta}) - P(e^{i\theta})\big)\right|^q |\cos(\alpha+\gamma(\theta))|^q \, d\theta \leqslant (R^n-1)^q \int_0^{2\pi} \left|\mathrm{Re}\big(P(e^{i\theta})\big)\right|^q d\theta.$$

Integrating this inequality both sides with respect to α from 0 to 2π, we obtain

$$\int_0^{2\pi}\int_0^{2\pi} \left|\big(P(Re^{i\theta}) - P(e^{i\theta})\big)\right|^q |\cos(\alpha+\gamma(\theta))|^q \, d\alpha \, d\theta \leqslant 2\pi(R^n-1)^q \int_0^{2\pi} \left|\mathrm{Re}\big(P(e^{i\theta})\big)\right|^q d\theta$$

which gives for every $q \geqslant 1$, $R \geqslant 1$ and α real,

$$\int_0^{2\pi} \left\{ \left|\big(P(Re^{i\theta}) - P(e^{i\theta})\big)\right|^q \int_0^{2\pi} \left| \frac{e^{2i(\alpha+\gamma(\theta))}+1}{2} \right|^q d\alpha \right\} d\theta \leqslant 2\pi(R^n-1)^q \int_0^{2\pi} \left|\mathrm{Re}\big(P(e^{i\theta})\big)\right|^q d\theta. \quad (40)$$

Using a well-known property of definite integrals, it follows from (40) that

$$\int_0^{2\pi} \left|\big(P(Re^{i\theta}) - P(e^{i\theta})\big)\right|^q d\theta \int_0^{2\pi} \left| \frac{e^{in\theta}+1}{2} \right|^q d\theta \leqslant 2\pi(R^n-1)^q \int_0^{2\pi} \left|\mathrm{Re}\big(P(e^{i\theta})\big)\right|^q d\theta,$$

from which the desired result follows immediately.

Proof of Theorem 2. This follows by taking $\mu = 1$ in Theorem 3.

Proof of Theorem 3. By hypothesis $P \in \mathbb{P}_{n,\mu}$ and $P(z) \neq 0$ for $|z| < k$ where $k \geqslant 1$, therefore, by Lemma 5, for each θ, $0 \leqslant \theta < 2\pi$, $q > 0$ and $R \geqslant 1$, we have

$$k^\mu |P(Re^{i\theta}) - P(e^{i\theta})| \leqslant |R^n P(e^{i\theta}/R) - P(e^{i\theta})| - (R^n-1)m(P,k).$$

This implies

$$k^\mu \left\{ |P(Re^{i\theta}) - P(e^{i\theta})| + \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k) \right\} \leqslant |R^n P(e^{i\theta}/R) - P(e^{i\theta})| - \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k). \quad (41)$$

Taking

$$A = |R^n P(e^{i\theta}/R) - P(e^{i\theta})|, \quad B = |P(Re^{i\theta}) - P(e^{i\theta})| \quad \text{and} \quad C = \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k)$$

in Lemma 8 and noting by (41) that for $k \geqslant 1$,

$$B + C \leqslant k^\mu (B + C) \leqslant (A - C) \leqslant A,$$

we get for every real α,

$$\begin{aligned}\Big|\Big\{|R^nP(e^{i\theta}/R) - P(e^{i\theta})| - \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k)\Big\}e^{i\alpha} \\ + \Big\{|P(Re^{i\theta}) - P(e^{i\theta})| + \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k)\Big\}\Big| \\ \leqslant \big||R^nP(e^{i\theta}/R) - P(e^{i\theta})|e^{i\alpha} + |P(Re^{i\theta}) - P(e^{i\theta})|\big|.\end{aligned}$$

This implies for each $q > 0$,

$$\int_0^{2\pi} |F(\theta)+e^{i\alpha}G(\theta)|^q \, d\theta \leqslant \int_0^{2\pi} \big||R^nP(e^{i\theta}/R)-P(e^{i\theta})|e^{i\alpha}+|P(Re^{i\theta})-P(e^{i\theta})|\big|^q \, d\theta, \quad (42)$$

where

$$F(\theta) = |P(Re^{i\theta}) - P(e^{i\theta})| + \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k)$$

and

$$G(\theta) = |R^nP(e^{i\theta}/R) - P(e^{i\theta})| - \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k).$$

Integrating both sides of (42) with respect to α from 0 to 2π, we get with the help of Lemma 7, for each $q > 0$, $R \geqslant 1$ and α real,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q \, d\theta \, d\alpha \\ &\leqslant \int_0^{2\pi}\int_0^{2\pi} \big||R^nP(e^{i\theta}/R) - P(e^{i\theta})|e^{i\alpha} + |P(Re^{i\theta}) - P(e^{i\theta})|\big|^q \, d\alpha \, d\theta \\ &= \int_0^{2\pi}\Big\{\int_0^{2\pi} \big||R^nP(e^{i\theta}/R) - P(e^{i\theta})|e^{i\alpha} + |P(Re^{i\theta}) - P(e^{i\theta})|\big|^q \, d\alpha\Big\} d\theta \\ &= \int_0^{2\pi}\Big\{\int_0^{2\pi} \big|(R^nP(e^{i\theta}/R) - P(e^{i\theta}))e^{i\alpha} + (P(Re^{i\theta}) - P(e^{i\theta}))\big|^q \, d\alpha\Big\} d\theta \\ &= \int_0^{2\pi}\Big\{\int_0^{2\pi} \big|(R^nP(e^{i\theta}/R) - P(e^{i\theta}))e^{i\alpha} + (P(Re^{i\theta}) - P(e^{i\theta}))\big|^q \, d\theta\Big\} d\alpha \\ &\leqslant (R^n-1)^q \int_0^{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \, d\alpha = 2\pi(R^n-1)^q \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta.\end{aligned} \quad (43)$$

Now for every real α and $t \geqslant t_0 \geqslant 1$, we have $|t+e^{i\alpha}| \geqslant |t_0+e^{i\alpha}|$, which implies

$$\int_0^{2\pi} |t+e^{i\alpha}|^q\,d\alpha \geqslant \int_0^{2\pi} |t_0+e^{i\alpha}|^q\,d\alpha, \quad q>0.$$

If $F(\theta) \neq 0$, we take $t = |G(\theta)|/|F(\theta)|$ and $t_0 = k^\mu$, $1 \leqslant \mu \leqslant n$, then by (41), $t \geqslant t_0 \geqslant 1$ and we get

$$\begin{aligned}\int_0^{2\pi} |F(\theta)+e^{i\alpha}G(\theta)|^q\,d\theta &= |F(\theta)|^q \int_0^{2\pi} \left|1+\frac{G(\theta)}{F(\theta)}e^{i\alpha}\right|^q d\alpha \\ &= |F(\theta)|^q \int_0^{2\pi} \left|\frac{G(\theta)}{F(\theta)}+e^{i\alpha}\right|^q d\alpha \\ &= |F(\theta)|^q \int_0^{2\pi} \left|\left|\frac{G(\theta)}{F(\theta)}\right|+e^{i\alpha}\right|^q d\alpha \\ &\geqslant |F(\theta)|^q \int_0^{2\pi} |k^\mu+e^{i\alpha}|^q\,d\alpha \\ &= \left\{|P(Re^{i\theta})-P(e^{i\theta})| + \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k)\right\}^q \int_0^{2\pi} |k^\mu+e^{i\alpha}|^q\,d\alpha.\end{aligned}$$

For $F(\theta)=0$, this inequality is trivially true. Using this in (43), we conclude that for each $q>0$, $R \geqslant 1$ and α real,

$$\int_0^{2\pi} |k^\mu+e^{i\alpha}|^q\,d\alpha \int_0^{2\pi} \left\{|P(Re^{i\theta})-P(e^{i\theta})| + \Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k)\right\}^q d\theta \leqslant 2\pi(R^n-1)^q \int_0^{2\pi} |P(e^{i\theta})|^q\,d\theta.$$

This gives for every real or complex number β with $|\beta| \leqslant 1$, $q>0$, $R \geqslant 1$ and α real,

$$\int_0^{2\pi} |k^\mu+e^{i\alpha}|^q\,d\alpha \int_0^{2\pi} \left|P(Re^{i\theta})-P(e^{i\theta}) + \beta\Big(\frac{R^n-1}{1+k^\mu}\Big)m(P,k)\right|^q d\theta \leqslant 2\pi(R^n-1)^q \int_0^{2\pi} |P(e^{i\theta})|^q\,d\theta,$$

which immediately leads to (12) and this completes the Proof of Theorem 3.

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(Received August 14, 1997)

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