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A GENERALIZATION OF PÓLYA'S INEQUALITY TO STOLARSKY AND GINI MEANS

C. E. M. PEARCE, J. PEČARIĆ AND J. ŠUNDE

(*communicated by Zs. Páles*)

Abstract. Pólya's inequality has recently been extended in a general way involving two geometric means. We show that further extensions to Stolarsky and Gini means are possible. The two occurrences of the mean in each inequality can involve different parameters. Corresponding discrete results are also derived.

1. Introduction

A celebrated inequality due to Pólya [8] states the following result.

THEOREM A. *If* $f : [0,1] \to \mathbf{R}$ *is a nonnegative and nondecreasing function, then*

$$\left(\int_0^1 x^{a+b} f(x)dx\right)^2 \geqslant \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^1 x^{2a} f(x)dx \int_0^1 x^{2b} f(x)dx.$$

The geometric mean $G(x,y)$ of nonnegative real numbers x, y is defined by $G(x,y) := (xy)^{1/2}$. The Pólya result can be expressed in terms of this as

$$\int_0^1 \left[\frac{d}{dx} G\left(x^{2a+1}, x^{2b+1}\right)\right] f(x)dx$$
$$\geqslant G\left(\int_0^1 \left(\frac{d}{dx} x^{2a+1}\right) f(x)dx, \int_0^1 \left(\frac{d}{dx} x^{2b+1}\right) f(x)dx\right).$$

This suggests the possibility of more general results. This idea has been taken up by Alzer [1], who derived the following.

THEOREM B. *Let* $f, g, h : [a,b] \to \mathbf{R}$ *be nonnegative, increasing functions such that* g, h *and* $\sqrt{gh}$ *are continuously differentiable on* $[a,b]$. *If* $g(a) = h(a)$ *and* $g(b) = h(b)$, *then*

$$\left(\int_a^b \left(G\left(g(x), h(x)\right)\right)' f(x)dx\right) \geqslant G\left(\int_a^b g'(x)f(x)dx, \int_a^b h'(x)f(x)dx\right).$$

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In this article we show that this result holds not only for geometric means but for two very general classes of means, the Stolarsky and Gini means. These include many commonly used means as particular cases. For instance, consider the Stolarsky mean $E_{r,s}(x,y)$. As noted by Leach and Sholander [4], this includes the arithmetic mean $A(x,y) := (x+y)/2$ for $(r,s) = (1,2)$, the geometric mean for $(r,s) = (0,0)$, the harmonic mean for $(r,s) = (-2,-1)$, the logarithmic mean for $(r,s) = (0,1)$, the identric mean for $(r,s) = (1,1)$, the power mean for $s = 2r$ and Galvani's mean for $s = 1$. We remark further that $G(x,y) = E_{r,-r}(x,y)$ for every real number r.

In Section 2 we generalize Theorem B to Stolarsky means and in Section 3 we present a corresponding discrete result. Sections 4 introduces the necessary preliminaries for comparable results for Gini means. These are then presented in Section 5.

2. A Stolarsky theorem, continuous version

For r, s real numbers and x, y positive numbers, the Stolarsky mean $E_{r,s}(x,y)$ is defined by $E_{r,s}(x,y) = x$ for $x = y$, and for $x \neq y$ by

$$E_{r,s}(x,y) = \begin{cases} \left(\dfrac{s(x^r - y^r)}{r(x^s - y^s)}\right)^{\frac{1}{r-s}} & \text{if } \ rs(r-s) \neq 0, \\ \left(\dfrac{x^r - y^r}{r(\ln x - \ln y)}\right)^{1/r} & \text{if } \ r \neq 0,\ s = 0 \\ \left(\dfrac{s(\ln x - \ln y)}{x^s - y^s}\right)^{-1/s} & \text{if } \ s \neq 0,\ r = 0 \\ e^{-1/r}\left(\dfrac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r - y^r}} & \text{if } \ r = s \neq 0 \\ \sqrt{xy} & \text{if } \ r = s = 0. \end{cases}$$

A fundamental question is when is it the case that

$$E_{r,s}(x,y) \leqslant E_{u,v}(x,y) \tag{2.1}$$

for all positive and distinct x, y?

This question has been solved by Leach and Sholander [4]. (See also Páles [6], which treats a more general question that subsumes this problem, and Páles [7], where a unified treatment of the comparison of Stolarsky and Gini means is given.) For clarity, we expand and reword their enunciation slightly.

LEMMA C. *Let $r,\ s,\ u,\ v$ be real numbers with $r \neq s$ and $u \neq v$.*

(*a*) *If either $0 \leqslant \min(r,s,u,v)$ or $\max(r,s,u,v) \leqslant 0$, then (2.1) holds for all distinct positive x, y if and only if*

$$r + s \leqslant u + v$$

and

$$e(r,s) \leqslant e(u,v),$$

where

$$e(\alpha,\beta) = \begin{cases} (\alpha-\beta)/\ln(\alpha/\beta), & \textit{for} \quad \alpha\beta > 0,\ \alpha \neq \beta \\ 0, & \textit{if} \quad \alpha\beta = 0. \end{cases} \tag{2.2}$$

(b) If $\min(r,s,u,v) < 0 < \max(r,s,u,v)$*, then (2.1) holds for all distinct positive* x, y *if and only if*

$$r + s \leqslant u + v$$

and

$$e(r,s) \leqslant e(u,v),$$

where

$$e(\alpha,\beta) = (|\alpha| - |\beta|)/(\alpha-\beta) \quad \textit{for} \quad \alpha \neq \beta. \tag{2.3}$$

First we use this result to engender a basic lemma that will be used to derive the main results of both this and the next section. We define sets $A, A^\star$ by

$$A = \{(r,s) | r+s \leqslant 3 \quad \text{and} \quad e(r,s) \leqslant e(1,2)\}$$

and

$$A^\star = \{(r,s) | r+s \geqslant 3 \quad \text{and} \quad e(r,s) \geqslant e(1,2)\} \tag{2.4}$$

(see Figure 1), where e is defined by (2.2) if $r,s \geqslant 0$ and by (2.3) if $\min(r,s) < 0$. In particular,

$$e(1,2) = \begin{cases} 1/\ln 2, & \text{if} \quad \min(r,s) \geqslant 0; \\ 1, & \text{if} \quad \min(r,s) < 0. \end{cases}$$

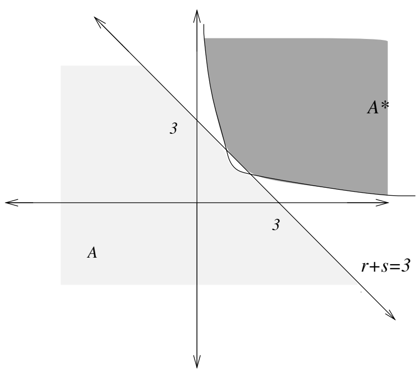

Figure 1.

We now establish our basic lemma.

LEMMA 2.1. *Let* r, s *be real numbers. If* $(r,s) \in A$ *then*

$$E_{r,s}(x,y) \leqslant E_{1,2}(x,y), \tag{2.5}$$

while if $(r,s) \in A^\star$*, then*

$$E_{r,s}(x,y) \geqslant E_{1,2}(x,y). \tag{2.6}$$

Proof. The first fact is immediate from Lemma C. We note that $\max(r,s,1,2) \leqslant 0$ cannot occur and so e is given by (2.2) if $r,s \geqslant 0$ and by (2.3) if $\min(r,s) < 0$. The second part follows similarly. □

REMARK 2.2. Lemma 2.1 is a generalization of the fact that $E_{r,s}(x,y)$ is a nondecreasing function of r and s. So (2.1) holds if $r \leqslant u$ and $s \leqslant v$. In particular, (2.5) holds if $r \leqslant 1, s \leqslant 2$ and (2.6) if $r \geqslant 1, s \geqslant 2$.

We now proceed to our first theorem.

THEOREM 2.3. *Suppose $g,h : [a,b] \to \mathbf{R}$ are positive, nondecreasing functions with continous first derivatives and $g(a) = h(a), g(b) = h(b)$.*

a) Let f be a nonnegative, nondecreasing, differentiable function on $[a,b]$. If (r,s), $(u,v) \in A$, then

$$E_{r,s}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \leqslant \int_a^b (E_{u,v}(g(t),h(t)))' f(t)dt. \tag{2.7}$$

If $(r,s) \in A^\star$ and $(u,v) \in A^\star$, the inequality is reversed.

b) Let f be a nonnegative, nonincreasing, differentiable function. If $(r,s) \in A$ and $(u,v) \in A^\star$, then (2.7) holds, while if $(r,s) \in A^\star$ and $(u,v) \in A$, the inequality is reversed.

Proof. $a)$ Suppose $(r,s),(u,v) \in A$. By Lemma 2.1

$$\begin{aligned}
&E_{r,s}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \\
&\quad \leqslant \frac{1}{2}\left(\int_a^b g'(t)f(t)dt + \int_a^b h'(t)f(t)dt\right) \\
&\quad = \frac{1}{2}(g(t)+h(t))f(t)\Big|_a^b - \int_a^b \frac{1}{2}(g(t)+h(t))\,df(t) \\
&\quad \leqslant \frac{1}{2}(g(t)+h(t))f(t)\Big|_a^b - \int_a^b E_{u,v}(g(t),h(t))\,df(t) \\
&\quad = \frac{1}{2}(g(t)+h(t))f(t)\Big|_a^b - E_{u,v}(g(t),h(t))f(t)\Big|_a^b \\
&\qquad\quad + \int_a^b (E_{u,v}(g(t),h(t)))' f(t)dt \\
&\quad = \int_a^b (E_{u,v}(g(t),h(t)))' f(t)dt.
\end{aligned}$$

If $(r,s),(u,v) \in A^\star$, we have trivially that the inequality is reversed.

b) Suppose $(r,s) \in A$ and $(u,v) \in A^\star$. Put $F = -f$. Then we have

$$\begin{aligned}
&E_{r,s}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \\
&\quad \leqslant \int_a^b \frac{1}{2}(g(t)+h(t))' f(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(g(t)+h(t)\right)f(t)\Big|_a^b + \int_a^b \left(g(t)+h(t)\right)dF(t) \\
&\leqslant \frac{1}{2}\left(g(t)+h(t)\right)f(t)\Big|_a^b + \int_a^b E_{u,v}\left(g(t),h(t)\right)dF(t) \\
&= \frac{1}{2}\left(g(t)+h(t)\right)f(t)\Big|_a^b - E_{u,v}\left(g(t),h(t)\right)f(t)\Big|_a^b \\
&\qquad + \int_a^b \left(E_{u,v}\left(g(t),h(t)\right)\right)' f(t)dt \\
&= \int_a^b \left(E_{u,v}\left(g(t),h(t)\right)\right)' f(t)dt.
\end{aligned}
$$

If $(r,s)\in A^\star$ and $(u,v)\in A$, the inequality is clearly reversed. □

COROLLARY 2.4. *Let g,h be defined as in Theorem 2.3.*

a) Let f be a nonnegative, nondecreasing, differentiable function on $[a,b]$. If $r,u \leqslant 1$ and $s,v \leqslant 2$, then (2.7) holds. If $r,u \geqslant 1$ and $s,v \geqslant 2$, then (2.7) is reversed.

b) Let f be a nonnegative, nonincreasing, differentiable function on $[a,b]$. If $r \leqslant 1 \leqslant u$ and $s \leqslant 2 \leqslant v$ then (2.7) holds. If $u \leqslant 1 \leqslant r$ and $v \leqslant 2 \leqslant s$, then (2.7) is reversed.

Proof. This follows from Theorem 2.3 and Remark 2.2. □

3. A Stolarsky theorem, discrete version

It is convenient to introduce the notation $\Delta a_i = a_{i+1} - a_i$.

THEOREM 3.1. *Suppose a and b are positive, nondecreasing n-tuples $(n \geqslant 2)$ such that $a_n = b_n$ and $a_1 = b_1$.*

a) Let w be a nonnegative, nondecreasing n-tuple. If $(r,s),(u,v)\in A$, then

$$
E_{r,s}\left(\sum_{j=1}^{n-1} w_j\Delta a_j, \sum_{j=1}^{n-1} w_j\Delta b_j\right) \leqslant \sum_{j=1}^{n-1} w_j \Delta E_{u,v}(a_j,b_j), \tag{3.1}
$$

while if $(r,s),(u,v)\in A^\star$, the inequality is reversed.

b) Let w be a nonnegative, nonincreasing n-tuple $(n \geqslant 2)$. If $(r,s)\in A$ and $(u,v)\in A^\star$, then (3.1) holds. If $(r,s)\in A^\star$ and $(u,v)\in A$, the inequality is reversed.

Proof. *a)* Let $(r,s),(u,v)\in A$. We have

$$
\begin{aligned}
&E_{r,s}\left(\sum_{j=1}^{n-1} w_j\Delta a_j, \sum_{j=1}^{n-1} w_j\Delta b_j\right) \\
&\qquad \leqslant E_{1,2}\left(\sum_{j=1}^{n-1} w_j\Delta a_i, \sum_{i=1}^{n-1} w_i\Delta b_i\right) = \frac{1}{2}\left\{\sum_{i=1}^{n-1} w_i\Delta a_i + \sum_{i=1}^{n-1} w_i\Delta b_i\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} - \sum_{i=2}^{n} \frac{a_i+b_i}{2} \Delta w_{i-1} \\
&\leqslant w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} - \sum_{i=2}^{n} E_{u,v}(a_i,b_i) \Delta w_{i-1} \\
&= w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} \\
&\qquad - \left\{ w_n E_{u,v}(a_n,b_n) - w_1 E_{u,v}(a_1,b_1) - \sum_{i=1}^{n-1} \Delta E_{u,v}(a_i,b_i) w_i \right\} \\
&= \sum_{i=1}^{n-1} w_i \Delta E_{u,v}(a_i,b_i).
\end{aligned}
$$

If $(r,s),(u,v) \in A^\star$, the inequality is clearly reversed.

b) Let $(r,s) \in A$ and $(u,v) \in A^\star$. Set $W_i = -w_i \ \ (i-1,\dots,n-1)$. We have

$$
\begin{aligned}
E_{r,s}&\left(\sum_{i=1}^{n-1} w_i \Delta a_i,\ \sum_{i=1}^{n-1} w_i \Delta b_i \right) \\
&\leqslant \sum_{i=1}^{n-1} w_i \Delta \left(\frac{a_i+b_i}{2} \right) \\
&= w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} + \sum_{i=2}^{n} w_i \frac{a_i+b_i}{2} \Delta W_{i-1} \\
&\leqslant w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} + \sum_{i=2}^{n} E_{u,v}(a_i,b_i) \Delta W_{i-1} \\
&= w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} \\
&\qquad - \left(w_n E_{u,v}(a_n+b_n) - w_1 E_{u,v}(a_1,b_1) - \sum_{i=1}^{n-1} \Delta E_{u,v}(a_i,b_i) w_i \right) \\
&= \sum_{i=1}^{n-1} w_i \Delta E_{u,v}(a_i,b_i).
\end{aligned}
$$

If $(r,s) \in A^\star$ and $(u,v) \in A$, the inequality is clearly reversed. □

As before, we can make the following deduction.

COROLLARY 3.2. *Suppose n–tuples a and b are as in Theorem 1.1.*

a) *Let w be a nonnegative, nondecreasing n-tuple. If $r,u \leqslant 1$ and $s,v \leqslant 2$, then* (3.1) *holds. If $r,u \geqslant 1$ and $s,v \geqslant 2$, then* (3.1) *is reversed.*

b) *Let w be a nonnegative, nonincreasing n–tuple. If $r \leqslant 1 \leqslant u$ and $s \leqslant 2 \leqslant v$, then* (3.1) *holds. If $r \geqslant 1 \geqslant u$ and $s \geqslant 2 \geqslant v$, then* (3.1) *is reversed.*

4. Gini means – preliminaries

Let $r, s \in \mathbf{R}$ be real numbers. The Gini mean of a positive n–vector $\mathbf{x} = (x_1, \ldots, x_n)$ with weights $\mathbf{w} = (w_1, \ldots, w_n)$ with coordinates in $\mathbf{R} = (0, \infty)$ is defined by

$$G_{r,s}(\mathbf{x};\mathbf{w}) = G_{r,s}(x_1, \ldots, x_n; \mathbf{w}) = \begin{cases} \left(\dfrac{w_1x_1^r + \ldots + w_nx_n^r}{w_1x_1^s + \ldots + w_nx_n^s}\right)^{\frac{1}{r-s}}, & \text{if} \quad r \neq s, \\ \exp\left(\dfrac{w_1x_1^r \ln x_1 + \ldots + w_nx_n^r \ln x_n}{x_1^r + \ldots + x_n^r}\right), & \text{if} \quad r = s, \end{cases} \tag{4.1}$$

(see Gini [3]). If $\mathbf{w} = (1, \ldots, 1)$ we write $G_{r,s}(\mathbf{x};\mathbf{w}) = G_{r,s}(\mathbf{x})$. We remark that $G_{r,s} = G_{s,r}$.

A comparison theorem of Daróczy and Losonczi [2] provides the basis for our arguments. See also Páles [7]. The comparison result of [2] may be expressed as follows.

PROPOSITION D. *Let r, s, u, v be real numbers. Then in order that*

$$G_{r,s}(\mathbf{x}) \leqslant G_{u,v}(\mathbf{x}) \tag{4.2}$$

hold for all $n \in N$ and $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n > 0$, it is necessary and sufficient that

$$\min(r,s) \leqslant \min(u,v) \quad \textit{and} \quad \max(r,s) \leqslant \max(u,v). \tag{4.3}$$

A simple consequence is as follows.

LEMMA 4.1. *Let r, s, u, v be real numbers satisfying (4.3). If n–vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ have all positive coordinates, then*

$$G_{r,s}(\mathbf{x};\mathbf{w}) \leqslant G_{u,v}(\mathbf{x};\mathbf{w}). \tag{4.4}$$

Proof. Replication of values x_i extends Proposition D to give (4.4) with positive integer weights. As the quotients in (4.1) are not changed by multiplication of numerator and denominator by the same factor, the result therefore further extends to positive rational weights. A limiting argument concludes the proof. □

The case $n = 2$ in (4.2) is of special interest. The following result is due to Páles [5].

LEMMA E. *Let r, s, u, v be arbitrary real numbers such that $r \neq s$ and $u \neq v$. We have the following.*

(*a*) *If $0 \leqslant \min(r,s,u,v)$, then a necessary and sufficient condition for*

$$G_{r,s}(x,y) \leqslant G_{u,v}(x,y) \quad \textit{for all positive} \quad x, y \tag{4.5}$$

is that

$$r + s \leqslant u + v \quad \textit{and} \quad m(r,s) \leqslant m(u,v), \tag{4.6}$$

where

$$m(\alpha,\beta) = \min(\alpha,\beta).$$

(*b*) *If* $\min(r,s,u,v) < 0 < \max(r,s,u,v)$, *then a necessary and sufficient condition for* (4.5) *is that* (4.6) *applies, where*

$$m(\alpha,\beta) = (|\alpha| - |\beta|)/(\alpha - \beta).$$

(*c*) *If* $\max(r,s,u,v) \leqslant 0$, *then a necessary and sufficient condition for* (4.5) *is that* (4.6) *applies, where*

$$m(\alpha,\beta) = \max(\alpha,\beta).$$

We shall consider the two special cases

$$G_{r,s}(x,y) \leqslant G_{0,1}(x,y) = A(x,y) \tag{4.7}$$

and

$$G_{r,s}(x,y) \geqslant G_{0,1}(x,y) = A(x,y). \tag{4.8}$$

Suppose, without loss of generality, that $r < s$. For (4.7) we set $u = 0$, $v = 1$ in Lemma E. From (4.6) we get

$$r + s \leqslant 1 \quad \text{and} \quad m(r,s) \leqslant m(0,1). \tag{4.9}$$

As $\max(r,s,0,1)$ cannot be $\leqslant 0$, we have only cases (a) and (b) in the definition of m.

Since $0 \leqslant \min(r,s,0,1)$ is equivalent to $r \geqslant 0$, we have

$$m(r,s) = r \quad \text{and} \quad m(0,1) = 0.$$

Applying this to (4.9) we get $r = 0$ and $s \leqslant 1$.

Similarly $\min(r,s,0,1) < 0 < \max(r,s,0,1)$ is equivalent to $r < 0$. Then

$$m(r,s) = \frac{|s| - |r|}{s - r} \quad \text{and} \quad m(0,1) = 1.$$

Using this in (4.9) we have $|s| - |r| \leqslant s - r$. If $s \geqslant 0$, this becomes $r \leqslant 0$. For $s < 0$, it becomes $s - r \geqslant 0$, which is obviously true.

We thus have that (4.7) holds in the case $r < s$ if $r \leqslant 0$ and $r + s \leqslant 1$. By symmetry, (4.7) holds if $(r,s) \in B$, where

$$B = \{(r,s) | r + s \leqslant 1 \quad \wedge \quad (r \leqslant 0 \vee s \leqslant 0)\} \tag{4.10}$$

(see Figure 2).

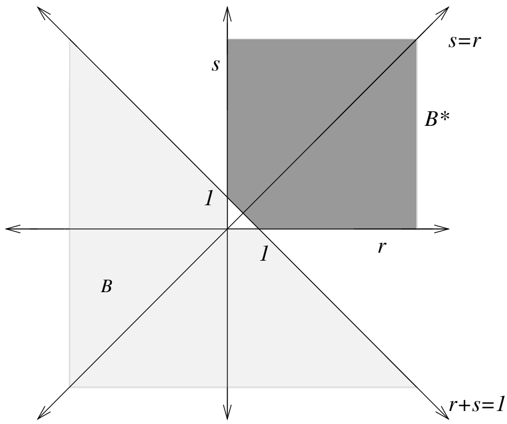

Figure 2.

Now consider (4.8). Take $u \leqslant v$ and put $r = 0$, $s = 1$, $u = r$, $v = s$ in (4.5). Then (4.6) becomes

$$r + s \geqslant 1 \quad \text{and} \quad m(r,s) \geqslant m(0,1), \tag{4.11}$$

where m is now defined by

$$m(\alpha,\beta) = \begin{cases} \min(\alpha,\beta) & \text{if} \quad r \geqslant 0 \\ \dfrac{|\alpha| - |\beta|}{\alpha - \beta}, & \text{if} \quad r < 0. \end{cases}$$

So for $r \geqslant 0$, (4.11) becomes $r \geqslant 0$, while for $r < 0$ we have $|s| - |r| \geqslant s - r$. The latter provides a contradiction both for $s \geqslant 0$ (which gives $r \geqslant 0$) and for $s < 0$ (which gives $s \leqslant r$).

Thus we have that (4.8) holds when $r < s$ if $r + s \geqslant 1$ and $r \geqslant 0$ applies. By symmetry, (4.8) holds if $(r,s) \in B^\star$, where

$$B^\star = \{(r,s) | r + s \geqslant 1, r \geqslant 0, s \geqslant 0\} \tag{4.12}$$

(see Figure 2).

Therefore we have the following special case of Lemma E.

LEMMA 4.2. *If* $(r,s) \in B$, *where B is defined by* (4.10), *then* (4.7) *holds, while if* $(r,s) \in B^\star$, *where* $B^\star$ *is defined by* (4.12), *then* (4.8) *applies.*

The following lemma, which compares $G_{r,s}(\mathbf{x};\mathbf{w})$ with

$$G_{0,1}(\mathbf{x};\mathbf{w}) = \frac{w_1x_1 + \ldots + w_nx_n}{w_1 + \ldots + w_n} \quad (:= A(\mathbf{x};\mathbf{w})),$$

is a simple consequence of Lemma 4.1.

LEMMA 4.3. (a) *Suppose* $(r,s) \in C$, *where* C *is defined by*

$$C = \{(r,s) | (r \leqslant 0 \wedge s \leqslant 1) \vee (r \leqslant 1 \wedge s \leqslant 0)\}$$

(see Figure 3). Then

$$G_{r,s}(\mathbf{x};\mathbf{w}) \leqslant G_{0,1}(\mathbf{x};\mathbf{w}).$$

(b) *Suppose* $(r,s) \in B^{\star}$*, where*

$$C^{\star} = \{(r,s)|(r \geqslant 0 \wedge s \geqslant 1) \vee (r \geqslant 1 \wedge s \geqslant 0)\}.$$

Then

$$G_{r,s}(\mathbf{x};\mathbf{w}) \geqslant G_{0,1}(\mathbf{x};\mathbf{w}).$$

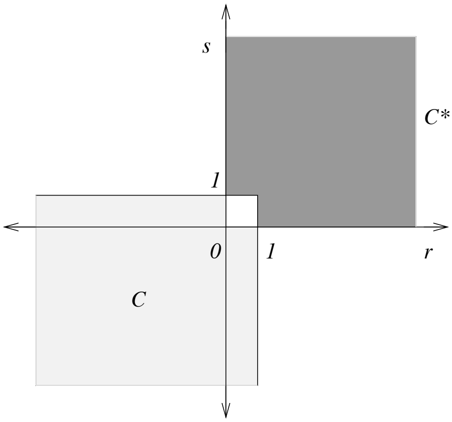

Figure 3.

5. Results involving Gini means

THEOREM 5.1. *Let* $g_1,\ldots,g_n : [a,b] \to \mathbf{R}$ *be positive, nondecreasing functions with continuous first derivatives and* $g_1(a) = \ldots = g_n(a), g_1(b) = \ldots = g_n(b)$*. Suppose* $\mathbf{w}$ *is a positive* n*-tuple.*

$a)$ *Let* f *be a nonegative, nondecreasing function on* $[a,b]$*. If* $(r,s),(u,v) \in B$*, then*

$$G_{r,s}\left(\int_a^b g_1'(t)f(t)dt,\ldots,\int_a^b g_n'(t)f(t)dt;\mathbf{w}\right) \leqslant \int_a^b (G_{u,v}(g_1(t),\ldots,g_n(t);\mathbf{w}))' f(t)dt. \tag{5.1}$$

If $(r,s),(u,v) \in B^{\star}$*, then the reverse inequality holds.*

$b)$ *Let* f *be a nonnegative, nonincreasing function. If* $(r,s) \in B$ *and* $(u,v) \in B^{\star}$*, then* (5.1) *holds, while if* $(r,s) \in B^{\star}$ *and* $(u,v) \in B$ *then the reverse inequality applies.*

The proof is the same as to that of Theorem 2.3, except in that we use Lemma 4.3 in place of Lemma 2.1.

In particular, we have the following.

COROLLARY 5.2. *Let g and h be positive, nondecreasing functions with continuous first derivative and $g(a) = h(a), g(b) = h(b)$.*

a) Let f be a nonnegative, nondecreasing function on $[a,b]$. If $(r,s),(u,v) \in B$, then

$$G_{r,s}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \leqslant \int_a^b G_{u,v}\left(g(t),h(t)\right)' f(t)dt. \tag{5.2}$$

If $(r,s),(u,v) \in B^\star$, then the reverse inequality holds.

b) Let f be a nonnegative, nonincreasing function. If $(r,s) \in B$ and $(u,v) \in B^\star$ then (5.2) holds, while if $(r,s) \in B^\star$ and $(u,v) \in B$, then the reverse inequality applies.

Similarly we can establish the following discrete analogues of the above results. We introduce the notation $\Delta a_{j,i} = a_{j,i+1} - a_{j,i}$.

THEOREM 5.3. *Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be positive, nondecreasing n–tuples such that $a_{1,1} = \ldots = a_{m,1}$ and $a_{1,n} = \ldots = a_{m,n}$ and let $\mathbf{w}$ be a positive n–tuple.*

a) Suppose $\mathbf{f}$ is a nonnegative, nondecreasing n–tuple. If $(r,s),(u,v) \in C$, then

$$G_{r,s}\left(\sum_{i=1}^{n-1} f_i \Delta a_{1,i}, \ldots, \sum_{i=1}^{n-1} f_i \Delta a_{m,i}; \mathbf{w}\right) \leqslant \sum_{i=1}^{n-1} f_i \Delta G_{u,v}(a_{1,i}, \ldots, a_{m,i}; \mathbf{w}). \tag{5.3}$$

If $(r,s),(u,v) \in C^\star$, then the reverse inequality holds.

b) Suppose $\mathbf{f}$ is a nonnegative nonincreasing n–tuple. If $(r,s) \in C$ and $(u,v) \in C^\star$ then (5.3) applies. If $(r,s) \in C^\star$ and $(u,v) \in C$, then the inequality is reversed.

COROLLARY 5.4. *Let a and b be positive, nondecreasing n–tuples such that $a_n = b_n$ and $a_1 = b_1$.*

a) Suppose $\mathbf{f}$ is a nonnegative, nondecreasing n–tuple. If $(r,s),(u,v) \in B$, then

$$G_{r,s}\left(\sum_{i=1}^{n-1} f_i \Delta a_i, \sum_{i=1}^{n-1} f_i \Delta b_i\right) \leqslant \sum_{i=1}^{n-1} f_i \Delta G_{u,v}(a_i, b_i). \tag{5.4}$$

If $(r,s),(u,v) \in B^\star$, then the inequality is reversed.

b) Suppose $\mathbf{f}$ is a nonegative, nondecreasing n–tuple. If $(r,s) \in B$ and $(u,v) \in B^\star$, then (5.4) applies. If $(r,s) \in B^\star$ and $(u,v) \in B$, then the reverse inequality holds.

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Charles E. M. Pearce
Department of Mathematics
University of Adelaide
Adelaide, SA 5005, Australia

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia

Jadranka Šunde
Department of Mathematics
University of Adelaide
Adelaide, SA 5005, Australia