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CERTAIN OPEN PROBLEMS OF QUASICONFORMAL THEORY RELATED TO INEQUALITIES

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Abstract. A list of the old and new problems of the quasiconformal theory related to the stability theorems is given.

1. Some estimates in the distortion theorems for the spatial quasiconformal mappings

We define the Grötzsch ring as the condenser in the following way:

$$R_{G,n}(r) = B(0,1) \setminus \{te_1 : 0 \leqslant t \leqslant r\}, \quad r \in (0,1),$$

where $B(0,1)$ is the unit ball in Euclidean space $\mathbf{R}^n$, $e_1 = (1,0,\dots,0)$.

The modulus of the Grötzsch ring $\Phi_n(r) = \operatorname{mod} R_{G,n}(r)$ has the following properties:

1) the function $\eta(r) = \Phi_n(r) + \ln r$ is decreasing for $r \in (0,1)$;
2) the limit $\lim_{r\to 0} \eta(r) = \ln \lambda_n$ is finite.

It is proved by M. Vuorinen (see [16]–[17] and also [3]) that every quasiconformal mapping $f : B(0,1) \to B(0,1)$, $f(0) = 0$, satisfies the inequality

$$|f(x)| \leqslant \lambda_n^{1-\alpha}|x|^{\alpha},$$

where $\alpha = (K(f))^{\frac{1}{1-n}}$, $K(f) = \max\{K_0(f), K_1(f)\}$ and $K_0(f)$, $K_1(f)$ are the outer and inner coefficients of quasiconformality (see [15]).

Since $\alpha \geqslant \dfrac{1}{K_0(f)}$, $|x| < 1$ and $\lambda_n \geqslant 4$ (see [1], [5], [15]–[17]) it follows that

$$|f(x)| \leqslant \lambda_n^{1-\frac{1}{K_0(f)}}|x|^{\frac{1}{K_0(f)}}. \tag{1}$$

Some interesting estimates are given also in [2]–[3].

Now, we use the infinitesimal idea. Let $u : \overline{B(0,1)} \to \mathbf{R}^n$ be a quasiconformal deformation such that

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1) $u(0) = 0$;
2) the scalar product $(u(x), x) \leqslant 0$ on the sphere $|x| = 1$;
3) for every unit vector v the scalar product $(Qu(x)v, v) \leqslant \frac{1}{n}$ almost everywhere.

Here the elements of the matrix $Qu(x)$ are defined by

$$q_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) - \frac{\delta_{ij}}{n}\operatorname{div} u, \quad i, j = 1, 2, \ldots, n.$$

Every such deformation generates the semiflow $\{\varphi_t\}_{t\geqslant 0}$ of quasiconformal mappings of the unit ball $\overline{B(0,1)}$ into itself. Since the outer coefficient satisfies the inequality $K_0(\varphi_t) \leqslant e^t$ and $\varphi_t(x) = x + tu(x) + o(t)$ we obtain from (1) the estimate

$$(u(x), x) \leqslant |x|^2 \ln \frac{\lambda_n}{|x|}. \tag{2}$$

QUESTION 1. Is the constant λ_n the best in the inequality (1)? Is this constant the best in the inequality (2)?

At least, it is true if the dimension $n = 2$. It is very useful to compare the exact constants in these inequalities when the dimension $n > 2$. If these constants will be different then we shall give the negative answer to Gehring's superposition problem for quasiconformal mappings of the unit ball onto itself.

One of the ways can be following. Every spatial quasiconformal deformation has an integral representation:

$$u(x) = a + \alpha \cdot x + Kx - 2(b, x)x + |x|^2 b + J(Qu)(x), \tag{3}$$

where a, $b \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$, K is a skew-symmetric transformation, J is an integral operator with a weak singularity (see [10]). For our calculations we take the integral representation from [12] which gives an estimate. From condition 2) on the unit sphere $|x| = 1$ we have the inequality:

$$\alpha - (b, x) \leqslant \big(J(Qu)(0) - J(Qu)(x), x\big), \tag{4}$$

since $u(0) = 0$.

Then for all x, $|x| = r$, from (3)–(4) we obtain

$$\begin{aligned}(u(x), x) &= \big(J(Qu)(x) - J(Qu)(0), x\big) + (\alpha - (b, x))r^2 \\ &\leqslant \big(J(Qu)(x) - J(Qu)(0), x\big) + \alpha(1 - r)r^2 \\ &\quad + \Big((Qu)\Big(\frac{x}{r}\Big) - J(Qu)(0), x\Big)r^2.\end{aligned} \tag{5}$$

Integrating (4) over the unit sphere we get an upper bound for the constant α, namely

$$\alpha \leqslant -\frac{1}{n\sigma_n}\int_{|x|=1} (J(Qu)(x), x)dS. \tag{6}$$

In (4) and (6) we have the equalities if and only if the scalar product $(u(x), x) = 0$ for all x, $|x| = 1$. Therefore we have to consider only such deformations.

QUESTION 2. Let f be an arbitrary K-quasiconformal mapping of the unit disc $|z| \leqslant 1$ onto itself and $f(0) = 0$. What is the best constant γ in the inequality

$$|f(z) - f(a)| \leqslant \gamma^{1-\frac{1}{K}} |z - a|^{\frac{1}{K}}?$$

This problem was considered by O. Lehto and K. Virtanen in [8, p. 68]. The upper bound for the constant γ is given by many authors. The last estimate $\gamma \leqslant 64$ is proved in [4]. The conjecture is that $\gamma = 16$. For its proof or disproof it is sufficient to find the sharp estimate for the integral

$$I(r) = \frac{r^2}{2\pi} \int_{|\zeta| \leqslant 1} \left| \frac{1}{(\zeta - z)(\zeta - z)} + \frac{\overline{a} + \overline{z} - \overline{az}\zeta}{(1 - \overline{z}\zeta)(1 - \overline{a}\zeta)\zeta} \right| d\xi \, d\eta$$

where $r = |z-a|$, $\zeta = \xi + i\eta$. The upper estimate has to be such $I(r) \leqslant r^2 \ln \frac{\gamma}{r}$. The appearance of this integral is reminiscent of plane quasiconformal deformations (see [13] for details). It is important to estimate the scalar product $(u(z) - u(a), z - a)$ over all vector fields $u : \overline{B(0,1)} \to \mathbf{C}$ such that $|u_{\overline{z}}| \leqslant 0.5$, $u(0) = 0$ and $(u(z), z) = 0$ if $|z| = 1$. Then the exact upper bound for the scalar product is equal to $I(r)$.

2. Stability of quasiconformal mappings and the maximal deviation

Let us consider the space $\mathbf{R}^n$, $n \geqslant 3$. We define the function of the maximal deviation by the equality

$$\mu(\varepsilon) = \sup_f \inf_\varphi \sup_{|x| \leqslant 1} |\varphi \circ f(x) - x|$$

where the supremum is taken over all $(1+\varepsilon)$-quasiregular mappings of the unit ball and the infimum is taken over the group of Möbius transformations. Now, we choose only the outer coefficient. The global result belongs to Ju. Reshetnyak [11]. It concludes in particular way that the function $\mu(\varepsilon) = O(\varepsilon)$ if $\varepsilon \to 0$ (the stability theorem without the estimates of derivatives). In other words there exist constants k_n, ε_0 such that $\mu(\varepsilon) \leqslant k_n \varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$.

QUESTION 3. What is the best constant k_n? (Lavrent'iev's problem.) What is the best constant ε_0? (Reshetnyak's problem.)

Why the Question 3 is interesting? In fact, the function μ satisfies conditions: $\mu = \infty$ or $0 \leqslant \mu \leqslant 1$. If $\mu(\varepsilon) < 1$ then every spatial $1 + \varepsilon$-quasiregular mapping has no branch points. Such mappings are homeomorphic on an arbitrary ball from a domain of definition (see [14]). Therefore, the radius of injectivity (see definition in [9]) of these mappings is equal to one.

QUESTION 4. Is the function $\mu = \mu(\varepsilon)$ semiadditive? (A function $\lambda : (a, \infty) \to \overline{\mathbf{R}}$ is called semiadditive function if $\lambda(t+s) \leqslant \lambda(t) + \lambda(s)$ for every $t, s \in (a, \infty)$. The properties of such functions are given in [6, chapter VII]).

Question 4 is important because an estimate for $\lim_{\varepsilon \to 0} \dfrac{\mu(\varepsilon)}{\varepsilon}$ (see [12]) is known. The intermediate step would be to study the monotonicity of the function $\ln \mu(\varepsilon) - \ln \varepsilon$.

I think that the maximal deviation is closely related to the old problem that is connected with the density of the image of a ball under a quasiconformal mapping in space. There exists a quasiconformal mapping $f : B(0,1) \to \mathbf{R}^n$ such that $\overline{f(B(0,1))} = \mathbf{R}^n$. Obviously, this is true if and only if

$$\mu(K(f) - 1) = \infty. \tag{7}$$

QUESTION 5. What is the value $\inf\{K(f)\}$, where the infimum is taken over all quasiconformal mappings $f : B(0,1) \to \mathbf{R}^n$ satisfying the condition (7)?

3. Extension of spatial quasiconformal mappings

Now, we don't consider the origin of this question. For small values of ε A. Kopylov proved (see [7]) that every $1+\varepsilon$-quasiconformal mapping $f : B(0,1) \to \mathbf{R}^n$ can be extended onto the exterior of the unit ball. We denote the quasiconformal extension by F. In the inequality $K_0(F) \leqslant 1 + c\varepsilon$ the value of c is not found in [7].

QUESTION 6. What is the smallest constant c in this inequality?

We shall put this question more simply by considering quasiconformal deformations. It is clear that every quasiconformal deformation of a unit ball can be extended onto the space $\mathbf{R}^n$, $n \geqslant 3$. Let U be an extension of u.

QUESTION 7. What is the smallest constant c in the inequality

$$(QU(x)v, v) \leqslant \frac{c}{n} \quad \text{if} \quad (Qu(x)v, v) \leqslant \frac{1}{n}$$

a.e. for every vector v?

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