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ON THE OSCILLATION OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. We shall discuss two powerful techniques, namely, the averaging method, and the inequalities method, which have been used for quite some time to establish the oscillations of second order differential equations, whereas their use in the study of difference equations is recent and deserves more attention.

1. Introduction

It is well known that the *average function* $A_p(t)$ defined by

$$A_p(t) \;=\; \frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-1} h(s)ds, \quad (p \geqslant 1) \tag{1.1}$$

plays a crucial role in proving the oscillation of solutions of the equation

$$y'' + h(t)g(y(t)) \;=\; 0, \quad t \geqslant t_0.$$

In fact, in the linear case important oscillation criteria of Wintner [26] and Hartman [7,8], and for the nonlinear case of Butler [2] involve the asymptotic behavior of $A_2(t)$ as $t \to \infty$. Other investigations making use of the average function $A_p(t)$ for particular values of p for the linear case include Coles [3], Coles and Willett [4], Hartman [9], Kamenev [11], Willett [25], and for the nonlinear case Kamenev [10], Kwong and Wong [12], Philos [15-19], Philos and Purnaras [20,21], Wong [28-31], Wong and Yeh [27,32]. Recently Naito [14] has improved most of these oscillation criteria by considering the general average function $A_p(t)$, $p \geqslant 1$.

While in the last ten years for the oscillation of solutions of difference equations hundreds of articles have appeared the only paper in which average function for difference equations has been touched is Erbe and Yan [5]. As a first contribution of this paper in Section 2, we shall show that the discrete version of the average function (1.1), which we call as *average sum* can be used to study the oscillatory behavior of solutions of the second order nonlinear difference equation

$$\Delta^2 y_k + A(k, y_k) \;=\; B(k, y_k, \Delta y_k), \quad k \in \mathbf{N} = \{0, 1, \cdots\} \tag{1.2}$$

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where Δ is the standard forward difference operator defined by $\Delta y_k = y_{k+1} - y_k$. The important features of our study are:

1. It generalizes the integral averaging techniques used in literature to discrete case.
2. It leads to new necessary conditions for the existence of a nonoscillatory solution of (1.2).
3. The contra positive form of the results obtained then gives rise to new oscillation criteria for (1.2).

In the work of Graef and Spikes [6], Kwong and Wong [13], and Wong and Agarwal [33], Volterra integral inequalities have been used successfully to obtain oscillatory criteria for the second order differential equations. As a second contribution of this paper in Section 3, we shall use Volterra discrete inequalities to offer sufficient conditions for the oscillation of all solutions of the difference equation

$$\Delta(a_k(\Delta y_k)^{\sigma}) + q_{k+1}f(y_{k+1}) \;=\; r_k, \quad k \in \mathbf{N} \tag{1.3}$$

where σ is a positive quotient of odd integers odd/odd, or even over odd integers $even/odd$, and $\{a_k\}$ is an eventually positive real sequence. Our results here extend the work of Thandapani et al [22,23], and Zhang and Chen [36]. We also remark that this technique has been recently employed in Thandapani et al [24] for the quasilinear difference equations of the type

$$\Delta(a_k|\Delta y_k|^{\sigma-1}\Delta y_k) + q_{k+1}f(y_{k+1}) \;=\; r_k, \quad k \in \mathbf{N}.$$

As usual by a solution of (1.2) ((1.3)) we mean a nontrivial sequence $y = \{y_k\}$ satisfying (1.2) ((1.3)) for $k \in \mathbf{N}$. A solution $\{y_k\}$ is said to be oscillatory if it is neither eventually positive nor negative, and nonoscillatory otherwise.

2. Oscillation via summation averages

With respect to the difference equation (1.2), we shall assume that there exist real sequences $\{\alpha_k\}$, $\{\beta_k\}$ and a function f : Re $\to$ Re such that

(i) $uf(u) > 0$ for all $u \neq 0$;

(ii) $f(u) - f(v) = g(u,v)\,(u-v)$ for $u, v \neq 0$, where g is a nonnegative function; and

(iii) $\dfrac{A(k,u)}{f(u)} \geqslant \alpha_k, \quad \dfrac{B(k,u,v)}{f(u)} \leqslant \beta_k$ for $u \neq 0,\ k \in \mathbf{N}$.

The 'average sum' which is the discrete analog of (1.1) for the difference equation (1.2) is defined as follows:

$$J_p(k) = \frac{1}{k^{(p-1)}} \sum_{\ell=0}^{k} (k-\ell)^{(p-1)}[\alpha_\ell - \beta_\ell], \tag{2.1}$$

where $p(\geqslant 1)$ is an integer. For this sum, we state the following two lemmas which will be used to obtain necessary conditions for the existence of a nonoscilltory solution of (1.2).

LEMMA 2.1. [34] *Let* p *be an integer such that* $p \geqslant 2$. *Further, suppose that* b_k *is defined for* $k \in \mathbf{N}$, *and*

$$\lim_{k\to\infty} \sum_{\ell=0}^{k} b_\ell = \sum_{\ell=0}^{\infty} b_\ell \in \mathrm{Re} \cup \{-\infty, \infty\}.$$

Then,

$$\lim_{k\to\infty} \frac{1}{k^{(p-1)}} \sum_{\ell=0}^{k} (k-\ell)^{(p-1)} b_\ell = (p-1)\theta' \sum_{\ell=0}^{\infty} b_\ell,$$

where $\theta' > 0$ *is given by*

$$\theta' = \sum_{j=0}^{p-2} (-1)^j \begin{pmatrix} p-2 \\ j \end{pmatrix} \frac{1}{j+1}.$$

LEMMA 2.2. [34] *Let*

$$\lim_{k\to\infty} J_2(k) = \lambda \in \mathrm{Re} \cup \{-\infty, \infty\}.$$

Then, for any integer $p \geqslant 3$,

$$\lim_{k\to\infty} J_p(k) = (p-1)(p-2)\theta''\lambda,$$

where $\theta'' > 0$ *is given by*

$$\theta'' = \sum_{j=0}^{p-3} (-1)^j \begin{pmatrix} p-3 \\ j \end{pmatrix} \frac{1}{j+2}.$$

We shall also need the following:

DEFINITION 2.1. We say that equation (1.2) is *strictly superlinear* if for all $k > 0$,

$$\left| \sum_{\ell=k}^{\infty} \frac{\Delta y_\ell}{f(y_\ell)} \right| < \infty \quad \text{and} \quad g(y_k, y_{k+1}) \left| \sum_{\ell=k}^{\infty} \frac{\Delta y_\ell}{f(y_\ell)} \right| \geqslant c(f) > 1;$$

equation (1.2) is said to be *strictly sublinear* if for all $T > 0,\ k > T+1$,

$$\left| \sum_{\ell=T}^{k-1} \frac{\Delta y_\ell}{f(y_\ell)} \right| < \infty \quad \text{and} \quad g(y_k, y_{k+1}) \left| \sum_{\ell=T}^{k-1} \frac{\Delta y_\ell}{f(y_\ell)} \right| \geqslant d(f) > 0;$$

and, equation (1.2) is *linear* if $f(u) \equiv u$ and for all $T > 0,\ k > T+1$,

$$\left| \sum_{\ell=T}^{k-1} \frac{\Delta y_\ell}{y_\ell} \right| > 0.$$

For the rest of this section, we shall assume that equation (1.2) is strictly superlinear, strictly sublinear, or linear.

THEOREM 2.3. *Let y be a nonoscillatory solution of (1.2) such that*

(A1) y is eventually monotone, i.e., Δy is eventually of fixed sign;

(A2) there exists a constant $M > 0$ such that $\dfrac{\Delta y_k}{f(y_k)}\dfrac{f(y_{k+1})}{\Delta y_{k+1}} \leqslant M$ for sufficiently large k;

(A3) $y\Delta y$ is eventually positive.

Then, either

(I) $\limsup_{k\to\infty} J_p(k) < \infty$ for any integer $p \geqslant 2$, or

(II) $\limsup_{k\to\infty} J_q(k) = -\infty$ where q is any integer satisfying

$$\begin{cases} q \geqslant 2, & \text{(1.2) is strictly sublinear} \\ q > 2 \ \text{ and } \ \dfrac{2M(q-1)}{(q-2)c(f)} < 1, & \text{(1.2) is strictly superlinear} \\ q > 2, & \text{(1.2) is linear} \end{cases} \tag{2.2}$$

holds. Further,

(a) the conditions (A1) and (A3) are not needed for (I) to hold;

(b) in the strictly sublinear case, only condition (A3) is required for (II) to hold;

(c) in the strictly superlinear and linear cases, only conditions (A1) and (A2) are needed for (II) to hold.

Proof. From Lemma 2.2, we see that if $\limsup_{k\to\infty} J_2(k) = \lambda$, then $\limsup_{k\to\infty} J_p(k) = (\textit{positive constant}) \times \lambda$ for any $p \geqslant 3$. Hence, in statement (I) of Theorem 2.3, it suffices to consider $p = 2$, and in statement (II) it suffices to consider $q = 2$ in the strictly sublinear case.

Let y be a nonoscillatory solution of (1.2), say, $y_k > 0$ for $k \geqslant T \geqslant 0$. We shall consider only this case because the proof for the case y is eventually negative is similar. Let $p \geqslant 2$. >From (1.2), (iii), and summation by parts [1], we find

$$\begin{aligned}
&\frac{1}{k^{(p-1)}}\sum_{\ell=T}^{k}(k-\ell)^{(p-1)}[\alpha_\ell - \beta_\ell] \\
&\qquad \leqslant -\frac{1}{k^{(p-1)}}\sum_{\ell=T}^{k}(k-\ell)^{(p-1)}\frac{\Delta^2 y_\ell}{f(y_\ell)} \\
&\qquad = -\frac{p-1}{k^{(p-1)}}\sum_{\ell=T}^{k-1}(k-\ell-1)^{(p-2)}\left[\sum_{\tau=T}^{\ell}\frac{\Delta^2 y_\tau}{f(y_\tau)}\right], \quad k \geqslant T+1 \\
&\qquad \leqslant \frac{\Delta y_T}{f(y_T)}\frac{p-1}{k^{(p-1)}}\sum_{\ell=T}^{k-1}(k-\ell-1)^{(p-2)} - \frac{p-1}{k^{(p-1)}}\sum_{\ell=T+1}^{k}(k-\ell)^{(p-2)}\frac{\Delta y_\ell}{f(y_\ell)} \\
&\qquad\quad - \frac{p-1}{k^{(p-1)}}\sum_{\tau=T}^{k-1}\sum_{\ell=\tau}^{k-1}(k-\ell-1)^{(p-2)}\frac{\Delta y_\tau \Delta y_{\tau+1} g(y_\tau, y_{\tau+1})}{f(y_\tau)f(y_{\tau+1})}
\end{aligned}$$

$$= \frac{\Delta y_T}{f(y_T)} \frac{(k-T)^{(p-1)}}{k^{(p-1)}} - \frac{p-1}{k^{(p-1)}} \sum_{\ell=T+1}^{k} (k-\ell)^{(p-2)} \frac{\Delta y_\ell}{f(y_\ell)}$$

$$- \frac{1}{k^{(p-1)}} \sum_{\ell=T}^{k-1} (k-\ell)^{(p-1)} \frac{\Delta y_\ell \Delta y_{\ell+1} g(y_\ell, y_{\ell+1})}{f(y_\ell) f(y_{\ell+1})}, \quad k \geqslant T+1. \tag{2.3}$$

Now, we define for $k \geqslant T+1$,

$$w_k = \begin{cases} \sum_{\ell=k}^{\infty} \frac{\Delta y_\ell}{f(y_\ell)}, \quad a=-1, & \text{(1.2) is strictly superlinear} \\ \sum_{\ell=T}^{k-1} \frac{\Delta y_\ell}{f(y_\ell)}, \quad a=1, & \text{(1.2) is strictly sublinear} \\ \sum_{\ell=T}^{k-1} \frac{\Delta y_\ell}{y_\ell}, \quad a=1, & \text{(1.2) is linear.} \end{cases} \tag{2.4}$$

In each case, we have $a\Delta w_k = \Delta y_k / f(y_k)$, $k \geqslant T+1$. Therefore, (2.3) can be rewritten as

$$\frac{1}{k^{(p-1)}} \sum_{\ell=T}^{k} (k-\ell)^{(p-1)} [\alpha_\ell - \beta_\ell] \leqslant \frac{\Delta y_T}{f(y_T)} \frac{(k-T)^{(p-1)}}{k^{(p-1)}}$$

$$- \frac{a(p-1)}{k^{(p-1)}} \sum_{\ell=T+1}^{k} (k-\ell)^{(p-2)} \Delta w_\ell - \frac{1}{k^{(p-1)}} \sum_{\ell=T}^{k-1} (k-\ell)^{(p-1)} \Delta w_\ell \Delta w_{\ell+1} g(y_\ell, y_{\ell+1}). \tag{2.5}$$

Let

$$S = \sum_{\ell=T}^{\infty} |\Delta w_\ell \Delta w_{\ell+1}| g(y_\ell, y_{\ell+1}). \tag{2.6}$$

We shall consider two mutually exclusive cases, namely, S is finite and S is infinite.

Case 1. Suppose that S is finite. We will show that (I) holds. As remarked earlier, it suffices to prove that $\limsup_{k\to\infty} J_2(k) < \infty$, or equivalently,

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{\ell=T}^{k} (k-\ell)[\alpha_\ell - \beta_\ell] < \infty.$$

Substituting $p=2$ in (2.5), for $k \geqslant T+1$ we get

$$\frac{1}{k} \sum_{\ell=T}^{k} (k-\ell)[\alpha_\ell - \beta_\ell] \leqslant \frac{\Delta y_T}{f(y_T)} \frac{k-T}{k} + \frac{a}{k} w_{T+1} - \frac{a}{k} w_{k+1} - \frac{1}{k} \sum_{\ell=T}^{k-1} (k-\ell) \Delta w_\ell \Delta w_{\ell+1} g(y_\ell, y_{\ell+1}). \tag{2.7}$$

If we can show that

$$\lim_{k\to\infty} \frac{|w_{k+1}|}{k} = 0, \tag{2.8}$$

then together with Lemma 2.1 ($p = 2$), it is clear from (2.7) that

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{\ell=T}^{k}(k-\ell)[\alpha_\ell - \beta_\ell] \leqslant \frac{\Delta y_T}{f(y_T)} - \theta' \sum_{\ell=T}^{\infty} \Delta w_\ell \Delta w_{\ell+1} g(y_\ell, y_{\ell+1}). \tag{2.9}$$

Since S is finite, the right side of (2.9) is finite, i.e., we are done. To prove (2.8), we shall consider two subcases, namely, (1.2) is strictly superlinear/sublinear and (1.2) is linear.

Case 1(a). Suppose that (1.2) is strictly superlinear/sublinear. Let $\tau \geqslant T+1$ be an arbitrary integer. We have

$$\begin{aligned} |w_{k+1}| &= \left\{ |w_\tau|^{1/2} + \sum_{\ell=\tau}^{k} \Delta\left[|w_\ell|^{1/2}\right]\right\}^2 \leqslant \left\{|w_\tau|^{1/2} + \sum_{\ell=\tau}^{k} \frac{|\Delta w_\ell|}{|w_\ell|^{1/2} + |w_{\ell+1}|^{1/2}}\right\}^2 \\ &\leqslant \left\{|w_\tau|^{1/2} + \sum_{\ell=\tau}^{k} \frac{|\Delta w_\ell|}{2\min\left\{|w_\ell|^{1/2}, |w_{\ell+1}|^{1/2}\right\}}\right\}^2. \end{aligned} \tag{2.10}$$

Using $(s+t)^2 \leqslant 2s^2 + 2t^2$, and Schwarz's inequality in (2.10), we obtain

$$\begin{aligned} |w_{k+1}| &\leqslant 2|w_\tau| + \frac{1}{2}\left\{\sum_{\ell=\tau}^{k} \frac{|\Delta w_\ell|}{\min\left\{|w_\ell|^{1/2}, |w_{\ell+1}|^{1/2}\right\}}\right\}^2 \\ &\leqslant 2|w_\tau| + \frac{1}{2}\left\{\sum_{\ell=\tau}^{k} \frac{|\Delta w_\ell|}{|\Delta w_{\ell+1}| g(y_\ell, y_{\ell+1}) \min\{|w_\ell|, |w_{\ell+1}|\}}\right\} \\ &\qquad \times \left\{\sum_{\ell=\tau}^{\infty} |\Delta w_\ell \Delta w_{\ell+1}| g(y_\ell, y_{\ell+1})\right\}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} 0 < \frac{|w_{k+1}|}{k} &\leqslant \frac{2|w_\tau|}{k} + \frac{1}{2}\,\frac{1}{k}\left\{\sum_{\ell=\tau}^{k} \frac{|\Delta w_\ell|}{|\Delta w_{\ell+1}|}\,\frac{1}{g(y_\ell, y_{\ell+1}) \min\{|w_\ell|, |w_{\ell+1}|\}}\right\} \\ &\qquad \times \left\{\sum_{\ell=\tau}^{\infty} |\Delta w_\ell \Delta w_{\ell+1}| g(y_\ell, y_{\ell+1})\right\}. \end{aligned} \tag{2.11}$$

Taking limit supremum in (2.11), applying discrete l'Hospital's rule [1], and using the condition (A2), we get

$$0 \leqslant \limsup_{k\to\infty} \frac{|w_{k+1}|}{k} \leqslant \frac{M}{2}\,\frac{\sum_{\ell=\tau}^{\infty} |\Delta w_\ell \Delta w_{\ell+1}| g(y_\ell, y_{\ell+1})}{\liminf_{k\to\infty} g(y_{k+1}, y_{k+2}) \min\{|w_{k+1}|, |w_{k+2}|\}}. \tag{2.12}$$

In view of Definition 2.1, we have

$$\liminf_{k\to\infty} g(y_{k+1}, y_{k+2}) \min\{|w_{k+1}|, |w_{k+2}|\} > 0,$$

and hence, since S is finite, by letting $\tau \to \infty$ in (2.12) we obtain $\limsup_{k\to\infty} |w_{k+1}|/k = 0$. This proves (2.8).

Case 1(b). Suppose that (1.2) is linear. Let $\tau \geqslant T+1$ be an arbitrary integer. As in Case 1(a), we obtain (2.12) with $g \equiv 1$. By Definition 2.1, we have

$$\liminf_{k\to\infty} \min\{|w_{k+1}|, |w_{k+2}|\} > 0,$$

using this together with the fact that S is finite, by letting $\tau \to \infty$ in (2.12) we obtain $\limsup_{k\to\infty} |w_{k+1}|/k = 0$, and therefore (2.8) is proved.

Case 2. Suppose that S is infinite. We will show that (II) holds. For this, we need to consider the following three subcases.

Case 2(a). Suppose that (1.2) is strictly sublinear. As noted earlier, it suffices to show that $\limsup_{k\to\infty} J_2(k) = -\infty$.

Let y be an eventually positive solution of (1.2) and let T be sufficiently large so that $y_k > 0$ and $y_k \Delta y_k > 0$, $k \geqslant T \geqslant 0$ (i.e., condition (A3) holds). Thus, w_k defined in (2.4) is positive in this case. By substituting $p = 2$ in (2.5), we obtain (2.7). In the right side of (2.7), as $k \to \infty$, the first term tends to $\Delta y_T / f(y_T)$, the second term vanishes, the third term $-w_{k+1}/k$ is negative, and the last term tends to $-\theta' S = -\infty$ (by Lemma 2.1, $p = 2$). Therefore, it is clear that $\limsup_{k\to\infty} J_2(k) = -\infty$.

Case 2(b). Suppose that (1.2) is strictly superlinear. Let y be an eventually positive solution of (1.2) and let T be sufficiently large so that $y_k > 0$, and Δy_k is of fixed sign for $k \geqslant T \geqslant 0$ (i.e., condition (A1) holds), and also condition (A2) holds for $k \geqslant T$. Let $q > 2$ and $2M(q-1) < (q-2)c(f)$. To show that $\limsup_{k\to\infty} J_q(k) = -\infty$, it suffices to prove that

$$\limsup_{k\to\infty} \frac{1}{k^{(q-1)}} \sum_{\ell=T}^{k} (k-\ell)^{(q-1)} [\alpha_\ell - \beta_\ell] = -\infty. \tag{2.13}$$

Let

$$S_1(k) = \frac{1}{k^{(q-1)}} \sum_{\ell=T+1}^{k} (k-\ell)^{(q-2)} \Delta w_\ell \tag{2.14}$$

and

$$S_2(k) = \frac{1}{k^{(q-1)}} \sum_{\ell=T}^{k-1} (k-\ell)^{(q-1)} \Delta w_\ell \Delta w_{\ell+1} g(y_\ell, y_{\ell+1}). \tag{2.15}$$

Then, (2.5) with $p = q$ can be rewritten as

$$\frac{1}{k^{(q-1)}} \sum_{\ell=T}^{k} (k-\ell)^{(q-1)} [\alpha_\ell - \beta_\ell] \leqslant \frac{\Delta y_T}{f(y_T)} \frac{(k-T)^{(q-1)}}{k^{(q-1)}} - a(q-1) S_1(k) - S_2(k), \ k \geqslant T+1. \tag{2.16}$$

We note that

$$\begin{aligned} |S_1(k)| &\leqslant \frac{1}{k^{(q-1)}} \sum_{\ell=T+1}^{k} (k-\ell)^{(q-2)} |\Delta w_\ell| \\ &= \frac{1}{k^{(q-1)}} \sum_{\ell=T+1}^{k} \left[(k-\ell)^{(q-1)}(k-\ell)^{(q-3)}\right]^{1/2} \left|\frac{k-\ell-q+3}{k-\ell-q+2}\right|^{1/2} |\Delta w_\ell| \\ &\leqslant \sqrt{2}\, \frac{1}{k^{(q-1)}} \sum_{\ell=T+1}^{k} \left[(k-\ell)^{(q-1)}(k-\ell)^{(q-3)}\right]^{1/2} |\Delta w_\ell|, \;\; k \geqslant T+1. \end{aligned}$$

Hence, by Schwarz's inequality

$$\begin{aligned} [S_1(k)]^2 &\leqslant 2\left[\frac{1}{k^{(q-1)}} \sum_{\ell=T}^{k-1} (k-\ell)^{(q-1)} \Delta w_\ell \Delta w_{\ell+1} g(y_\ell, y_{\ell+1})\right] \\ &\quad \times \left[\frac{1}{k^{(q-1)}} \sum_{\ell=T+1}^{k} (k-\ell)^{(q-3)} \frac{\Delta w_\ell}{\Delta w_{\ell+1} g(y_\ell, y_{\ell+1})}\right] \\ &\leqslant 2MS_2(k) \left[\frac{1}{k^{(q-1)}} \sum_{\ell=T+1}^{k} (k-\ell)^{(q-3)} \frac{1}{g(y_\ell, y_{\ell+1})}\right], \;\; k \geqslant T+1 \end{aligned} \tag{2.17}$$

where we have used condition (A2) in the last inequality. Since Definition 2.1 implies $1/g(y_\ell, y_{\ell+1}) \leqslant |w_\ell|/c(f)$, it follows from (2.17) that

$$\begin{aligned} [S_1(k)]^2 &\leqslant \frac{2M}{c(f)} S_2(k) \frac{1}{k^{(q-1)}} \sum_{\ell=T+1}^{k} (k-\ell)^{(q-3)} |w_\ell| \\ &= \frac{2M}{c(f)} S_2(k) \frac{1}{k^{(q-1)}} \left[\left.\frac{-(k-\ell+1)^{(q-2)}|w_\ell|}{q-2}\right|_{T+1}^{k+1} + \sum_{\ell=T+1}^{k} \frac{(k-\ell)^{(q-2)}}{q-2} \Delta|w_\ell|\right] \\ &\leqslant \frac{2M}{c(f)} S_2(k) \frac{1}{k^{(q-1)}(q-2)} \left[(k-T)^{(q-2)}|w_{T+1}| + k^{(q-1)}|S_1(k)|\right], \;\; k \geqslant T+1 \end{aligned}$$

where in the last inequality, in view of (A1), we have used the fact that w_ℓ is of fixed sign for $\ell \geqslant T$. Now, the above relation leads to the following quadratic inequality in $|S_1(k)|/S_2(k)$

$$\left[\frac{S_1(k)}{S_2(k)}\right]^2 \leqslant \frac{2M}{(q-2)c(f)} \left[\frac{(k-T)^{(q-2)}|w_{T+1}|}{k^{(q-1)}S_2(k)} + \frac{|S_1(k)|}{S_2(k)}\right], \;\; k \geqslant T+1$$

from which we find

$$\frac{|S_1(k)|}{S_2(k)} \leqslant \frac{1}{2}\left[\frac{2M}{(q-2)c(f)} + \sqrt{D_k}\right], \;\; k \geqslant T+1 \tag{2.18}$$

where

$$D_k = \left[\frac{2M}{(q-2)c(f)}\right]^2 + \frac{8M}{(q-2)c(f)} \frac{(k-T)^{(q-2)}|w_{T+1}|}{k^{(q-1)}S_2(k)}.$$

By Lemma 2.1, $S_2(k)$ tends to $(q-1)\theta' S(=\infty)$ as $k \to \infty$. Thus,

$$D_k \to \left[\frac{2M}{(q-2)c(f)}\right]^2 \textit{ as } k \to \infty.$$

Taking limit supremum in (2.18), we get

$$\limsup_{k\to\infty} \frac{|S_1(k)|}{S_2(k)} \leqslant \frac{2M}{(q-2)c(f)}.$$

Hence, there exists a T_1 such that

$$|S_1(k)| \leqslant \frac{2M}{(q-2)c(f)}\, S_2(k),\ k \geqslant T_1. \tag{2.19}$$

Using (2.19) in (2.16), we obtain for $k \geqslant T_1$,

$$\begin{aligned} &\frac{1}{k^{(q-1)}}\sum_{\ell=T}^{k}(k-\ell)^{(q-1)}[\alpha_\ell - \beta_\ell] \\ &\quad\leqslant \frac{\Delta y_T}{f(y_T)}\frac{(k-T)^{(q-1)}}{k^{(q-1)}} + (q-1)\,\frac{2M}{(q-2)c(f)}\,S_2(k) - S_2(k) \\ &\quad= \frac{\Delta y_T}{f(y_T)}\frac{(k-T)^{(q-1)}}{k^{(q-1)}} + \left[\frac{2M(q-1)}{(q-2)c(f)} - 1\right] S_2(k). \end{aligned} \tag{2.20}$$

Clearly, the right side of (2.20) tends to $-\infty$ as $k \to \infty$ and (2.13) is proved.

Case 2(c). Suppose that (1.2) is linear. Again, let y be an eventually positive solution of (1.2) and let T be sufficiently large so that $y_k > 0$, and Δy_k is of fixed sign for $k \geqslant T \geqslant 0$ (i.e., condition (A1) holds), and further condition (A2) holds for $k \geqslant T$.

Following the same argument as in Case 2(b), we get (2.17). Since $g \equiv 1$, for $k \geqslant T+1$, we find

$$[S_1(k)]^2 \leqslant 2MS_2(k)\,\frac{1}{k^{(q-1)}}\sum_{\ell=T+1}^{k}(k-\ell)^{(q-3)} = 2MS_2(k)\,\frac{1}{k^{(q-1)}}\frac{(k-T)^{(q-2)}}{q-2},$$

or

$$|S_1(k)| \leqslant \left[\frac{2M}{q-2}\frac{(k-T)^{(q-2)}}{k^{(q-1)}}\,S_2(k)\right]^{1/2},\ k \geqslant T+1. \tag{2.21}$$

Using (2.21) in (2.16), we get for $k \geqslant T+1$,

$$\begin{aligned} \frac{1}{k^{(q-1)}}\sum_{\ell=T}^{k}(k-\ell)^{(q-1)}[\alpha_\ell - \beta_\ell] &\leqslant \frac{\Delta y_T}{f(y_T)}\frac{(k-T)^{(q-1)}}{k^{(q-1)}} \\ &\quad + (q-1)\left[\frac{2M}{q-2}\frac{(k-T)^{(q-2)}}{k^{(q-1)}}S_2(k)\right]^{1/2} - S_2(k). \end{aligned} \tag{2.22}$$

It is clear that the right side of (2.22) tends to $-\infty$ as $k \to \infty$. Hence, (2.13) is proved. □

COROLLARY 2.4. *Suppose that the following hold*

$(I)'$ $\limsup_{k\to\infty} J_p(k) = \infty$ *for some integer* $p \geqslant 2$;

$(II)'$ $\limsup_{k\to\infty} J_q(k) > -\infty$ *for some integer* q *satisfying (2.1).*

Then, (1.2) does not have any nonoscillatory solutions satisfying (A1) – (A3).

Proof. This is the contra positive form of Theorem 2.3. □

COROLLARY 2.5. *Suppose that the following hold*

$(III)'$ $\limsup_{k\to\infty} J_q(k) = \infty$ *for some integer* q *satisfying (2.1).*

Then, (1.2) does not have any nonoscillatory solutions satisfying (A1) – (A3).

Proof. This is the particular case of Corollary 2.4 when $p = q$. □

COROLLARY 2.6. *Suppose in addition to* $(I)'$ *the following hold*

$(II)''$ $\liminf_{k\to\infty} J_r(k) > -\infty$ *for some integer* $r \geqslant 2$.

Then, (1.2) does not have any nonoscillatory solutions satisfying (A1) – (A3).

Proof. By Lemma 2.2 condition $(II)''$ implies that $\liminf_{k\to\infty} J_2(k) > -\infty$. Again it follows from Lemma 2.2 that

$$\limsup_{k\to\infty} J_q(k) \geqslant \liminf_{k\to\infty} J_q(k) > -\infty$$

for any q satisfying (2.1). Hence, in particular we get condition $(II)'$. The result is now obvious from Corollary 2.4. □

3. Oscillation via inequalities

Let $\alpha, \beta \in \mathbf{N}$, $\mathbf{N}_\beta = \{\beta, \beta+1, \cdots\}$, $\mathbf{N}_\beta^\alpha = \{\beta, \beta+1, \cdots, \alpha\}$, and for notational simplicity, let $w_k = a_k(\Delta y_k)^\sigma$. In what follows, we shall assume that f : Re $\to$ Re and satisfies the assumptions (i) and (ii) of Section 2.

LEMMA 3.1. *Let the function* $K(k,s,y) : \mathbf{N}_{k_0} \times \mathbf{N}_{k_0} \times \mathrm{Re}^+ \to \mathrm{Re}$ *be such that for each fixed* k, s, *the function* $K(k,s,\cdot)$ *is nondecreasing. Furthermore, let* $\{p_k\}$ *be a given sequence and* $\{u_k\}$, $\{v_k\}$ *be sequences satisfying, for* $k \in \mathbf{N}_{k_0}$,

$$u_k \geqslant (\leqslant)\, p_k + \sum_{s=k_0}^{k-1} K(k,s,u_s) \quad \textit{and} \quad v_k = p_k + \sum_{s=k_0}^{k-1} K(k,s,v_s).$$

Then, $u_k \geqslant (\leqslant)\, v_k$ *for all* $k \in \mathbf{N}_{k_0}$.

Proof. The proof is by induction and is obvious. □

As an application of Lemma 3.1, we have

LEMMA 3.2. [35] *Let* $\sigma = odd/odd$. *Suppose that* $\{y_k\}$ *is a positive (negative) solution of (1.3) for* $k \in \mathbf{N}_{k_0}^{\alpha}$, *and there exists* $k_1 \in \mathbf{N}_{k_0}^{\alpha}$ *and* $m > 0$ *such that*

$$-\frac{a_{k_0}(\Delta y_{k_0})^{\sigma}}{f(y_{k_0})} + \sum_{s=k_0}^{k-1}\left[q_{s+1} - \frac{r_s}{f(y_{s+1})}\right] + \sum_{s=k_0}^{k_1-1}\frac{a_s(\Delta y_s)^{\sigma+1}g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})} \geqslant m \tag{3.1}$$

for all $k \in \mathbf{N}_{k_1}^{\alpha}$. *Then,*

$$a_k(\Delta y_k)^{\sigma} \leqslant (\geqslant)\ -mf(y_{k_1}),\ k \in \mathbf{N}_{k_1}^{\alpha}. \tag{3.2}$$

THEOREM 3.3. *Let* $\sigma = odd/odd$, *and*

$$\sum_{s=0}^{\infty}|r_s| < \infty, \tag{3.3}$$

$$-\infty < \sum_{s=k_0}^{\infty} q_{s+1} < \infty, \tag{3.4}$$

$$g(u,v) \geqslant \mu > 0 \text{ for all } u,v \neq 0, \tag{3.5}$$

$$\sum^{\infty}\frac{1}{a_s^{1/\sigma}} = \infty, \tag{3.6}$$

$$\sum^{\infty}\frac{1}{a_s} = \infty, \tag{3.7}$$

and let $\{y_k\}$ *be a nonoscillatory solution of (1.3) such that* $\liminf_{k\to\infty}|y_k| > 0$, *and there exists* $L > 0$ *such that*

$$|\Delta y_k| \begin{cases} \geqslant L^{\frac{1}{\sigma-1}}, & \sigma < 1 \\ \leqslant \infty, & \sigma = 1 \\ \leqslant L^{\frac{1}{\sigma-1}}, & \sigma > 1. \end{cases} \tag{3.8}$$

Then,

$$\sum_{s=k_0}^{\infty}\frac{a_s(\Delta y_s)^{\sigma+1}g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})} < \infty, \tag{3.9}$$

$$\lim_{k\to\infty}\frac{a_k(\Delta y_k)^{\sigma}}{f(y_k)} = 0, \tag{3.10}$$

and

$$\frac{a_k(\Delta y_k)^{\sigma}}{f(y_k)} = \sum_{s=k}^{\infty}\frac{a_s(\Delta y_s)^{\sigma+1}g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})} + \sum_{s=k}^{\infty}\left[q_{s+1} - \frac{r_s}{f(y_{s+1})}\right] \tag{3.11}$$

for sufficiently large k.

Proof. Since $\liminf_{k\to\infty}|y_k|>0$, there exist $k_1\geqslant k_0$ and $m_1,m_2>0$ such that $|y_k|\geqslant m_1$ and $|f(y_k)|\geqslant m_2$ for $k\in\mathbf{N}_{k_1}$. Then, it follows from (3.3) that

$$\left|\sum_{s=k_1}^{k}\frac{r_s}{f(y_{s+1})}\right|\leqslant\sum_{s=k_1}^{k}\left|\frac{r_s}{f(y_{s+1})}\right|\leqslant\frac{1}{m_2}\sum_{s=k_1}^{k}|r_s|\leqslant m_3,\quad k\in\mathbf{N}_{k_1} \tag{3.12}$$

where m_3 is a finite positive constant.

Suppose that (3.9) does not hold. Then, in view of (3.4) and (3.12), we see that (3.1) is satisfied for $k\in\mathbf{N}_{k_1}$ if k_1 is sufficiently large. Suppose that $\{y_k\}$ is positive for $k\in\mathbf{N}_{k_1}$. Applying Lemma 3.2, we obtain

$$\Delta y_k\leqslant[-mf(y_{k_1})]^{1/\sigma}\frac{1}{a_k^{1/\sigma}},\ k\in\mathbf{N}_{k_1}. \tag{3.13}$$

Summing (3.13) from k_1 to $(k-1)$, we get

$$y_k\leqslant y_{k_1}-[mf(y_{k_1})]^{1/\sigma}\sum_{s=k_1}^{k-1}\frac{1}{a_s^{1/\sigma}}. \tag{3.14}$$

By (3.6) the right side of (3.14) tends to $-\infty$ as $k\to\infty$ whereas the left side is positive. The case when $\{y_k\}$ is negative for $k\in\mathbf{N}_{k_1}$ follows a similar argument. Hence, (3.9) is proved.

Next, to prove (3.10) and (3.11), we note that (1.3) can be written as

$$\frac{\Delta w_k}{f(y_{k+1})}=\frac{r_k}{f(y_{k+1})}-q_{k+1}. \tag{3.15}$$

Then, it follows from (ii) and (3.15) that

$$\Delta\left[\frac{w_k}{f(y_k)}\right]=\frac{r_k}{f(y_{k+1})}-q_{k+1}-\frac{w_k(\Delta y_k)g(y_{k+1},y_k)}{f(y_k)f(y_{k+1})}. \tag{3.16}$$

We sum (3.16) from k_0 to $(k-1)$, to obtain

$$\frac{a_k(\Delta y_k)^\sigma}{f(y_k)}=\frac{a_{k_0}(\Delta y_{k_0})^\sigma}{f(y_{k_0})}-\sum_{s=k_0}^{k-1}\left[q_{s+1}-\frac{r_s}{f(y_{s+1})}\right]-\sum_{s=k_0}^{k-1}\frac{a_s(\Delta y_s)^{\sigma+1}g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}. \tag{3.17}$$

In view of (3.4), (3.12) and (3.9), it follows from (3.17) that $\beta=\lim_{k\to\infty}a_k(\Delta y_k)^\sigma/f(y_k)$ exists. Letting $k\to\infty$ in (3.17) and changing k_0 to k provides

$$\frac{a_k(\Delta y_k)^\sigma}{f(y_k)}=\beta+\sum_{s=k}^{\infty}\left[q_{s+1}-\frac{r_s}{f(y_{s+1})}\right]+\sum_{s=k}^{\infty}\frac{a_s(\Delta y_s)^{\sigma+1}g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}. \tag{3.18}$$

Hence, (3.10) and (3.11) are proved if we can show that $\beta=0$.

Case 1. Suppose that $\beta < 0$. Then, (3.4), (3.12) and (3.9), respectively, for $k \in \mathbf{N}_{k_1}$ imply

$$\left|\sum_{s=k}^{\infty} q_{s+1}\right| \leqslant -\frac{\beta}{6}, \quad \left|\sum_{s=k}^{\infty} \frac{r_s}{f(y_{s+1})}\right| \leqslant -\frac{\beta}{6}, \quad \text{and} \quad \left|\sum_{s=k_1}^{\infty} \frac{a_s(\Delta y_s)^{\sigma+1} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})}\right| \leqslant -\frac{\beta}{6}. \tag{3.19}$$

Next, let $k = k_0$ in (3.18) to obtain

$$\frac{a_{k_0}(\Delta y_{k_0})^{\sigma}}{f(y_{k_0})} = \beta + \sum_{s=k_0}^{\infty}\left[q_{s+1} - \frac{r_s}{f(y_{s+1})}\right] + \sum_{s=k_0}^{\infty} \frac{a_s(\Delta y_s)^{\sigma+1} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})}. \tag{3.20}$$

Using (3.20) and also the inequalities (3.5), (3.19), we find

$$\begin{aligned}
&-\frac{a_{k_0}(\Delta y_{k_0})^{\sigma}}{f(y_{k_0})} + \sum_{s=k_0}^{k-1}\left[q_{s+1} - \frac{r_s}{f(y_{s+1})}\right] + \sum_{s=k_0}^{k_1-1} \frac{a_s(\Delta y_s)^{\sigma+1} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} \\
&= -\beta - \sum_{s=k}^{\infty}\left[q_{s+1} - \frac{r_s}{f(y_{s+1})}\right] - \sum_{s=k_1}^{\infty} \frac{a_s(\Delta y_s)^{\sigma+1} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} \\
&\geqslant -\beta + \frac{\beta}{6} + \frac{\beta}{6} + \frac{\beta}{6} = -\frac{\beta}{2} \equiv m > 0, \quad k \in \mathbf{N}_{k_1},
\end{aligned}$$

i.e., (3.1) is satisfied. Hence, we can apply Lemma 2.2 and obtain a contradiction as earlier.

Case 2. Suppose that $\beta > 0$. From the definition of β, we may assume that

$$\frac{w_k}{f(y_k)} = \frac{a_k(\Delta y_k)^{\sigma}}{f(y_k)} \geqslant \frac{\beta}{2}, \ k \in \mathbf{N}_{k_1}. \tag{3.21}$$

Now, using (3.21), (3.5) and (3.8) we find for $k \in \mathbf{N}_{k_1}$,

$$\begin{aligned}
\frac{w_k g(y_{k+1}, y_k)}{a_k(\Delta y_k)^{\sigma-1} f(y_{k+1})} &= \frac{w_k g(y_{k+1}, y_k)}{a_k(\Delta y_k)^{\sigma-1}[f(y_{k+1}) - f(y_k) + f(y_k)]} \\
&\geqslant \frac{\frac{w_k}{f(y_k)} g(y_{k+1}, y_k)}{\frac{w_k}{f(y_k)} g(y_{k+1}, y_k) + a_k|\Delta y_k|^{\sigma-1}} \geqslant \frac{\frac{\beta}{2}\mu}{\frac{\beta}{2}\mu + a_k|\Delta y_k|^{\sigma-1}} \\
&\geqslant \begin{cases} \dfrac{\frac{\beta}{2}\mu}{\frac{\beta}{2}\mu + a_k L}, & \sigma \neq 1 \\[2ex] \dfrac{\frac{\beta}{2}\mu}{\frac{\beta}{2}\mu + a_k}, & \sigma = 1. \end{cases}
\end{aligned} \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$\sum_{s=k_1}^{\infty}\frac{a_s(\Delta y_s)^{\sigma+1}g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})} = \sum_{s=k_1}^{\infty}\frac{w_s^2 g(y_{s+1},y_s)}{a_s(\Delta y_s)^{\sigma-1}f(y_s)f(y_{s+1})}$$
$$\geqslant \begin{cases} \dfrac{\beta}{2}\displaystyle\sum_{s=k_1}^{\infty}\frac{\frac{\beta}{2}\mu}{\frac{\beta}{2}\mu+a_sL}, & \sigma\neq 1 \\ \frac{\beta}{2}\displaystyle\sum_{s=k_1}^{\infty}\frac{\frac{\beta}{2}\mu}{\frac{\beta}{2}\mu+a_s}, & \sigma=1. \end{cases} \tag{3.23}$$

By (3.7) the right side of (3.23) is infinite whereas the left side is finite by (3.9). □

We note that if (3.3) and (3.4) hold, then

$$h_0(k)=\sum_{s=k}^{\infty}(q_{s+1}-\ell|r_s|),\quad k\in\mathbf{N}_{k_0}$$

is finite for any positive constant ℓ. Assume that $h_0(k)\geqslant 0$ for sufficiently large k. Define, for a positive integer m and a positive constant K, the following series

$$h_1(k)=\sum_{s=k}^{\infty}\frac{[h_0(s)]^2}{a_s+Kh_0(s)}\quad\text{and}\quad h_{m+1}(k)=\sum_{s=k}^{\infty}\frac{[h_0(s)+Kh_m(s)]^2}{a_s+K[h_0(s)+Kh_m(s)]}.$$

CONDITION (H). For every $K>0$, there exists a positive integer M such that $h_m(k)$ is finite for $m=1,2,\cdots,M-1$ and $h_M(k)$ is infinite.

THEOREM 3.4. *Let* $\sigma=odd/odd$. *Suppose that (3.3) – (3.7) and (H) hold. Let* $\{y_k\}$ *be any solution of (1.3) such that (3.8) holds. Then,* $\{y_k\}$ *is either oscillatory or satisfies* $\liminf_{k\to\infty}|y_k|=0$.

Proof. Suppose on the contrary that $\{y_k\}$ is a nonoscillatory solution of (1.3) and $\liminf_{k\to\infty}|y_k|>0$. Hence, by Theorem 3.3, $\{y_k\}$ satisfies (3.9) – (3.11). Furthermore, there exists $k_1\geqslant k_0$ and $m_1,m_2>0$ such that $|y_k|\geqslant m_1$ and $|f(y_k)|\geqslant m_2$ for $k\in\mathbf{N}_{k_1}$. Hence, from (3.11) we find

$$\frac{w_k}{f(y_k)}=\frac{a_k(\Delta y_k)^{\sigma}}{f(y_k)}\geqslant\sum_{s=k}^{\infty}\frac{a_s(\Delta y_s)^{\sigma+1}g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}+h_0(k) \tag{3.24}$$
$$\geqslant h_0(k),\ k\in\mathbf{N}_{k_1}. \tag{3.25}$$

It follows from (3.25), (3.5) and (3.8) that

$$\frac{w_k^2 g(y_{k+1},y_k)}{a_k(\Delta y_k)^{\sigma-1} f(y_k) f(y_{k+1})} = \frac{\left[\frac{w_k}{f(y_k)}\right]^2 g(y_{k+1},y_k)}{\frac{w_k}{f(y_k)}\, g(y_{k+1},y_k) + a_k(\Delta y_k)^{\sigma-1}}$$

$$\geqslant \frac{[h_0(k)]^2\mu}{h_0(k)\mu + a_k|\Delta y_k|^{\sigma-1}}$$

$$\geqslant \left\{ \begin{array}{ll} \dfrac{[h_0(k)]^2\mu}{h_0(k)\mu + a_k L}, & \sigma \neq 1 \\ \dfrac{[h_0(k)]^2\mu}{h_0(k)\mu + a_k}, & \sigma = 1 \end{array} \right\}$$

$$= \frac{K[h_0(k)]^2}{K h_0(k) + a_k}, \quad k \in \mathbf{N}_{k_1}$$

where

$$K = \begin{cases} \dfrac{\mu}{L}, & \sigma \neq 1 \\ \mu, & \sigma = 1. \end{cases}$$

Therefore,

$$\sum_{s=k}^{\infty} \frac{a_s(\Delta y_s)^{\sigma+1} g(y_{s+1},y_s)}{f(y_s) f(y_{s+1})} = \sum_{s=k}^{\infty} \frac{w_s^2 g(y_{s+1},y_s)}{a_s(\Delta y_s)^{\sigma-1} f(y_s) f(y_{s+1})}$$
$$\geqslant \sum_{s=k}^{\infty} \frac{K[h_0(s)]^2}{K h_0(s) + a_s} = K h_1(k), \quad k \in \mathbf{N}_{k_1}. \tag{3.26}$$

If $M = 1$ in (H), then the right side of (3.26) is infinite. This is a contradiction to (3.9).

Next, it follows from (3.24) and (3.26) that

$$\frac{w_k}{f(y_k)} \geqslant h_0(k) + K h_1(k), \quad k \in \mathbf{N}_{k_1}$$

and by using a similar technique, we obtain

$$\sum_{s=k}^{\infty} \frac{a_s(\Delta y_s)^{\sigma+1} g(y_{s+1},y_s)}{f(y_s) f(y_{s+1})} \geqslant \sum_{s=k}^{\infty} \frac{K[h_0(s) + K h_1(s)]^2}{K[h_0(s) + K h_1(s)] + a_s} = K h_2(k), \quad k \in \mathbf{N}_{k_1}. \tag{3.27}$$

If $M = 2$ in (H), then the right side of (3.27) is infinite. This again contradicts (3.9). A similar argument yields a contradiction for any integer $M > 2$. This completes the proof of the theorem. □

Finally, we state three results for the case $r_k \equiv 0$. In these results we shall use the equation number $(\cdot)_0$ to denote the case $r_k \equiv 0$.

THEOREM 3.5. [35] *Let $\sigma = odd/odd$. Suppose that (3.4) – (3.7) and (H) hold. Let $\{y_k\}$ be any solution of $(1.3)_0$ such that (3.8) holds. Then, $\{y_k\}$ is oscillatory.*

THEOREM 3.6. [35] *Let $\sigma = odd/odd$. Let $\{y_k\}$ be any solution of $(1.3)_0$ such that (3.8) holds. Suppose that (3.4) – (3.7) hold, and*

$$a_k|\Delta y_k|^{\sigma-1} + \mu h_0(k) > 0, \quad k \in \mathbf{N}_{k_0},$$

and

$$\begin{cases} \displaystyle\sum_{s=k_0}^{\infty} \frac{[h_0^+(s)]^2}{a_s + \frac{\mu}{L}\, h_0(s)} = \infty, & \sigma \neq 1 \\ \displaystyle\sum_{s=k_0}^{\infty} \frac{[h_0^+(s)]^2}{a_s + \mu h_0(s)} = \infty, & \sigma = 1 \end{cases}$$

where $h_0^+(s) = \max\{h_0(s), 0\}$. Then, $\{y_k\}$ is oscillatory.

THEOREM 3.7. [35] *Suppose*

$$\sum_{s=k_0}^{\infty} q_{s+1} = \infty.$$

(a) If $\sigma = odd/odd$ and (3.6) holds, then all solutions of $(1.3)_0$ are oscillatory.
(b) If $\sigma = even/odd$, then a solution $\{y_k\}$ of $(1.3)_0$ is either oscillatory or $\{\Delta y_k\}$ is oscillatory.

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