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SOME NEW OPIAL–TYPE INEQUALITIES

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Abstract. In [3] Agarwal, Pečarić and Brnetić improved many known Opial-type inequalities in n independent variables. Here we shall use the same basic idea as used in [3] to improve some Opial-type inequalities in one variable proved by Agarwal and Pang [1]. We shall also make some remarks on the results proved in [3] and give some simple generalizations.

1. Introduction

First, we state the results we shall improve:

THEOREM A. [1, Lemma 9.1.] *Let $\lambda \geqslant 1$ be a given real number and let p be a non-negative and continuous function on $[0,h]$. Further, let x be an absolutely continuous function on $[0,h]$, with $x(0) = x(h) = 0$. Then, the following inequality holds:*

$$\int_0^h p(t)|x(t)|^\lambda dt \leqslant \frac{1}{2}\left(\int_0^h \big(t(h-t)\big)^{\frac{\lambda-1}{2}} p(t)dt\right)\int_0^h |x'(t)|^\lambda dt. \tag{1}$$

For $p(t) = const.$ the inequality (1) reduces to

$$\int_0^h |x(t)|^\lambda dt \leqslant \frac{h^\lambda}{2} B\left(\frac{\lambda+1}{2}, \frac{\lambda+1}{2}\right)\int_0^h |x'(t)|^\lambda dt, \tag{2}$$

where B is the beta function.

THEOREM B. [1, Theorem 9.2.] *Assume that*

(*i*) *l, m, μ and λ are non-negative real numbers such that $\frac{1}{\mu} + \frac{1}{\lambda} = 1$ and $l\mu \geqslant 1$,*
(*ii*) *q is a non-negative and continuous function on $[0,h]$,*
(*iii*) *x_1 and x_2 are absolutely continuous functions on $[0,h]$, with $x_1(0) = x_1(h) = x_2(0) = x_2(h) = 0$.*

Then, the following inequality holds

$$\int_0^h q(t)\big(|x_1(t)|^l|x_2'(t)|^m + |x_2(t)|^l|x_1'(t)|^m\big)dt \leqslant \left(\frac{1}{2}\int_0^h \big(t(h-t)\big)^{\frac{l\mu-1}{2}} q^\mu(t)dt\right)^{\frac{1}{\mu}} \times$$
$$\times \int_0^h \Big(\frac{1}{\mu}(|x_1'(t)|^{l\mu} + |x_2'(t)|^{l\mu}) + \frac{1}{\lambda}(|x_1'(t)|^{m\lambda} + |x_2'(t)|^{m\lambda})\Big)dt.$$

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THEOREM C. [1, Theorem 10.1.] *Let r_k, $k = 0,\dots,n-1$, and l be non-negative real numbers such that $l\alpha \geqslant 1$, where $\alpha = \sum_{k=0}^{n-1} r_k$ and let p be a non-negative continuous function on $[0,h]$. Further, let $x \in C^{(n-1)}[0,h]$ be such that $x^{(i)}(0) = x^{(i)}(h) = 0$, $i = 0,\dots,n-1$, and $x^{(n-1)}$ is absolutely continuous. Then, the following inequality holds*

$$\int_0^h p(t)\Big(\prod_{k=0}^{n-1} |x^{(k)}(t)|^{r_k}\Big)^l dt \leqslant \frac{1}{2\alpha}\Big(\int_0^h \big(t(h-t)\big)^{\frac{l\alpha-1}{2}} p(t)dt\Big)\times \times \sum_{k=0}^{n-1} r_k \int_0^h |x^{(k+1)}(t)|^{l\alpha} dt.$$

2. The main results

An improvement of Theorem A is the following:

THEOREM 1. *Let $\lambda \geqslant 1$ be a given real number and let p be a non-negative and continuous function on $[0,h]$. Further, let x be an absolutely continuous function on $[0,h]$, with $x(0) = x(h) = 0$. Then, the following inequality holds:*

$$\int_0^h p(t)|x(t)|^\lambda dt \leqslant \Big(\int_0^h \big(t^{1-\lambda} + (h-t)^{1-\lambda}\big)^{-1} p(t)dt\Big) \int_0^h |x'(t)|^\lambda dt. \tag{3}$$

Proof. As in [1] from the hypotheses of theorem we have

$$x(t) = \int_0^t x'(s)ds\,, \quad x(t) = -\int_t^h x'(s)ds.$$

Using Hölder's inequality with indices λ and $\frac{\lambda}{\lambda-1}$, we obtain

$$|x(t)|^\lambda \leqslant t^{\lambda-1}\int_0^t |x'(s)|^\lambda ds, \tag{4}$$

and, similarly,

$$|x(t)|^\lambda \leqslant (h-t)^{\lambda-1}\int_t^h |x'(s)|^\lambda ds. \tag{5}$$

Multiplying (4) by $t^{1-\lambda}$ and (5) by $(h-t)^{1-\lambda}$ and adding these inequalities, we find

$$\big(t^{1-\lambda} + (h-t)^{1-\lambda}\big)|x(t)|^\lambda \leqslant \int_0^h |x'(t)|^\lambda dt,$$

i.e.

$$|x(t)|^\lambda \leqslant \big(t^{1-\lambda} + (h-t)^{1-\lambda}\big)^{-1}\int_0^h |x'(t)|^\lambda dt. \tag{6}$$

Now multiplying (6) by p and integrating on $[0,h]$ we obtain the inequality (3).

REMARK 1. By the harmonic-geometric inequality, we have

$$2\left(t^{1-\lambda}+(h-t)^{1-\lambda}\right)^{-1} \leqslant \left(t^{\lambda-1}(h-t)^{\lambda-1}\right)^{\frac{1}{2}}.$$

Hence, it is clear that (3) improves (1).

COROLLARY 2. *Let* $\lambda \geqslant 1$ *be a given real number and let* x *be an absolutely continuous function on* $[0,h]$, *with* $x(0)=x(h)=0$. *Then, the following inequality holds:*

$$\int_0^h |x(t)|^\lambda dt \leqslant h^\lambda I(\lambda) \int_0^h |x'(t)|^\lambda dt, \tag{7}$$

where

$$I(\lambda)=\int_0^1 \left(s^{1-\lambda}+(1-s)^{1-\lambda}\right)^{-1} ds.$$

Proof. By putting $p(t)=const.$ in (3) we obtain

$$\int_0^h |x(t)|^\lambda dt \leqslant \left(\int_0^h \left(t^{1-\lambda}+(h-t)^{1-\lambda}\right)^{-1} dt\right) \int_0^h |x'(t)|^\lambda dt.$$

The inequality (7) is now clear.

COROLLARY 3. *Let* $\lambda \geqslant 1$ *be a given real number and let* x *be an absolutely continuous function on* $[0,h]$, *with* $x(0)=x(h)=0$. *Then, the following inequalities hold:*

$$\int_0^h |x(t)|^2 dt \leqslant \frac{h^2}{6} \int_0^h |x'(t)|^2 dt, \tag{8}$$

$$\int_0^h |x(t)|^3 dt \leqslant \frac{3\pi-8}{24} h^3 \int_0^h |x'(t)|^3 dt, \tag{9}$$

$$\int_0^h |x(t)|^4 dt \leqslant \frac{20\sqrt{3}\pi-81}{1215} h^4 \int_0^h |x'(t)|^4 dt. \tag{10}$$

Proof. It is a special case of Corollary 2 for $\lambda=2,3,4$. The integrals $I(2)=\frac{1}{6}$, $I(3)=\frac{3\pi-8}{24}$ and $I(4)=\frac{20\sqrt{3}\pi-81}{1215}$ are computed easily.

REMARK 2. It is interesting to compare this results with the inequalities which follow from Theorem A by taking $p(t)=const.$ and $\lambda=2,3,4$. From (2), the inequalities corresponding to (8) - (10) will have corresponding constants $\frac{\pi h^2}{16}$, $\frac{h^3}{12}$ and $\frac{3\pi h^4}{256}$.

Now we shall improve the result of Theorem B.

THEOREM 4. *Assume that*

(*i*) l, m, μ *and* λ *are non-negative real numbers such that* $\frac{1}{\mu}+\frac{1}{\lambda}=1$ *and* $l\mu \geqslant 1$,

(*ii*) q *is a non-negative and continuous function on* $[0,h]$,

(*iii*) x_1 *and* x_2 *are absolutely continuous functions on* $[0,h]$, *with* $x_1(0)=x_1(h)=x_2(0)=x_2(h)=0$.

Then, the following inequality holds

$$\int_0^h q(t)\big(|x_1(t)|^l|x_2'(t)|^m+|x_2(t)|^l|x_1'(t)|^m\big)dt \leqslant \left(\int_0^h \big(t^{1-l\mu}+(h-t)^{1-l\mu}\big)^{-1}q^\mu(t)dt\right)^{\frac{1}{\mu}}\times$$
$$\times\int_0^h \Big(\frac{1}{\mu}(|x_1'(t)|^{l\mu}+|x_2'(t)|^{l\mu})+\frac{1}{\lambda}(|x_1'(t)|^{m\lambda}+|x_2'(t)|^{m\lambda})\Big)dt. \quad (11)$$

Proof. From Hölder's inequality with indices μ and λ we have

$$\int_0^h q(t)|x_1(t)|^l|x_2'(t)|^m dt \leqslant \left(\int_0^h q^\mu(t)|x_1(t)|^{l\mu}dt\right)^{\frac{1}{\mu}}\left(\int_0^h |x_2'(t)|^{m\lambda}dt\right)^{\frac{1}{\lambda}}.$$

Now, from (3) we find

$$\int_0^h q(t)|x_1(t)|^l|x_2'(t)|^m dt \leqslant \left(\int_0^h \big(t^{1-l\mu}+(h-t)^{1-l\mu}\big)^{-1}q^\mu(t)dt\right)^{\frac{1}{\mu}}\times$$
$$\times\left(\int_0^h |x_1'(t)|^{l\mu}dt\right)^{\frac{1}{\mu}}\left(\int_0^h |x_2'(t)|^{m\lambda}dt\right)^{\frac{1}{\lambda}},$$

and from Young's inequality it follows that

$$\int_0^h q(t)|x_1(t)|^l|x_2'(t)|^m dt \leqslant \left(\int_0^h \big(t^{1-l\mu}+(h-t)^{1-l\mu}\big)^{-1}q^\mu(t)dt\right)^{\frac{1}{\mu}}\times$$
$$\times\int_0^h \Big(\frac{1}{\mu}|x_1'(t)|^{l\mu}+\frac{1}{\lambda}|x_2'(t)|^{m\lambda}\Big)dt. \quad (12)$$

Similarly, we obtain

$$\int_0^h q(t)|x_1'(t)|^m|x_2(t)|^l dt \leqslant \left(\int_0^h \big(t^{1-l\mu}+(h-t)^{1-l\mu}\big)^{-1}q^\mu(t)dt\right)^{\frac{1}{\mu}}\times$$
$$\times\int_0^h \Big(\frac{1}{\lambda}|x_1'(t)|^{m\lambda}+\frac{1}{\mu}|x_2'(t)|^{l\mu}\Big)dt. \quad (13)$$

An addition of (12) and (13) gives the inequality (11).

Now we will establish Opial-type inequality involving higher order derivatives which improves the result of Theorem C.

THEOREM 5. *Let* r_k, $k=0,\dots,n-1$, *and* l *be non-negative real numbers such that* $l\alpha \geqslant 1$, *where* $\alpha=\sum_{k=0}^{n-1} r_k$ *and let* p *be a non-negative continuous function on* $[0,h]$. *Further, let* $x\in C^{(n-1)}[0,h]$ *be such that* $x^{(i)}(0)=x^{(i)}(h)=0$, $i=0,\dots,n-1$, *and* $x^{(n-1)}$ *is absolutely continuous. Then, the following inequality holds*

$$\int_0^h p(t)\left(\prod_{k=0}^{n-1}|x^{(k)}(t)|^{r_k}\right)^l dt \leqslant \frac{1}{\alpha}\left(\int_0^h \big(t^{1-l\alpha}+(h-t)^{1-l\alpha}\big)^{-1}p(t)dt\right)\times$$
$$\times\sum_{k=0}^{n-1} r_k\int_0^h |x^{(k+1)}(t)|^{l\alpha}dt. \quad (14)$$

Proof. Using some well-known elementary inequalities, we have

$$\left(\prod_{k=0}^{n-1}|x^{(k)}(t)|^{r_k}\right)^l = \left(\prod_{k=0}^{n-1}|x^{(k)}(t)|^{\frac{r_k}{\alpha}}\right)^{l\alpha} \leqslant \left(\sum_{k=0}^{n-1}\frac{r_k}{\alpha}|x^{(k)}(t)|\right)^{l\alpha}$$
$$\leqslant \sum_{k=0}^{n-1}\frac{r_k}{\alpha}|x^{(k)}(t)|^{l\alpha}. \tag{15}$$

The inequality (14) now follows from (15) and (3).

Let $\Omega = \prod_{i=1}^{n}[a_i, b_i]$ and let $x = (x_1, \ldots, x_n)$ be a general point in Ω and dx stands for $dx_1 \ldots dx_n$. Let $G_n(a_i)$ stands for geometric mean and $H_n(a_i)$ stands for harmonic mean of $a_1, \ldots, a_n$, and generally, let $M^{[k]}(a_i)$ stands for mean of order k of $a_1, \ldots, a_n$, i.e. $M^{[k]}(a_i) = \left(\frac{\sum_{i=1}^{n} a_i^k}{n}\right)^{\frac{1}{k}}$, $k \neq 0$.

We denote by $G(\Omega)$ the class of sufficiently smooth real functions which vanish on the boundary $\partial\Omega$ of Ω. Further, let $u(x;t_i)$ stands for $u(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n)$ and define

$$||grad \quad u(x)||_\mu = (\sum_{i=1}^{n}|\frac{\partial}{\partial x_i}u(x)|^\mu)^{\frac{1}{\mu}}.$$

The generalization of Corollary 2 for the case of several independent variables is given in [3]. Let's repeate this result:

THEOREM D. [3, Theorem 1] *Let λ, $\mu \geqslant 1$ be real numbers and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^\lambda dx \leqslant K(\lambda,\mu)\int_\Omega ||grad \quad u(x)||_\mu^\lambda dx,$$

where

$$K(\lambda,\mu) = \frac{1}{n}I(\lambda)C(\frac{\lambda}{\mu})H_n((b_i-a_i)^\lambda),$$
$$I(\lambda) = \int_0^1\left(s^{1-\lambda}+(1-s)^{1-\lambda}\right)^{-1}ds,$$
$$C(\alpha) = 1, \textit{ for } \alpha \geqslant 1 \quad \textit{and} \quad C(\alpha) = n^{1-\alpha}, \textit{ for } 0 \leqslant \alpha \leqslant 1.$$

Moreover, the following theorem is a consequence of Theorem D:

THEOREM 6. *Let $\mu_k > 0$, $\lambda_k \geqslant 1$, $k = 1, \ldots, m$, be real numbers such that $\sum_{k=1}^{m}\frac{\mu_k}{\lambda_k} = 1$ and let $u_k \in G(\Omega)$, $k = 1, \ldots, m$. Then, the following inequality holds*

$$\int_\Omega\prod_{k=1}^{m}|u_k(x)|^{\mu_k}dx \leqslant \frac{M^{[-2]}(b_i-a_i)}{\sqrt{6n}}\sum_{k=1}^{m}\frac{\mu_k}{\lambda_k}(\int_\Omega|u(x)|^{2(\lambda_k-1)}dx)^{\frac{1}{2}}\times$$
$$\times(\int_\Omega||grad \quad u(x)||_2^2dx)^{\frac{1}{2}}. \tag{16}$$

Proof. By applying Cauchy-Schwarz inequality and the result of Theorem D for $\lambda = 2$ and $\mu = 2$, we find

$$\int_\Omega |u(x)|^{\lambda_k} dx \leqslant \left(\int_\Omega |u(x)|^{2(\lambda_k-1)} dx\right)^{\frac{1}{2}} \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}$$
$$\leqslant \frac{M^{[-2]}(b_i - a_i)}{\sqrt{6n}} \left(\int_\Omega |u(x)|^{2(\lambda_k-1)} dx\right)^{\frac{1}{2}} \left(\int_\Omega ||grad \quad u(x)||_2^2 dx\right)^{\frac{1}{2}}. \tag{17}$$

The inequality (16) now follows from (17) by using weighted arithmetic-geometric inequality.

REMARK 3. In [3, Remark 12] a special case of Theorem 6 (for $m = 2$ and $\mu_1 = \mu_2 = 1$) is proved. In [4] Agarwal and Sheng obtained the same type of inequality for $m = 2$ and with the right-hand side of the inequality (16) multiplied by $\sqrt{\frac{3\pi}{8}}$ and the term $G_n(b_i - a_i)$ instead of $M^{[-2]}(b_i - a_i)$. On the other hand, in [5], Cheung obtained the same type of inequality with the right-hand side of the inequality (16) multiplied by $\frac{\sqrt{6}}{2}$, the term $max\{b_i - a_i : i = 1, \ldots, m\}$ instead of $M^{[-2]}(b_i - a_i)$ and the term μ_k instead of μ_k/λ_k.

REFERENCES

[1] R. P. AGARWAL AND P. Y. H. PANG, *Remarks on the generalization of Opial's inequality*, Journal of Math.Analysis and Appl. **190** (1995), 559–577.

[2] ———, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London,, 1995.

[3] R. P. AGARWAL, J. PEČARIĆ AND I. BRNETIĆ, *Improved integral inequalities in n independent variables*, Computers Math. Applic. **33** (8) (1997), 27–38.

[4] R. P. AGARWAL AND Q. SHENG, *Sharp integral inequalities in n independent variable*, Nonlinear Analysis **26** (2) (1996), 179–210.

[5] W. S. CHEUNG, *On integral inequalities of the Sobolev type*, Acquationes Mathematicae **49** (1995), 153–159.

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