

NORMS AND DETERMINANTS OF PRODUCTS OF LOGARITHMIC FUNCTIONS OF POSITIVE SEMI-DEFINITE OPERATORS

MITSURU UCHIYAMA

(communicated by T. Ando)

Abstract. Let A, B be bounded positive semi-definite operators (matrices) on a Hilbert space. We will show

$$\|\log(1 + A) \log(1 + B)\| \leq \{\log(1 + \|AB\|^{\frac{1}{2}})\}^2,$$

and

$$\|\log(1 + B) \log(1 + A) \log(1 + B)\| \leq \{\log(1 + \|BAB\|^{\frac{1}{3}})\}^3.$$

Further we will prove the corresponding determinantal inequalities.

Let A and B be bounded positive semi-definite operators (or matrices) on a Hilbert space. The following inequalities are known ([3], [2, Section 9.2]):

$$\|A^a B^a\| \leq \|AB\|^a \text{ and } \|B^a A^a B^a\| \leq \|BAB\|^a \quad (0 \leq a \leq 1).$$

These mean

$$\|f(A)f(B)\| \leq f(\|AB\|) \text{ and } \|f(B)f(A)f(B)\| \leq f(\|BAB\|),$$

where $f(t) = t^a$. We want to get similar inequalities for $f(t) = \log(1 + t)$. But since there is a positive number b such that $\{\log(1 + b)\}^2 > \log(1 + b^2)$, the inequality is not valid even for the case $A = B = bI$. We can write, however, the above inequalities also in the following forms:

$$\|f(A)f(B)\| \leq \{f(\|AB\|^{\frac{1}{2}})\}^2 \text{ and } \|f(B)f(A)f(B)\| \leq \{f(\|BAB\|^{\frac{1}{3}})\}^3,$$

where $f(t) = t^a$.

The aim of this short note is to show that the inequalities of these types hold for $f(t) = \log(1 + t)$.

Mathematics subject classification (1991): Primary 47A30; Secondary 15A15, 15A45.

Key words and phrases: Norm, determinant, positive semi-definite operator, logarithmic function, operator monotone function, majorization.

The author would like to express his thanks to Professor T. Ando and the referee. This research was partially supported by Grant-in-Aid for Scientific Research.

THEOREM 1. *If A, B are bounded positive semi-definite operators, then*

$$\|\log(1 + A)\log(1 + B)\| \leq \{\log(1 + \|AB\|^{\frac{1}{2}})\}^2, \tag{1}$$

and

$$\|\log(1 + B)\log(1 + A)\log(1 + B)\| \leq \{\log(1 + \|BAB\|^{\frac{1}{3}})\}^3. \tag{2}$$

Actually we will establish more general inequalities for a class of functions. Further we will show

THEOREM 2. *If A, B are positive semi-definite matrices, then*

$$\det[\log(1 + A)\log(1 + B)] \leq \det[\log(1 + |AB|^{\frac{1}{2}})]^2, \tag{3}$$

and

$$\det[\log(1 + B)\log(1 + A)\log(1 + B)] \leq \det[\log(1 + (BAB)^{\frac{1}{3}})]^3. \tag{4}$$

Recall that a real-valued continuous function $f(t)$ on $[0, \infty)$ is said to be *operator monotone* if $f(A) \geq f(B)$ whenever $A \geq B \geq 0$. Löwner's theorem says that $f(t)$ is operator monotone on $[0, \infty)$ if and only if it has an analytic extension $f(z)$ to the upper half plane such that $Im f(z) \geq 0$ for $Im z > 0$.

A consequence of this characterization is that if $f(t) \geq 0$ is operator monotone, so is $f(t^p)^{\frac{1}{p}}$ ($0 < p \leq 1$).

Examples of operator monotone functions are t^p ($0 < p \leq 1$), $\frac{t}{s+t}$ with $s > 0$ and $\log(1 + t)$.

As one of the special properties common to these functions $f(t)$ we single out the property that $\log f(e^x)$ is concave on $(-\infty, \infty)$.

We can easily check this property for t^p ($0 < p \leq 1$), $\frac{t}{s+t}$ with $s > 0$ and $\log(1 + t)$ by calculating the second derivatives. But not all non-negative operator monotone function has this property as seen in $1 + t^{\frac{1}{2}}$.

In the following theorems, (1) and (2) of Theorem 1 will be shown in much generalized forms.

THEOREM 3. *Let $0 \leq f(t)$ be an operator monotone function on $[0, \infty)$ such that $\log f(e^x)$ is concave on $(-\infty, \infty)$. Then for every $A, B \geq 0$, and for every $p, q > 0$,*

$$\|f(B^{\frac{1}{q}})^q f(A^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q\| \leq f(\|BAB\|^{\frac{1}{2p+2q}})^{2p+2q}, \tag{5}$$

$$\|f(A^{\frac{1}{p}})^p f(B^{\frac{1}{q}})^q\| \leq f(\|AB\|^{\frac{1}{p+q}})^{p+q}. \tag{6}$$

Proof. Since $\log f(e^x)$ is concave on $(-\infty, \infty)$ we can see

$$f(s^{\frac{1}{p}})^p f(t^{\frac{1}{q}})^q \leq f((st)^{\frac{1}{p+q}})^{p+q} \quad (s, t > 0; p, q > 0). \tag{7}$$

This implies

$$f(t^{\frac{1}{q}})^q f\left(\left(\frac{S}{2}\right)^{\frac{1}{2p}}\right)^{2p} f(t^{\frac{1}{q}})^q \leq f\left(s^{\frac{1}{2p+2q}}\right)^{2p+2q} \quad (s, t > 0; p, q > 0).$$

First, we assume $p \geq \frac{1}{2}$. If B is invertible, by the above, we have

$$f(B^{\frac{1}{q}})^q f((sB^{-2})^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q \leq f(s^{\frac{1}{2p+2q}})^{2p+2q}.$$

Here we set $s = \|BAB\|$. Then since $A \leq sB^{-2}$ and $f(t^{\frac{1}{2p}})^{2p}$ is an operator monotone function for $p \geq \frac{1}{2}$, we have

$$f(A^{\frac{1}{2p}})^{2p} \leq (f(sB^{-2})^{\frac{1}{2p}})^{2p}.$$

Therefore, it follows from the above inequality that

$$f(B^{\frac{1}{q}})^q f(A^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q \leq f(\|BAB\|^{\frac{1}{2p+2q}})^{2p+2q}.$$

Thus we obtain (5) for invertible B . For general $B \geq 0$, approximate it by $B + \delta I$ and let $\delta \downarrow 0$. Consequently we have shown (5) for $p \geq \frac{1}{2}$.

Next, we assume $0 < p < \frac{1}{2}$. By (7) we have

$$f(s^{\frac{1}{2p}})^{2p} f(t^{\frac{1}{1-2p}})^{1-2p} \leq f(st) \quad (s, t > 0),$$

from which it follows that

$$f(A^{\frac{1}{2p}})^{2p} \leq f(t^{\frac{1}{1-2p}})^{-1+2p} f(tA) \quad (t > 0).$$

Using (5) for $p = \frac{1}{2}$, we get

$$\begin{aligned} \|f(B^{\frac{1}{q}})^q f(A^{\frac{1}{2p}})^{2p} f(B^{\frac{1}{q}})^q\| &\leq f(t^{\frac{1}{1-2p}})^{-1+2p} \|f(B^{\frac{1}{q}})^q f(tA) f(B^{\frac{1}{q}})^q\| \\ &\leq f(t^{\frac{1}{1-2p}})^{-1+2p} f(\|tBAB\|^{\frac{1}{1+2q}})^{1+2q}. \end{aligned}$$

By setting $t = \|BAB\|^{\frac{1-2p}{2p+2q}}$, we can get (5) for $0 < p < \frac{1}{2}$.

Observe that (6) is equivalent to

$$\|f(B^{\frac{1}{q}})^q f(A^{\frac{1}{p}})^{2p} f(B^{\frac{1}{q}})^q\| \leq f(\|BA^2B\|^{\frac{1}{2p+2q}})^{2p+2q},$$

which follows from (5) with A replaced by A^2 . Thus we conclude the proof. \square

The special cases of (5) with $p = \frac{1}{2}, q = 1$ and of (6) with $p = q = 1$ have simpler forms:

$$\|f(B)f(A)f(B)\| \leq f(\|BAB\|^{\frac{1}{3}})^3, \tag{8}$$

$$\|f(A)f(B)\| \leq f(\|AB\|^{\frac{1}{2}})^2. \tag{9}$$

With $f(t) = \log(1+t)$ (8) and (9) reduce to (2) and (1) of Theorem 1, respectively. Further applying (9) to $f(t) = \frac{t}{t+1}$ we have

COROLLARY 4.

$$\|(A + I)^{-1}AB(I + B)^{-1}\| \leq \frac{\|AB\|}{(1 + \|AB\|^{\frac{1}{2}})^2}.$$

If $0 \leq f(t)$ is a non-constant operator monotone function on $[0, \infty)$ such that $\log f(e^x)$ is concave on $(-\infty, \infty)$, then necessarily $f(0) = 0$ and the inverse function f^{-1} is defined on $[0, \|f\|_\infty)$. Therefore the following is an immediate consequence of (8) and (9).

COROLLARY 5. Let $0 \leq f(t)$ be a non-constant operator monotone function on $[0, \infty)$ such that $\log f(e^x)$ is concave on $(-\infty, \infty)$. Then for $A, B \geq 0$

$$f^{-1}(\|AB\|^{1/2}) \leq \|f^{-1}(A)f^{-1}(B)\|^{1/2},$$

and

$$f^{-1}(\|BAB\|^{1/3}) \leq \|f^{-1}(B)f^{-1}(A)f^{-1}(B)\|^{1/3}.$$

Since the inverse function of $\log(1+t)$ is $\exp t - 1$, this implies

COROLLARY 6. For $A, B \geq 0$

$$\exp(\|AB\|^{1/2}) - 1 \leq \|(\exp A - 1)(\exp B - 1)\|^{1/2},$$

and

$$\exp(\|BAB\|^{1/3}) - 1 \leq \|(\exp B - 1)(\exp A - 1)(\exp B - 1)\|^{1/3}.$$

Remark here that the following inequality is known for Hermitian A, B (see [2, p. 261])

$$\|\exp(A+B)\| \leq \|\exp A \exp B\|.$$

In the sequel we consider $N \times N$ positive semi-definite matrices. In the following theorem, (3) of Theorem 2 will be shown in much generalized forms. Then (4) will follow as a corollary.

THEOREM 7. Let $f(t)$ be a (not necessarily operator monotone) non-negative function on $[0, \infty)$ such that $\log f(e^x)$ is concave on $(-\infty, \infty)$. Then for every $N \times N$ matrices $A, B \geq 0$ and for every $p, q > 0$

$$\det\{f(A^{\frac{1}{p}})^p f(B^{\frac{1}{q}})^q\} \leq \det f(|AB|^{\frac{1}{p+q}})^{p+q}, \quad (10)$$

where $|X| = (X^*X)^{\frac{1}{2}}$.

Proof. Arrange the eigenvalue of each matrix $X \geq 0$ in decreasing order

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_N(X).$$

First assume that both A, B are positive definite. A. Horn's theorem (see [2, p.72]) says that

$$\prod_{i=1}^k \lambda_i(|AB|) \leq \prod_{i=1}^k \lambda_i(A)\lambda_i(B) \quad (k = 1, 2, \dots, N-1)$$

and

$$\prod_{i=1}^N \lambda_i(|AB|) = \det(A) \det(B) = \prod_{i=1}^N \lambda_i(A)\lambda_i(B).$$

These mean the sequence $\{\log \lambda_i(|AB|)\}_{i=1}^N$ is majorized by $\{\log(\lambda_i(A)\lambda_i(B))\}_{i=1}^N$. Then according to a general theorem on majorization (see [1], [4], [2, Section 2.3]) if $g(e^x)$ is convex on $(-\infty, \infty)$, then

$$\sum_{i=1}^N g(\lambda_i(|AB|)) \leq \sum_{i=1}^N g(\lambda_i(A)\lambda_i(B)).$$

Since for any $\mu > 0$ the function $-\log f(e^{\frac{x}{\mu}})^\mu$ is convex on $(-\infty, \infty)$ by assumption, we have

$$-\sum_{i=1}^N \log f((\lambda_i(|AB|))^{\frac{1}{\mu}})^\mu \leq -\sum_{i=1}^N \log f((\lambda_i(A)\lambda_i(B))^{\frac{1}{\mu}})^\mu,$$

from which it follows that

$$\prod_{i=1}^N f((\lambda_i(A)\lambda_i(B))^{\frac{1}{\mu}})^\mu \leq \prod_{i=1}^N f((\lambda_i(|AB|))^{\frac{1}{\mu}})^\mu.$$

Here we set $\mu = p + q$. Then by (7) we get

$$\begin{aligned} \prod_{i=1}^N f(\lambda_i(A)^{\frac{1}{p}})^p f(\lambda_i(B)^{\frac{1}{q}})^q &\leq \prod_{i=1}^N f((\lambda_i(A)\lambda_i(B))^{\frac{1}{p+q}})^{p+q} \\ &\leq \prod_{i=1}^N f(\lambda_i(|AB|)^{\frac{1}{p+q}})^{p+q}. \end{aligned}$$

This implies

$$\prod_{i=1}^N \lambda_i(f(A^{\frac{1}{p}})^p) \lambda_i(f(B^{\frac{1}{q}})^q) \leq \prod_{i=1}^N \lambda_i(f(|AB|^{\frac{1}{p+q}}))^{p+q},$$

from which it follows that

$$\det f(A^{\frac{1}{p}})^p \cdot \det f(B^{\frac{1}{q}})^q \leq \det f(|AB|^{\frac{1}{p+q}})^{p+q}.$$

Thus we have proved (10) for invertible A, B . For general $A, B \geq 0$, approximate them by $A + \delta I, B + \delta I$ respectively and let $\delta \downarrow 0$. \square

Now (3) of Theorem 2 follows from (10) with $f(t) = \log(1 + t)$ and $p = q = 1$.

COROLLARY 8. *Let $f(t)$ be a (not necessarily operator monotone) non-negative function on $[0, \infty)$ such that $\log f(e^x)$ is concave on $(-\infty, \infty)$. Then for matrices $A_1, \dots, A_n \geq 0$ and for real numbers $p_1, \dots, p_n > 0$*

$$\prod_{i=1}^n \det f(A_i^{\frac{1}{p_i}})^{p_i} \leq \det f(|\prod_{i=1}^n A_i|^{\frac{1}{p_1+\dots+p_n}})^{p_1+\dots+p_n}. \tag{11}$$

Proof. Since $||A_1 A_2|A_3| = |A_1 A_2 A_3|$, by using Theorem 7, we can prove the assertion by induction on n . \square

(4) of Theorem 2 follows from (11) with $n = 3, A_1 = A_3 = B, A_2 = A$ and $p_1 = p_2 = p_3 = 1$.

REFERENCES

- [1] T. ANDO, *Majorizations, and inequalities in matrix theory*, Linear Algebra Appl. **199** (1994), 17–67.
- [2] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1996.
- [3] T. FURUTA, *Norm inequalities equivalent to Löwner-Heinz theorem*, Rev. Math. Phys. **1** (1989), 135–137.
- [4] F. HIAI, *Log-majorizations and norm inequalities for exponential operators*, in *Linear Operators*, Banach center Publications **38** (1997), 119–181.

(Received November 18, 1997)

Mitsuru Uchiyama
Department of Mathematics
Fukuoka University of Education
Munakata, Fukuoka, 811-4192
Japan
e-mail: uchiyama@fukuoka-edu.ac.jp