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OPERATOR FUNCTIONS ASSOCIATED WITH THE GRAND FURUTA INEQUALITY

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Abstract. We discuss the monotonicity of operator functions associated with the grand Furuta inequality, some of which are considered under the chaotic order $\log A \geqslant \log B$. In some restricted cases, several known operator inequalities related to the Furuta inequality will appear as corollaries of our results.

1. Introduction

A capital letter means a (bounded linear) operator on a Hilbert space H. An operator T is said to be positive (in symbol : $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$. Also an operator T is denoted by $T > 0$ if T is positive and invertible.

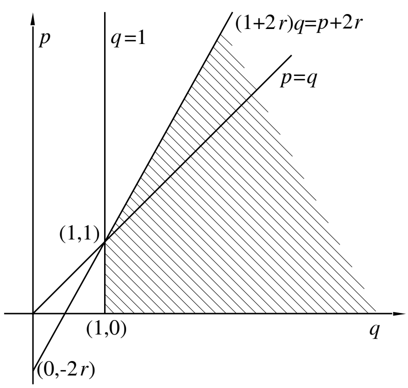
First of all, we begin with the Furuta inequality [14], one of the greatest developments in operator inequalites and a historical extension of the Löwner-Heinz inequality.

THE FURUTA INEQUALITY. *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

$$(A^r A^p A^r)^{1/q} \geqslant (A^r B^p A^r)^{1/q} \qquad (1)$$

holds for p and q such that $p \geqslant 0$ and $q \geqslant 1$ with

$$(1+2r)q \geqslant p+2r. \qquad (*)$$



The domain $(*)$ is expressed in the right.

Alternative proofs of it are given in [2,15,23] and also one page proof is shown in [16]. The best possibility of the conditions on p, q and r in the Furuta inequality

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is discussed in [27]. Furthermore it gave us many nice applications and became deeper: (I) It has been discussed as the monotonicity of operator functions associated with itself. (II) An attempt to extend the domain in which it holds has been done by Yoshino [29], Tanahashi [28], Furuta and ourselves [5,6,7,24]. The former is initiated by Furuta himself [17,18,19] and in succession is discussed in [3,4,11,12,13].

We now have to mention the grand Furuta inequality [19,13] as a top of such examples related to (I) and (II) above. It interpolates the Furuta inequality with the Ando-Hiai inequality [1] as extremal cases $t=0$ and $t=1$ with $r=s$.

THE GRAND FURUTA INEQUALITY. *If $A \geqslant B \geqslant 0$ and A is invertible, then for each $p \geqslant 1$ and $t \in [0,1]$,*

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-r/2} \tag{2}$$

is a decreasing function of both $r \geqslant t$ and $s \geqslant 1$.

Very recently, Furuta [20] proposed parallelism related to his inequality in the frame of the grand Furuta inequality, which covers our results [6,7] contained in (II) above. Motivated by this, we consider the monotonicity of the operator functions associated with the grand Furuta inequality. These operator functions are expressed by using the α-power mean $\sharp_\alpha$, a typical operator mean in the Kubo-Ando theory [25]. For instance, the following result will be shown:

If $\log A \geqslant \log B$ for $A,B>0$, then for each $t \leqslant 0$, $p \geqslant 0$ and $q \in \mathbb{R}$

$$F(r,s) = A^{-r} \sharp_{\frac{q+r}{(p-t)s+r}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant \max\{0,-q\}$ and $s \geqslant \max\{0, \frac{q}{p-t}\}$.

Note that the assumption $\log A \geqslant \log B$ for $A,B>0$ is weaker than $A \geqslant B > 0$. This order is called the chaotic order, cf. [4].

2. Parallelism by Furuta

For the sake of convenience, we recall the Löwner-Heinz inequality [21,26]:

$$A \geqslant B \geqslant 0 \text{ ensures } A^\alpha \geqslant B^\alpha \text{ for all } \alpha \in [0,1]. \tag{3}$$

Now the Löwner-Heinz inequality (3) corresponds to the α-power mean $\sharp_\alpha$ by virtue of the Kubo-Ando theory [25] on operator means as follows :

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$$

for $0 \leqslant \alpha \leqslant 1$ and $A,B \geqslant 0$. For convenience, $\natural_s$ for $s \in \mathbb{R}$ is defined by

$$A \natural_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}$$

for all $A,B>0$, as in [19].

As stated in [23,24], the Furuta inequality is understood as follows: If $A \geqslant B \geqslant 0$, then for each $t \leqslant 0$,

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \leqslant B \leqslant A \tag{1'}$$

holds for $p \geqslant 1$.

In [29], Yoshino initiated an attempt to extend the domain in which the Furuta inequality holds. In succession, we proved in [24]:

If $A \geqslant B > 0$, $0 \leqslant t < p$ and $\frac{1}{2} \leqslant p \leqslant 1$, then

$$A^t \natural_{\frac{1-t}{p-t}} B^p \leqslant B \leqslant A. \tag{4}$$

If $A \geqslant B > 0$ and $0 \leqslant t < p \leqslant \frac{1}{2}$, then

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \leqslant B^{2p} \leqslant A^{2p}. \tag{5}$$

Based on (4) and (5), we discussed the monotonicity of operator functions related to (4) and (5) in [6,7]. Very recently, Furuta [20] proposed parallelism related to these inequalities in the frame of the grand Furuta inequality, which consists of three theorems. We want to note that they clarifies the utility of the grand Furuta inequality. We rephrase them in terms of α-power mean:

THEOREM A. *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \leqslant 0$, $\alpha \in [0,1]$ and $p \geqslant 1$*

$$F(r,s) = A^{-r} \sharp_{\frac{(1-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s \tag{6}$$

is a decreasing function of both $r \geqslant 0$ and s such that $(p-t)s \geqslant (1-t)\alpha$.

THEOREM B. *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each p and t such that $p > t \geqslant 0$, $1 \geqslant p \geqslant \frac{1}{2}$ and $\alpha \in [0,1]$,*

$$G(r,s) = A^{-r} \sharp_{\frac{(1-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s \tag{7}$$

is a decreasing function of both $r \geqslant 0$ and s such that $(p-t)s \geqslant (1-t)\alpha$.

THEOREM C. *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each p and t such that $\frac{1}{2} \geqslant p > t \geqslant 0$, and $\alpha \in [0,1]$,*

$$H(r,s) = A^{-r} \sharp_{\frac{(2p-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s \tag{8}$$

is a decreasing function of both $r \geqslant 0$ and s such that $(p-t)s \geqslant (2p-t)\alpha$.

By the use of operator mean, we make clear their proofs a little bit. The following fact is basic to prove them.

THEOREM D. [17,3] *If* $A \geqslant B > 0$*, then*

$$f_1(p,r) = A^{-r} \natural_{\frac{1+r}{p+r}} B^p$$

is a decreasing function of both $p \geqslant 1$ *and* $r \geqslant 0$.

Proof of Theorem A. First of all, we note that $A \natural_{\alpha\beta} B = A \natural_\alpha (A \natural_\beta B)$. Then it follows from the Furuta inequality $(1')$ that

$$A^t \natural_{\frac{(1-t)\alpha}{p-t}} B^p = A^t \natural_\alpha (A^t \natural_{\frac{1-t}{p-t}} B^p) \leqslant A^t \natural_\alpha A = A^{(1-t)\alpha+t},$$

so that

$$B_1 = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{(1-t)\alpha}{p-t}} \leqslant A^{(1-t)\alpha} = A_1.$$

Since

$$\Big(1 + \frac{r}{(1-t)\alpha}\Big)/\Big(\frac{(p-t)s}{(1-t)\alpha} + \frac{r}{(1-t)\alpha}\Big) = \frac{(1-t)\alpha + r}{(p-t)s + r},$$

$(1')$ implies again that

$$\begin{aligned} F(r,s) &= A^{-r} \natural_{\frac{(1-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s \\ &= A_1^{-\frac{r}{(1-t)\alpha}} \natural_{\frac{(1-t)\alpha+r}{(p-t)s+r}} B_1^{\frac{(p-t)s}{(1-t)\alpha}} \leqslant A_1 = A^{(1-t)\alpha}. \end{aligned}$$

Hence Theorem D ensures that $F(r,s)$ is a decreasing function of both $\frac{r}{(1-t)\alpha} \geqslant 0$ and $\frac{(p-t)s}{(1-t)\alpha} \geqslant 1$, and so the statement is proved.

Theorems B and C are shown by similar way to Theorem A, in which $(1')$ must be replaces by (4) and (5) respectively.

3. Results

For $A, B > 0$, we denote by $A \gg B$ if $\log A \geqslant \log B$. Theorem D is improved in [11; Cor. 9] as follows :

THEOREM E. *If* $A \gg B$ *and* $q \in \mathbb{R}$*, then*

$$f_q(p,r) = A^{-r} \natural_{\frac{q+r}{p+r}} B^p$$

is a decreasing function of both $r \geqslant 0, p \geqslant 0$ *with* $-r \leqslant q \leqslant p$.

In the below, we use Theorem E instead of Theorem D; and another tool is the following Furuta's type characterization of the chaotic order, see [8,9,10]:

THEOREM F. *If* $A \gg B$, *then*

$$(A^r B^p A^r)^{\frac{1}{q}} \leqslant A^{\frac{p+2r}{q}}$$

for $r \geqslant 0, p \geqslant 0$ *and* $q \geqslant 1$ *with* $2rq \geqslant p+2r$.

By these tools, we generalize Theorem A under the chaotic order as follows:

THEOREM 1. *If* $A \gg B$, *then for each* $p \geqslant 0 \geqslant t$ *with* $p \neq t$ *and* $q \in \mathbb{R}$

$$E_q(r,s) = A^{-r} \sharp_{\frac{q+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant \max\{0, -q\}$ *and* $s \geqslant \max\{0, \frac{q}{p-t}\}$.

Proof. Putting $B_1 = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}}$ and $p_1 = (p-t)s$, we have

$$E_q(r,s) = A^{-r} \sharp_{\frac{q+r}{p_1+r}} B_1^{p_1}.$$

We here point out that $B_1 \ll A$. As a matter of fact, if $t=0$, then $B_1 = B \ll A$ by the assumption. In the case $t<0$, we can apply Theorem F to given p, $r = -\frac{t}{2}$ and $q = \frac{t-p}{t}$ because $2rq = p-t = p+2r$; hence we have

$$B_1^{-t} = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{-t}{p-t}} \leqslant A^{-t},$$

so that $B_1 \ll A$. Therefore it follows from Theorem E that $E_q(r,s) = A^{-r} \sharp_{\frac{q+r}{p_1+r}} B_1^{p_1}$ is decreasing for $r \geqslant 0$ and $p_1 \geqslant 0$ with $-r \leqslant q \leqslant p_1$. Since the monotonicity on p_1 is equivalent to that on s, we have the conclusion.

We take $q = (2p-t)\alpha$ or $q = (1-t)\alpha$; we have the following two corollaries and the latter is an extension of Theorem A, in which a given α is not restricted in $[0,1]$.

COROLLARY 1.1. *If* $A \gg B$, *then for each* $p \geqslant 0 \geqslant t$ *with* $p \neq t$ *and* $\alpha \geqslant 0$

$$F_{2p}(r,s) = A^{-r} \sharp_{\frac{(2p-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant 0$ *and* $s \geqslant \frac{(2p-t)\alpha}{p-t}$.

COROLLARY 1.2. *If* $A \gg B$, *then for each* $p \geqslant 0 \geqslant t$ *with* $p \neq t$ *and* $\alpha \geqslant 0$

$$F_1(r,s) = A^{-r} \sharp_{\frac{(1-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant 0$ *and* $s \geqslant \frac{(1-t)\alpha}{p-t}$.

Though Theorem 1 can be discussed under the chaotic order, we cannot do it in the following theorem because of the use of the Löwner-Heinz inequality. But the conclusion is the same as that of Theorem 1. In this sense, it is a variant of Theorem A.

THEOREM 2. *If $A \geqslant B > 0$, then for each t,p with $0 \leqslant t < p \leqslant 1$ and $q \in \mathbb{R}$*

$$E_q(r,s) = A^{-r} \natural_{\frac{q+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant \max\{0,-q\}$ and $s \geqslant \max\{0, \frac{q}{p-t}\}$.

Proof. As in the proof of Theorem 1, putting $B_1 = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}}$ and $p_1 = (p-t)s$, we have

$$E_q(r,s) = A^{-r} \natural_{\frac{q+r}{p_1+r}} B_1^{p_1}.$$

Since $p \in [0,1]$, we have $A^{p-t} \geqslant B_1^{p-t}$ and so $A \gg B_1$. Hence Theorem E implies the conclusion.

As in Theorem 1, we have the following corollaries which are actually extensions of Theorems C and B respectively.

COROLLARY 2.1. *If $A \geqslant B > 0$, then for each t,p such that $0 \leqslant t < p \leqslant 1$ and $\alpha \geqslant 0$*

$$F_{2p}(r,s) = A^{-r} \natural_{\frac{(2p-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant 0$ and $s \geqslant \frac{(2p-t)\alpha}{p-t}$.

COROLLARY 2.2. *If $A \geqslant B > 0$, then for each t,p such that $0 \leqslant t < p \leqslant 1$ and $\alpha \geqslant 0$*

$$F_1(r,s) = A^{-r} \natural_{\frac{(1-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant 0$ and $s \geqslant \frac{(1-t)\alpha}{p-t}$.

For the third result, we recall the following inequality which is a key in the grand Furuta inequality, see [13].

THEOREM G. *If $A \geqslant B > 0$, then*

$$(A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leqslant B \leqslant A$$

for $p \geqslant 1$, $s \geqslant 1$ and $t \in [0,1]$.

THEOREM 3. *If $A \geqslant B > 0$, then for each $t \neq p$ with $0 \leqslant t \leqslant 1 \leqslant p$ and $q \in \mathbb{R}$*

$$E_{q-t}(r,s) = A^{-r} \natural_{\frac{q-t+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant \max\{t, t-q\}$ and $s \geqslant \max\{1, \frac{q-t}{p-t}\}$.

Proof. We put $p_1 = (p-t)s+t$, $r_1 = r-t$ and $B_1 = (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}}$. Then we have $p_1 \geqslant p \geqslant 1$ by $s \geqslant 1$, $r_1 \geqslant 0$ and $B_1 \leqslant B \leqslant A$ by Theorem G. Hence it follows from Theorem E that

$$A^{-r_1} \natural_{\frac{q+r_1}{p_1+r_1}} B_1^{p_1}$$

is a decreasing function of both $r_1 \geqslant 0, p_1 \geqslant 0$ with $-r_1 \leqslant q \leqslant p_1$. Since

$$A^{-r_1} \natural_{\frac{q+r_1}{p_1+r_1}} B_1^{p_1} = A^{-r+t} \natural_{\frac{q-t+r}{(p-t)s+r}} (A^t \natural_s B^p) = A^{\frac{t}{2}} E_{q-t}(r,s) A^{\frac{t}{2}},$$

$E_{q-t}(r,s)$ is also a decreasing function of both $r \geqslant t$ and $s \geqslant 1$ with $t-r \leqslant q \leqslant (p-t)s+t$, so that r and s are taken over $r \geqslant \max\{t, t-q\}$ and $s \geqslant \max\{1, \frac{q-t}{p-t}\}$.

By taking either $q=(2p-t)\alpha+t$ or $q=(1-t)\alpha+t$ in Theorem 3, we have the following corollaries respectively.

COROLLARY 3.1. *If $A \geqslant B > 0$, then for each $t \neq p$ such that $0 \leqslant t \leqslant 1 \leqslant p$ and $\alpha \geqslant 0$*

$$F_{2p}(r,s) = A^{-r} \sharp_{\frac{(2p-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant t$ and $s \geqslant \max\{1, \frac{(2p-t)\alpha}{p-t}\}$.

COROLLARY 3.2. *If $A \geqslant B > 0$, then for each $t \neq p$ such that $0 \leqslant t \leqslant 1 \leqslant p$ and $\alpha \geqslant 0$*

$$F_1(r,s) = A^{-r} \sharp_{\frac{(1-t)\alpha+r}{(p-t)s+r}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s$$

is a decreasing function of both $r \geqslant t$ and $s \geqslant \max\{1, \frac{(1-t)\alpha}{p-t}\}$.

4. Applications

In this section, we give some applications of results in the preceding section, which are closely related to the Furuta inequality and extensions of our results in [6,7].

THEOREM 4. *Suppose that $A \geqslant B > 0$, $0 \leqslant t < p \leqslant 1$ and $\alpha \geqslant 0$. Then the following statements hold:*

$$f(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{(2p-t)\alpha+t}{\beta}} \tag{1}$$

is a decreasing function of $\beta \geqslant (2p-t)\alpha+t$, and in particular

$$f(\beta) \leqslant A^t \natural_{\frac{(2p-t)\alpha}{p-t}} B^p \quad \textit{for} \quad \beta \geqslant (2p-t)\alpha+t.$$

$$g(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{(1-t)\alpha+t}{\beta}} \tag{2}$$

is a decreasing function of $\beta \geqslant (1-t)\alpha+t$, and in particular

$$g(\beta) \leqslant A^t \natural_{\frac{(1-t)\alpha}{p-t}} B^p \quad \textit{for} \quad \beta \geqslant (1-t)\alpha+t.$$

Proof. We put $r=t$ in Corollary 2.1. Then it implies that

$$A^{\frac{t}{2}}F_{2p}(t,s)A^{\frac{t}{2}} = (A^t \natural_s B^p)^{\frac{(2p-t)\alpha+t}{(p-t)s+t}}$$

is a decreasing function of $s \geqslant \frac{(2p-t)\alpha}{(p-t)}$. Therefore, if we replace s by $\frac{\beta-t}{p-t}$ for $\beta \geqslant (2p-t)\alpha+t$, then

$$(A^t \natural_s B^p)^{\frac{(2p-t)\alpha+t}{(p-t)s+t}} = f(\beta)$$

and so $f(\beta)$ is decreasing for $\beta \geqslant (2p-t)\alpha + t$. Consequently, we have

$$f(\beta) \leqslant f((2p-t)\alpha+t) = A^t \natural_{\frac{(2p-t)\alpha}{(p-t)}} B^p$$

for all $\beta \geqslant (2p-t)\alpha + t$.

Similarly (2) is proved by using Corollary 2.2.

COROLLARY 4.1. *Suppose that* $A \geqslant B > 0$, $0 \leqslant t < p \leqslant 1$ *and* $\alpha \geqslant 0$. *Then the following inequalities hold:*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{2p}{\beta}} \leqslant A^t \natural_{\frac{2p-t}{p-t}} B^p \quad \text{for} \quad \beta \geqslant 2p \tag{1}$$

and in particular

$$(A^t \natural_{\frac{1-t}{p-t}} B^p)^{2p} \leqslant A^t \natural_{\frac{2p-t}{p-t}} B^p \quad \text{for} \quad 0 \leqslant t < p \leqslant \frac{1}{2}.$$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leqslant B^p \quad \text{for} \quad \beta \geqslant p \tag{2}$$

and in particular

$$(A^t \natural_{\frac{2p-t}{p-t}} B^p)^{\frac{1}{2}} \leqslant B^p \quad \text{and} \quad (A^t \natural_{\frac{1-t}{p-t}} B^p)^{p} \leqslant B^p \quad \text{for} \quad 0 \leqslant t < p \leqslant 1.$$

Proof. We have (1) (resp. (2)) by taking $\alpha = 1$ (resp. $\alpha = \frac{p-t}{2p-t}$) in Theorem 4 (1).

COROLLARY 4.2. *Suppose that* $A \geqslant B > 0$ *and* $0 \leqslant t < p \leqslant 1$. *Then*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leqslant A^t \natural_{\frac{1-t}{p-t}} B^p \quad \text{for} \quad \beta \geqslant 1$$

and in particular

$$(A^t \natural_{\frac{2p-t}{p-t}} B^p)^{\frac{1}{2p}} \leqslant A^t \natural_{\frac{1-t}{p-t}} B^p \quad \text{for} \quad \frac{1}{2} \leqslant p \leqslant 1.$$

Proof. We have the desired inequality by taking $\alpha = 1$ in Theorem 4 (2).

THEOREM 5. *Suppose that* $A \geqslant B > 0$, $0 \leqslant t \leqslant 1 \leqslant p$ $(t \neq p)$ *and* $\alpha \geqslant 0$. *Then the following statements hold:*

$$f(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{(2p-t)\alpha+t}{\beta}} \tag{1}$$

is a decreasing function of $\beta \geqslant \max\{p, (2p-t)\alpha + t\}$, *and in particular*

$$f(\beta) \leqslant A^t \natural_{\frac{(2p-t)\alpha}{p-t}} B^p \quad \text{for} \quad \beta \geqslant (2p-t)\alpha + t \geqslant p.$$

$$g(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{(1-t)\alpha+t}{\beta}} \tag{2}$$

is a decreasing function of $\beta \geqslant \max\{p, (1-t)\alpha + t\}$, *and in particular*

$$g(\beta) \leqslant A^t \natural_{\frac{(1-t)\alpha}{p-t}} B^p \quad \textit{for} \quad \beta \geqslant (1-t)\alpha + t \geqslant p.$$

Proof. We put $r = t$ in Corollary 3.1. Then it implies that

$$A^{\frac{t}{2}} F_{2p}(t,s) A^{\frac{t}{2}} = (A^t \natural_s B^p)^{\frac{(2p-t)\alpha+t}{(p-t)s+t}}$$

is a decreasing function of $s \geqslant \max\{1, \frac{(2p-t)\alpha}{p-t}\}$. Therefore, if we replace s by $\frac{\beta-t}{p-t}$ for $\beta \geqslant \max\{p, (2p-t)\alpha + t\}$, then

$$(A^t \natural_s B^p)^{\frac{(2p-t)\alpha+t}{(p-t)s+t}} = f(\beta)$$

and so $f(\beta)$ is decreasing for $\beta \geqslant \max\{p, (2p-t)\alpha + t\}$. Consequently, we have

$$f(\beta) \leqslant f((2p-t)\alpha + t) = A^t \natural_{\frac{(2p-t)\alpha}{(p-t)}} B^p$$

for all $\beta \geqslant (2p-t)\alpha + t$.

Similarly (2) is proved by using Corollary 3.2.

COROLLARY 5.1. *Suppose that* $A \geqslant B > 0$ *and* $0 \leqslant t \leqslant 1 \leqslant p$ $(t \neq p)$. *Then the following inequalities hold:*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{2p}{\beta}} \leqslant A^t \natural_{\frac{2p-t}{p-t}} B^p \quad \textit{for} \quad \beta \geqslant 2p. \tag{1}$$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leqslant B^p \quad \textit{for} \quad \beta \geqslant p, \tag{2}$$

and in particular

$$(A^t \natural_{\frac{2p-t}{p-t}} B^p)^{\frac{1}{2}} \leqslant B^p.$$

Proof. If we take $\alpha = 1$ in Theorem 5 (1), then

$$f(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{2p}{\beta}} \leqslant A^t \natural_{\frac{2p-t}{p-t}} B^p \quad \text{for} \quad \beta \geqslant 2p,$$

so that (1) is shown. Similarly we have (2) by taking $\alpha = \frac{p-t}{2p-t}$ in Theorem 5 (1).

COROLLARY 5.2. *Suppose that* $A \geqslant B > 0$, $0 \leqslant t \leqslant 1$ *and* $t < p$. *Then the following inequalities hold:*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leqslant (A^t \natural_{\frac{2p-t}{p-t}} B^p)^{\frac{1}{2}} \leqslant B^p \quad \textit{for} \quad \beta \geqslant 2p \tag{1}$$

and

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leqslant B^p \quad \textit{for} \quad \beta \geqslant p. \tag{2}$$

Proof. The former follows from Corollaries 4.1 (1) and 5.1 (1), i.e.,

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{2p}{\beta}} \leqslant A^t \natural_{\frac{2p-t}{p-t}} B^p \quad \text{for} \quad \beta \geqslant 2p.$$

Hence the Löwner-Heinz inequality ensures the conclusion (1) by the help of Corollary 5.1 (2). On the other hand, the latter (2) follows from Corollaries 4.1 (2) and 5.1 (2).

ADDENDUM. After we have written this paper, we are informed by Professor T. Furuta that an extension of Theorem 3 has been already obtained in the following; Takayuki Furuta, Takeaki Yamazaki and Masahiro Yanagida, *Operator functions implying generalized Furuta inequality*, preprint.

ADDED IN PROOF. It appears in this journal, **1** (1998), 123–130.

REFERENCES

[1] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequality*, Linear Algebra Appl., **197** (1994), 113-131.

[2] M. FUJII, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67-72.

[3] M. FUJII, T. FURUTA AND E. KAMEI, *Operator functions associated with Furuta's inequality*, Linear Algebra Appl., **149** (1991), 91-96.

[4] ———, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl., **179** (1993), 161-169.

[5] ———, *Complements to the Furuta inequality*, Proc. Japan Acad., **70** (1994), Ser. A ,239-242.

[6] ———, *Complements to the Furuta inequality, III*, Math. Japon., **45** (1997), 25-32.

[7] M. FUJII, J.-F. JIANG AND E. KAMEI, *Complements to the Furuta inequality, IV*, Math. Japon., **45** (1997), 511-518.

[8] ———, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc. **125** (1997), 3655–3658.

[9] ———, *Characterization of chaotic order and its application to Furuta's type operator inequalities*, Linear Multilinear Alg., to appear.

[10] M. FUJII, J.-F. JIANG, E. KAMEI AND K. TANAHASHI, *A characterization of chaotic order and a problem*, J. Ineq. Appl., to appear.

[11] M. FUJII, J.-F. JIANG AND E. KAMEI, *A geometrical structure in the Furuta inequality, II*, Nihonkai Math. J. **8** (1997), 37–46.

[12] M. FUJII AND E. KAMEI, *A geometrical structure in the Furuta inequality*, Math Japon., **43** (1996), 83-89; 389 (for Correction).

[13] ———, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 2751-2756.

[14] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0, p \geqslant 0, q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85-88.

[15] ———, *A proof via operator means of an order preserving inequality*, Linear Algebra Appl., **113** (1989), 129-130.

[16] ———, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126.

[17] ———, *Two operator functions with monotone property*, Proc. Amer. Math. Soc., **111** (1991), 511-516.

[18] ———, *Applications of order preserving operator inequalities*, Operator Theory: Advances and Applications, **59** (1992), 180-190.

[19] ———, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **219** (1995), 139-155.

[20] ———, *Parallelism related to the inequality "$A \geqslant B \geqslant 0$ ensures $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geqslant 1$ and $r \geqslant 0$"*, Math. Japon., **45** (1997), 203-209.

[21] E. HEINZ, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415-438.

[22] J.-F. JIANG, E. KAMEI AND M. FUJII, *The monotonicity of operator functions associated with the Furuta inequality*, Math. Japon. **46** (1997), 337–343.
[23] E. KAMEI, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883-886.
[24] ———, *Complements to the Furuta inequality, II*, Math. Japon., **45** (1997), 15-23.
[25] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (1980), 205-224.
[26] K. LÖWNER, *Über monotone Matrixfunktionen*, Math. Z., **38** (1934), 177-216.
[27] K. TANAHASHI, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141-146.
[28] ———, *The Furuta inequality in case of negative powers*, RIMS, **903** (1995), 78-96.
[29] T. YOSHINO, *Introduction to Operator Theory*, Pitman Research Notes in Math. Ser., 300, Longman Scientific and Technical, 1993.

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