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STABILITY OF INTEGRODIFFERENTIAL SYSTEMS OF NONCONVOLUTION TYPE

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Abstract. A diagonal dominance criterion for the stability of linear Volterra integrodifferential equations of nonconvolution type is given. An alternative method using Liapunov functionals is also introduced.

1. Introduction

Consider the following Volterra system of convolution type

$$x'(t) = Ax(t) + \int_0^t K(t-s)\, x(s)\, ds \tag{1}$$

where $A = (a_{ij})$ is $n \times n$ real matrix, $K(t) = (k_{ij}(t)) \epsilon L^1[0, \infty)$.

The basic theorem for stability is due to Miller [9] which states that the zero solution of Eq. (1) is asymptotically stable [2] iff

$$det(zI - A - K^*(z)) \neq 0, \text{ for } Re\, z \geqslant 0 \tag{2}$$

where

$$K^*(z) = \int_0^\infty e^{-zt} K(t)\, dt$$

is the Laplace transform of $K(t)$.

For the scalar case $(n = 1)$, Brauer [1] used condition (2) to establish criteria for the asymptotic stability of Eq. (1). This was extended by Jordan [7] to systems $(n > 1)$. Recently, Krisztin [8] improved these results for both scalar equations and systems.

Now consider the following infinite delay system of convolution type

$$x'(t) = Ax(t) + \int_{-\infty}^t K(t-s)x(s)\, ds. \tag{3}$$

In this case there are no available necessary and sufficient conditions such as Miller's [9] for the asymptotic stability of the zero solution of Eq. (3). However, for

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the scalar case $(n = 1)$, Gopalsamy [6] was able to extend some of Brauer's results to Eq. (3).

THEOREM 1. [6] *Assume that*

$$\int_0^\infty t|K(t)|dt < \infty, \int_0^\infty |K(t)|dt < \infty,$$

$$A < 0, \text{ and } |A| > \int_0^\infty |K(t)|\, dt. \tag{4}$$

Then for bounded initial functions, the zero solution of (3) *is asymptotically stable. If instead of* (4) *we have*

$$A + \int_0^\infty |K(t)|\, dt \geqslant 0, \tag{5}$$

then the zero solution of Eq. (3) *is not asymptotically stable.*

For linear systems of nonconvolution type, we can no longer use Miller's Condition (2) since it is based on Laplace transforms. Burton [2] and Elaydi and Sivasundaram [5] used Liapunov functions to investigate the stability of such systems. In this paper, we give two different schemes to study the stability of the zero solution for infinite delay systems. In the first approach we give a column dominance criterion for stability which would extend the results in [1], [5], [7], [8] to this class of equations. When specified for the convolution case, these results improve those obtained by Jordan [7] and Krisztin [8]. If this scheme fails, we introduce another approach that utilizes a certain Liapunov functional. This approach has been used for the finite delay case in [5]. Here we improve the results obtained in [5] and extend them to the infinite delay case.

Consider now the following Volterra system of nonconvolution type

$$x'(t) = A(t)x(t) + \int_{-\infty}^t K(t,s)x(s)\, ds, \tag{6}$$

where $A(t) = (a_{ij}(t))$ is a $n \times n$ matrix function continuous and bounded on $\mathbf{R}$, and $K(t,s) = (k_{ij}(t,s))$ is a $n \times n$ matrix function such that for all $t_o \geqslant 0$,

$$\int_{-\infty}^{t_0} \int_{t_0}^\infty |K_{ij}(u,s)|du\, ds < \infty, \; 1 \leqslant i, j \leqslant n. \tag{7}$$

Note that if $K(t,s) = K(t-s)$, then Condition (7) reduces to

$$\int_0^\infty t|K_{ij}(t)|dt < \infty, \; 1 \leqslant i, j \leqslant n \tag{8}$$

Following [2], [9], $x(t,t_0,\varphi)$, $t \geqslant t_0 \geqslant 0$, denotes the solution of Eq. (6) with an initial function $\varphi : (-\infty, t_0] \to \mathbf{R}^n$, which is assumed to be bounded and continuous. The norm on the initial function φ is given by $||\varphi|| = \sup\{|\varphi(t)| : -\infty \leqslant t \leqslant t_0\}$.

For definitions of various stability, the reader is referred to [2].

2. Column Dominance Criterion

Throughout the section we will adopt the following notations and definitions:

$$b_{ij}(t) = \int_t^{\infty} |K_{ij}(u,t)|\, du, \tag{9}$$

$$|x| = \sum_{i=1}^{n} |x_i| \tag{10}$$

$$\sum_j{}' a_{ji} = \sum_{j=1}^{n} a_{ji} - a_{ii}. \tag{11}$$

THEOREM 2. *Assume that for* $1 \leqslant i \leqslant n,\ t \geqslant t_0$.

$$a_{ii}(t) + \sum_j{}' |a_{ji}(t)| + \sum_{j=1}^{n} b_{ji}(t) \leqslant 0. \tag{12}$$

Then the zero solution of Eq. (6-7) *is stable.*

Proof. Define a Liapunov functional as

$$V(t,x(\cdot)) = \sum_{i=1}^{n} \left[|x_i(t)| + \sum_{j=1}^{n} \int_{-\infty}^{t} \int_t^{\infty} |k_{ij}(u,s)|\, du\, |x_j(s)|\, ds \right]. \tag{13}$$

Then

$$\begin{aligned} V'_{(7)}(t,x(\cdot)) &\leqslant \sum_{i=1}^{n} \left[sgn x_i(t) \sum_{j=1}^{n} a_{ij}(t) x_j(t) + \sum_{j=1}^{n} \int_t^{\infty} |k_{ij}(u,t)| du |x_j(t)| \right] \\ &\leqslant \sum_{i=1}^{n} \left[a_{ii}(t)|x_i| + \sum_j{}' |a_{ij}(t)||x_j(t)| + \sum_{j=1}^{n} \int_t^{\infty} |k_{ij}(u,t)| du |x_j(t)| \right] \\ &\leqslant \sum_{i=1}^{n} \left[a_{ii}(t)|x_i(t)| + \sum_j{}' |a_{ji}(t)|\, |x_j(t)| + \sum_{j=1}^{n} \int_t^{\infty} |k_{ji}(u,t)| du |x_i(t)| \right] \\ &\leqslant \sum_{i=1}^{n} \left[a_{ii}(t) + \sum_j{}' |a_{ji}(t)| + b_{ji}(t) \right] |x_i(t)|. \end{aligned} \tag{14}$$

Using Condition (12), and Inequality (14) we obtain

$$V'_{(6)}(t,x(\cdot)) \leqslant 0. \tag{15}$$

Now

$$V(t_0,\varphi(\cdot)) = \sum_{i=1}^{n} \left[|\varphi_i(t_0)| + \sum_{j=1}^{n} \int_{-\infty}^{t_0} \int_{t_0}^{-\infty} |k_{ij}(u,s)| du |\varphi_j(s)| ds \right] \leqslant \gamma ||\varphi|| \tag{16}$$

where

$$\gamma = 1 + \sum_{j=1}^{n} \int_{-\infty}^{t_0} \int_{t_0}^{-\infty} |k_{ij}(u,s)| du\, ds.$$

Thus from (14), (16), (17) we obtain

$$x(t, t_0, \varphi| \leqslant V(t, x(\cdot)) \leqslant V(t_0, \varphi(\cdot)) \leqslant \gamma ||\varphi||$$

This implies that the zero solution of Eq. (6-7) is stable.

THEOREM 3. *The zero solution of Eq.* (6-7) *is asymptotically stable if for some* $\delta > 0$, *and* $1 \leqslant i \leqslant n$,

$$a_{ii}(t) + \sum_{j}^{\prime} |a_{ji}(t)| + \sum_{j=1}^{n} b_{ji}(t) \leqslant -\delta. \tag{17}$$

Moreover, the zero solution of Eq. (6-7) *is not asymptotically stable if* $k_{ij}(t,s) \geqslant 0, 1 \leqslant i,j \leqslant n$, *and*

$$\sum_{i=1}^{n} (a_{ji}(t) + b_{ji}(t)) > 0. \tag{18}$$

Proof. By theorem 2, the zero solution of (6-7) is stable. Using the Liapunov functional (13) and Condition (17) in Inequality (14) we have

$$V'_{(7)}(t, x(\cdot)) \leqslant -\delta |x(t)|. \tag{19}$$

This implies that

$$\int_{t_0}^{t} |x(s)| ds \leqslant \frac{1}{\delta} \left[V(t_0, \varphi(\cdot)) \right],$$

and consequently

$$\int_{t_0}^{\infty} |x(s)| ds \leqslant M, \text{ for some } M > 0. \tag{20}$$

Using Inequalities (20) and (7) in Eq. (6) (after integrating), one obtains

$$\int_{t_0}^{\infty} |x'(s)| ds \leqslant L, \text{ for some } L > 0. \tag{21}$$

From Inequality(18), it follows that there is a sequence $\{t_i\}$ with $t_i \to \infty$ and $\sum_{j=1}^{n} |x_j(t_i)| \to 0$ as $t_i \to \infty$. Claim that $\sum_{j=1}^{n} |x_j(t)| \to 0$ as $t \to \infty$. For if not, then there exists a sequence $\{s_i\}$ with $s_i \to \infty$ and $\theta > 0$ with $\sum_{j=1}^{n} |x_j(s_i)| > \theta$ for all i. There is a positive integer i_0 such that

$$\sum_{j=1}^{n} |x_j(t_i)| < \frac{\theta}{2}, \text{ for } i \geqslant i_0.$$

Thus

$$\sum_{j=1}^{n} \int_{t_0}^{\infty} |x'_j(t)| dt > \sum_{i=1}^{\infty} \frac{\theta}{2} = \infty,$$

which contradicts Inequality (21). Hence the zero solution of Eq. (6-7) is asymptotically stable.

To prove the second part of the theorem, let $x(t, t_0, \varphi)$ be a solution of Eq. (6-7) with $\varphi_i(t) > 1$, for $t\epsilon(-\infty, t_0)$, and $1 \leqslant i \leqslant n$. Claim that $x_i(t) > 1$ for all $t \geqslant t_0$, $1 \leqslant i \leqslant n$. If not, let $x_r(t_*)$ be the first component with the smallest $t_*\epsilon(t_0, \infty)$ such that $x_r(t_*) = 1$.

Then

$$\begin{aligned} 0 \leqslant x_r'(t_*) &= \sum_{j=1}^{n} \left(a_{rj}(t_*)x_j(t_*) + \int_{-\infty}^{t_*} k_{rj}(t_*, s)x_j(s)\, ds \right) \\ &\geqslant \sum_{j=1}^{n} (a_{rj}(t_*) + b_{rj}(t_*)) > 0, \end{aligned}$$

which is absurd. This completes the proof of the second part of the theorem.

It is straightforward to show that the preceding result is valid for systems of the form

$$x'(t) = A(t)x(t) + \int_0^t K(t, s)\, x(s)\, ds.$$

Furthermore, one may extend Theorem 2 to systems of the form

$$x'(t) = \sum_{r=1}^{\infty} A_r(t)x(t - h_r) + \int_{-\infty}^{t} K(t, s)\, x(s)\, ds \tag{22}$$

where $A_r(t) = (a_{ij}^r(t))$ is an $n \times n$ matrix function continuous and bounded on $\mathbf{R}$ with $\sum_{r=0}^{\infty} |A_r(t)| < \infty$, and $K(t, s)$ satisfies Condition (7). It is assumed that $\{h_r\}$ is a sequence of real numbers such that $h_0 = 0 < h_1 < h_2 < \ldots$.

THEOREM 4. *The zero solution of Eq.* (22-7) *is asymptotically stable if for some* $\delta > 0$, $1 \leqslant j \leqslant n$,

$$a_{ii}^0(t) + \sum_{j}{}' a_{ji}^0(t) + \sum_{r=1}^{\infty} \left(\sum_{j=1}^{n} c_{ji}^r + b_{ji}(t) \right) \leqslant -\delta$$

where $b_{ji}(t)$ *as defined in* (9) *and* $c_{ji}^r = \sup_j |a_{ji}^r(s)|$, *for* $r \leqslant 1$.

Proof. Here we use the Liapunov functional

$$\begin{aligned} V(t, x(\cdot)) &= \sum_{i=1}^{n} \left[|x_i(t)| + \sum_{j=1}^{n} \sum_{r=1}^{\infty} \int_{t-h_r}^{t} c_{ij}^r |x_j(s)| ds \right] \\ &+ \sum_{j=1}^{n} \int_{-\infty}^{t} \int_{t}^{\infty} |k_{ij}(u, s)| du |x_j(s)| ds. \end{aligned}$$

The proof then proceeds in a similar fashion to that employed in the proving Theorem 2; and will thus be omitted.

3. Alternative Conditions for Stability

In this section we use the Euclidean norm for vectors and the corresponding operator norm for matrices. It is assumed that for all $t_0 \geqslant 0$,

$$\int_{-\infty}^{t_0} \int_{t_0}^{\infty} |K(u,s)| du\, ds \leqslant M, \text{ for some } M > 0 \tag{23}$$

$$|A(t)| \leqslant \lambda, \text{ for some } \lambda > 0 \tag{24}$$

and

$$\mu(A(t)) \leqslant -\rho, \text{ for some } \rho > 0 \tag{25}$$

where

$$\mu(A(t)) = \lim_{h \to 0^+} \frac{|I + hA(t)| - 1}{h}$$

is the Lozinskii norm of $A(t)$ (Coppel [3] which is equal to $\lambda_{max}[(A^T(t) + A(t))]/2$. Here λ_{max} denotes the maximum eigenvalue and $A^T(t)$ is the transpose of $A(t)$. For the equation

$$y'(t) = A(t)y(t) \tag{26}$$

it is known [3] that

$$|y_0| \cdot \exp\left(-\int_{t_0}^{t} \mu(-A(s)ds)\right) \leqslant |y(t,t_0,y_0)| \leqslant |y_0| \cdot \exp\left(\int_{t_0}^{t} \mu(A(s))ds\right). \tag{27}$$

Let $\Phi(t,t_0)$, $t \geqslant t_0 \geqslant 0$ be the fundamental matrix of Eq. (26) with $\Phi(t,t_0) = I$. Let

$$G(t) = \int_{t}^{\infty} \Phi^T(s,t)\Phi(s,t)ds, \tag{28}$$

where Φ^T denotes the transpose of Φ.

Under Assumption (23), $G(t)$ is defined for all $t \geqslant t_0 \geqslant 0$.

The following two theorems improve Theorems 2.1 and 2.4 in [5] and extend them to infinite delay equations.

THEOREM 5. *The zero solution of Eq.* (6, 23-25) *is stable if*

$$\int_{t}^{\infty} |K(u,t)| du \leqslant \sqrt{\rho^3/\lambda},\ t \geqslant t_0 \geqslant 0. \tag{29}$$

Proof. To prove this theorem we use the following Liapunov functional

$$V(t,x(\cdot)) = \langle G(t)x(t),x(t))\rangle^{1/2} + \frac{\sqrt{2\lambda}}{2\rho} \int_{-\infty}^{t} \int_{t}^{\infty} |K(u,s)| du |x(s)| ds \tag{30}$$

where $\langle \cdot,\cdot \rangle$ is the Euclidean inner product.

Using (25) and (27) in (28) we obtain

$$|G(t)| \leqslant \frac{1}{2\rho}. \tag{31}$$

This implies that

$$\langle\, G(t)x(t), x(t)\rangle^{\,1/2} \leqslant \frac{|x|}{\sqrt{2\rho}}. \tag{32}$$

Moreover

$$\begin{aligned}
\langle\, G(t)x(t), x(t)\rangle &= \int_t^\infty x^T(t)\Phi^T(s,t)\Phi(s,t)\; ds \\
&= \int_t^\infty |x(s,t,x(t))|^2\; ds \\
&\geqslant |x(t)|^2 \int_t^\infty \exp\left(-2\int_t^s \mu(-A(r)dr)\right)\; ds \text{ (using (27))} \\
&\geqslant \frac{|x(t)|^2}{2\lambda}
\end{aligned} \tag{33}$$

After some computations one may show that

$$\begin{aligned}
V'_{(6)}(t,x(\cdot)) &= \left[-|x(t)|^2/2 + \langle\, G(t)x(t),\; \int_{-\infty}^t K(t,s)x(s)\; ds\rangle\right] \\
&\times\; \langle\, G(t)x(t),\; x(t)\rangle^{\,1/2} \\
&+\; \left(\frac{\sqrt{2\lambda}}{2\rho}\right)\left[\int_t^\infty |K(u,t)|du|x(t)| - \int_{-\infty}^t |K(t,s)||x(s)|ds\right].
\end{aligned} \tag{34}$$

Using (31), (32), (33) in (34) one obtains

$$V'_{(6)}(t,x(\cdot)) \leqslant \left[-\sqrt{\rho/2} + (\frac{\sqrt{2\lambda}}{2\rho})\int_t^\infty |K(u,t)|du\right]\; |x(t)| \tag{35}$$

$$V'_{(6)}(t,x(\cdot)) \leqslant 0. \tag{36}$$

From (32), (36), and (23), it follows that

$$\begin{aligned}
|x(t,t_0,\varphi)|/\sqrt{2\rho} &\leqslant V(t,x)\cdot)) \\
&\leqslant V(t_0,\varphi(\cdot)) \\
&\leqslant \langle\, G(t_0)\varphi,\varphi\rangle^{\,1/2} + \left(\sqrt{2\lambda}/2\rho\right)\int_{-\infty}^{t_0}\int_{t_0}^\infty |K(u,s)|du|\varphi(s)|ds \\
&\leqslant \left(\frac{1}{\sqrt{2\rho}} + \left(\sqrt{2\lambda}\right)M/2\rho\right)||\varphi||.
\end{aligned} \tag{37}$$

For $\epsilon > 0$, let $\delta < \epsilon/\sqrt{2\lambda}(1/\sqrt{2\rho} + \sqrt{2\lambda})M/2\rho)]$. The $||\varphi|| < \delta$ implies by Inequality (37) that $|x(t,t_0,\varphi| < \epsilon$ for $t \geqslant t_0$, which establishes stability of the zero solution of Eq. (6).

THEOREM 6. *The zero solution of Eq.* (6, 23-25) *is asymptotically stable if*

$$\int_t^\infty |K(u,t)|du < \frac{1}{\nu}\frac{\sqrt{\rho^3}}{\lambda}, \text{ for some } \nu > 1. \tag{38}$$

Proof. Employing (38) in (35) one obtains

$$V'_{(6)}(t,x(\cdot)) \leqslant -\gamma|x(t)|, \text{ for some } \gamma > 0. \tag{39}$$

Using the argument used in the proof of Theorem 3, one may establish that the zero solution of Eq. (6, 23-25) is asymptotically stable.

REMARK. Note that in, the convolution case, Condition (37) is equivalent to the stability condition

$$\int_0^\infty |K(s)|ds < |A|$$

used in Elaydi and Sivasundaram [5]. Thus Theorem 6 is an extension of the results in [5] to nonautonomous systems of convolution type.

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