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INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

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Abstract. For a polynomial $p(z)$ of degree n, having a zero of order k ($\geqslant 1$) at β, we have obtained

$$\max_{|z|=1}\left|\frac{p(z)}{(z-\beta)^k}\right| \leqslant \left(\frac{n-k+1}{1+|\beta|}\right)^k \max_{1\leqslant l\leqslant n-k+1}|p(v_l')|,$$

v_1', $v_2', \ldots, v_{n-k+1}'$ being the roots of $z^{n-k+1}+e^{i\gamma(n-k+1)}=0$, with $\gamma=\arg\beta$ ($\gamma=0$ for $\beta=0$), thereby extending the previously known estimate (i. e. $\max_{|z|=1}\left|\frac{p(z)}{z-\beta}\right| \leqslant \frac{n}{1+\beta}\max_{1\leqslant i\leqslant n}|p(z_i)|$, $\beta\geqslant 0$, z_1, $z_2,\ldots,z_n$ being the roots of $z^n+1=0$).

1. Introduction and statement of results

While thinking about Schwarz's lemma and its various implications, Rahman and Mohammad [2] thought of obtaining a bound for

$$\max_{|z|=1}\left|\frac{p(z)}{z-a}\right|,$$

$p(z)$ being a polinomial of degree at most n, with $\max_{|z|=1}|p(z)|=1$ and $p(a)=0$ for a fixed a on the unit circle, and proved

THEOREM A. *If $p(z)$ is a polynomial of degree n such that $|p(z)|\leqslant 1$ on the unit circle and $p(1)=0$, then for $|z|\leqslant 1$,*

$$\left|\frac{p(z)}{z-1}\right| \leqslant \frac{n}{2}. \tag{1.1}$$

The example $\dfrac{z^n-1}{2}$ shows that the bound in (1.1) is precise.

Aziz [1] improved the inequality (1.1) and obtained

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THEOREM B. *Let $p(z)$ be a polynomial of degree n such that $p(1) = 0$. If z_1, $z_2, \dots, z_n$ are the zeros of $z^n + 1$, then*

$$\max_{|z|=1}\left|\frac{p(z)}{z-1}\right| \leqslant \frac{n}{2}\max_{1\leqslant i\leqslant n}|p(z_i)|. \tag{1.2}$$

The result is best possible with equality in (1.2) for $p(z) = z^n - 1$.

As a corollary of Theorem B, Aziz [1] obtained

THEOREM C. *Let $p(z)$ be a polynomial of degree n such that $p(\beta) = 0$, where β is an arbitrary non-negative real number. If z_1, $z_2, \dots, z_n$ are the zeros of $z^n + 1$, then*

$$\max_{|z|=1}\left|\frac{p(z)}{z-\beta}\right| \leqslant \frac{n}{1+\beta}\max_{1\leqslant i\leqslant n}|p(z_i)|. \tag{1.3}$$

We consider polynomials having a zero of order k, at an arbitrary point β of the plane and obtain the following extension of Theorem C.

THEOREM 1. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = (z-\beta)^k q(z), \quad k \geqslant 1 \quad \text{and } \beta \text{ is arbitrary.} \tag{1.4}$$

Then

$$\max_{|z|=1}\left|\frac{p(z)}{(z-\beta)^k}\right| \leqslant \left(\frac{n-k+1}{1+|\beta|}\right)^k \max_{1\leqslant l\leqslant n-k+1}|p(v_l')|, \tag{1.5}$$

where v_1', $v_2', \dots, v_{n-k+1}'$ are the roots of

$$z^{n-k+1} + e^{i\gamma(n-k+1)} = 0, \tag{1.6}$$

and

$$\gamma = \begin{cases} \arg\beta, & \beta \neq 0, \\ 0, & \beta = 0. \end{cases}$$

REMARK 1. By taking $k = 1$ and letting $z \to \beta$ in (1.5), we obtain

$$|p'(\beta)| \leqslant \frac{n}{1+|\beta|}\max_{1\leqslant i\leqslant n}|p(z_i')|, \quad 0 \leqslant |\beta| \leqslant 1,$$

where z_1', $z_2', \dots, z_n'$ are the roots of

$$z^n + e^{i\gamma n} = 0.$$

The inequality is sharp for $|\beta| = 1$.

2. Proof of Theorem 1

We firstly assume that $\beta \geqslant 0$. Now let $p^*(z) = (z-1)^k q(z)$. Then, as

$$\frac{p^*(z)}{(z-1)^{k-1}} = (z-1)q(z) = T(z), \tag{2.1}$$

say, we have by Theorem B

$$\max_{|z|=1}\left|\frac{T(z)}{z-1}\right| \leqslant \left(\frac{n-k+1}{2}\right) \max_{1\leqslant l\leqslant n-k+1} |T(v_l)|, \tag{2.2}$$

where v_1, $v_2, \ldots, v_{n-k+1}$ are the roots of

$$z^{n-k+1}+1=0.$$

Further by (2.1), we have

$$\begin{aligned}
|T(v_l)| &= \frac{|p^*(v_l)|}{|v_l-1|^{k-1}} = \frac{1}{|v_l-1|^{k-1}} \cdot \frac{|p^*(v_l)|}{|p(v_l)|} \cdot |p(v_l)| \\
&\leqslant \left(\frac{n-k+1}{2}\right)^{k-1} \left|\frac{v_l-1}{v_l-\beta}\right|^k |p(v_l)| \\
&\leqslant \left(\frac{n-k+1}{2}\right)^{k-1} \left(\frac{2}{1+\beta}\right)^k |p(v_l)|,
\end{aligned}$$

which, by (2.1) and (2.2), implies

$$\max_{|z|=1} |q(z)| = \max_{|z|=1}\left|\frac{T(z)}{z-1}\right| \leqslant \left(\frac{n-k+1}{1+\beta}\right)^k \max_{1\leqslant l\leqslant n-k+1} |p(v_l)|,$$

i. e.

$$\max_{|z|=1}\left|\frac{p(z)}{(z-\beta)^k}\right| \leqslant \left(\frac{n-k+1}{1+\beta}\right)^k \max_{1\leqslant l\leqslant n-k+1} |p(v_l)|. \tag{2.3}$$

Now if β is an arbitrary complex number, with $\beta = |\beta|e^{i\gamma}$, then we have

$$\begin{aligned}
\max_{|z|=1}\left|\frac{p(z)}{(z-\beta)^k}\right| &= \max_{|z|=1}\left|\frac{p(ze^{i\gamma})}{(z-|\beta|)^k}\right| \\
&\leqslant \left(\frac{n-k+1}{1+|\beta|}\right)^k \max_{1\leqslant l\leqslant n-k+1} |p(v_l e^{i\gamma})|, \quad (\text{by } (2.3)) \\
&= \left(\frac{n-k+1}{1+|\beta|}\right)^k \max_{1\leqslant l\leqslant n-k+1} |p(v_l')|, \quad (\text{by } (1.6)),
\end{aligned}$$

which completes the proof of Theorem 1.

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