Mathematical
Inequalities
& Applications
Volume 1, Number 3 (1998), 431–436

ON THE EXISTENCE OF PERIODIC SOLUTIONS OF A CERTAIN CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. A second order nonlinear differential equation is considered. Leray-Schauder principle is used to show the existence of periodic solutions. The results obtained are then applied to a specific example, where a computer program based on the fourth order Runge-Kutta method and Newton-Raphson algorithm is used to compute the periodic solutions.

Introduction. One of the powerful methods of proving the existence of periodic solutions of nonlinear nonoutonomous ordinary differential equations is the use of Green's function (see, for example [IV.] and [VII.]). In the case Green's function can not be contructed explicitly, or that, there is no need for an explicit expression of the Green's function, some theoretical tools are used to prove its existence (see for example [V.], [VI.] and [VIII]).

In this paper we consider a second order nonlinear differential equation. First we shall prove the existence of Green's function. Then with the aid of the Green's function, we construct an integral equation. To show the existence of periodic solutions of the differential equation, one can then prove the existence of the solution of the integral equation.

We consider the nonlinear differential equation

$$x'' + c(t)x' + f(t,x) = e(t)\ . \tag{1}$$

Where $c(t)$ and $e(t)$ are continues functions for $t \in [0,\omega]$ and $f(t,x)$ is continuous on $[0,\omega] \times \mathbb{R}$. In addition we assume all initial value problems corresponding to Eq. (1) can be extended to $[0,\omega]$.

THEOREM 1. *Assume*

i) $|f(t,x)| \leqslant \gamma|x| + \beta,\ t \in [0,\omega],\ |x| < \infty$
γ*, and* β *are non-negative constants.*

ii) $\gamma(\frac{\omega}{\pi})^2 + \gamma_1(\frac{\omega}{\pi}) < 1$,
$\gamma_1 = \max|c(t)|$

Mathematics subject classification (1991): 34C25, 34A05.

Key words and phrases: Periodic solutions, differential equations.

This research has been partially supported by a grant from Institute for Studies in Theoretical Physics and Mathematics (IPM).

Paper MIA-01-41

Then Eq. (1) *possesses a solution satisfying*

$$x^{(i)}(0)+x^{(i)}(\omega)=0,\ i=1,2. \tag{2}$$

Proof. First we give an estimate on the magnitude of the solutions of the problem

$$\begin{aligned} &x''+c(t)x'=\mu[e(t)-f(t,s)],\ \mu\in[0,1],\\ &x^{(i)}(0)+x^{(i)}(\omega)=0 \quad i=0,1. \end{aligned} \tag{3}$$

Here we make use of Wirtinger's inequalities in the following form: If $x(t)$ is a function of class $C^{n-1}[0,\omega]$, such that $x(t+\omega)+x(t)=0$, for all t, then

$$\begin{aligned} &||x^{(i-1)}(t)||_2 \leqslant (\tfrac{\omega}{\pi})^{n-i+1}||x^{(n)}(t)||_2,\ i=1,2,\dots,n,\\ &||\,.\,||_2=[\int_0^{\omega}|\,.\,|^2dt]^{1/2}. \end{aligned} \tag{4}$$

Now suppose $x(t)$ is a solution of the problem (3), then

$$|x''(t)|\leqslant \gamma_1|x'(t)|+\mu\{|e(t)|+\gamma|x(t)|+\beta\}.$$

Using Minkowski's inequality

$$||x''(t)||_2\leqslant \gamma_1||x'(t)||_2+\mu\{||e(t)||_2+\gamma||x(t)||_2+\beta\sqrt{\omega}\},$$

it follows from Wirtinger's inequality

$$||x''(t)||_2\leqslant \gamma_1(\frac{\omega}{\pi})||x''(t)||_2+\mu\{||e(t)||_2+\gamma(\frac{\omega}{\pi})^2||x''(t)||_2+\beta\sqrt{\omega}\},$$

where

$$[1-\gamma_1\frac{\omega}{\pi}-\mu\gamma(\frac{\omega}{\pi})^2]||x''(t)||_2\leqslant \mu\{||e(t)||_2+\beta\sqrt{\omega}\}.$$

By assumption (ii) of Theorem 1 and noting that $0\leqslant\mu\leqslant 1$, we get

$$||x''(t)||_2\leqslant \frac{||e(t)||_2+\beta\sqrt{\omega}}{1-\gamma_1(\frac{\omega}{\pi})-\gamma(\frac{\omega}{\pi})^2}\,. \tag{5}$$

Next we write

$$x^{(i-1)}(t)=x^{(i-1)}(0)+\int_0^t x^{(i)}(\tau)d\tau \quad i=1,2.$$

For $t=\omega$

$$x^{(i-1)}(\omega)=x^{(i-1)}(0)+\int_0^{\omega} x^{(i)}(\tau)d\tau \quad i=1,2.$$

Making use of Eq. (2) we have

$$x^{(i-1)}(0)=-x^{(i-1)}(\omega)=\frac{1}{2}\int_0^{\omega}x^{(i)}(\tau)d\tau.$$

Finally we obtain

$$x^{(i-1)}(t)=-\frac{1}{2}\int_0^t x^{(i)}(\tau)d\tau-\frac{1}{2}\int_0^{\omega}x^{(i)}(\tau)d\tau=-\frac{1}{2}\int_0^{\omega}x^{(i)}(\tau)d\tau\,,$$

which gives us

$$|x^{(i-1)}(t)| \leqslant \frac{1}{2}\int_0^{\omega} |x^{(i)}(\tau)|d\tau.$$

Now we have the estimate.

$$|x^{(i-1)}(t)| \leqslant \frac{1}{2}\sqrt{\omega}||x^{(i)}(t)||_2,\ i=1,2$$

Combining Wirtinger's inequality and (5)

$$|x^{(i-1)}(t)| \leqslant \frac{1}{2}\sqrt{\omega}(\frac{\omega}{\pi})^{3-i}\mu\frac{||e(t)_2||+\beta\sqrt{\omega}}{1-\gamma_1(\frac{\omega}{\pi})-\gamma(\frac{\omega}{\pi})^2}.$$

For $\mu = 0$ we obtain

$$x^{(i-1)}(t) = 0 \quad t\in[0,\omega], \quad i=1,2.$$

It follows that the homogeneous equation

$$x''(t)+c(t)x'(t)=0$$

has only a trivial solution which satisfies conditions (2). This proves the existence of Greens function $g(t,s)$ for problem (3).

Clearly the auxiliary problem (3) is equivalent to

$$x(t) = \mu\int_0^{\omega} g(t,s)[e(s)-f(s,x(s))]ds. \tag{6}$$

Next we consider the space $C^2[0,\omega]$ normed by

$$||x||_{c^2} = \max|x^{(i-1)}(t)|,\ t\in[0,\omega],\ i=1,2.$$

Let B_ρ be the space

$$B_\rho = \{x(t)\in C^2[a,b] : ||x||_{c^2}\leqslant\rho\}$$

where

$$\rho = \max\{\frac{1}{2}\sqrt{\omega}(\frac{\omega}{\pi})^{3-i}\frac{||e(t)||+\beta\sqrt{\omega}}{1-\gamma_1(\frac{\omega}{\pi})-\gamma(\frac{\omega}{\pi})^2}\},\ i=1,2.$$

Considering the space

$$S_R = \{x(t)\in c^2[0,\omega] : ||x||_{c^2} = R\},$$

it follows that for $R>\rho$, arbitrary, Eq. (6) has no solution on S_R. Hence by Leray-Schauder principle and the complete continuity of the operator

$$(Lx)(t) = \mu\int_0^{\omega} g(t,s)[e(s)-f(s,x(s))]ds, \tag{7}$$

Eq. (6) has at least a solution in the open sphere $\{x;||x||_{c^2}<R\}$, and as a result, there exists a solution in B_ρ. Therefore, it is shown that problem (3) has, at least, a solution for $\mu = 1$.

COROLLARY 1. *Under the hypotheses of Theorem 1, and*

iii) $c_i(t)$ *is an* ω *-periodic function*

iv) $e(t)$ *is* 2ω *-periodic, i.e.,*
$e(t+2\omega) \equiv e(t)$, *and* $e(t+\omega) = -e(t)$

v) $f(t+\omega) \equiv f(t,x)$, *and also,* $f(t,-x) = f(t,x)$,

it can be shown that Eq. (1) *has a* 2ω *-periodic solution*

Proof. Let $\bar{x}(t)$ be a 2ω -periodic extension of $x(t)$ defined by

$$\bar{x}(t) = \begin{cases} x(t), & 0 \leqslant t \leqslant \omega, \\ -x(t+\omega), & -\omega \leqslant t \leqslant 0. \end{cases}$$

It is easily verified that $\bar{x}(t) \in C^2[-\omega,\omega]$. In addition, using the assumptions (iii) and (iv), one can show $\bar{x}(t)$ is a solution of Eq. (1) which satisfies the boundary conditions

$$\bar{x}^{(i)}(\omega) = \bar{x}^{(i)}(-\omega) \quad i = 0, 1.$$

In addition we note

$$\int_o^{2\omega} \bar{x}(t)at = \int_0^{\omega} \bar{x}(t)dt + \int_{\omega}^{2\omega} \bar{x}(t)dt\ ,$$

which gives us

$$\int_o^{2\omega} \bar{x}(t)at = \int_0^{\omega} \bar{x}(t)dt + \int_0^{\omega} \bar{x}(t+\omega)dt = 0.$$

That is, the solution $\bar{x}(t)$ has a zero mean value.

Using Theorem 1 and Corollary 1, one can easily obtain similar results for the equation

$$x''(t) + \varphi(x,x')x' + f(t,x) = e(t). \tag{8}$$

THEOREM 2. *If in addition to the hypotheses (i), (ii), (iv) and (v) of Theorem 1 and Corollary 1, one assumes* $0 \leqslant |\varphi(x,x')| \leqslant M_1$ *and*

$$\gamma(\frac{\omega}{\pi})^2 + M_1(\frac{\omega}{\pi}) < 1\ ,$$

Then Eq. (8) has a 2ω *-periodic solution with zero mean value, i.e.,* $\displaystyle\int_0^{2\omega} x(t)dt = 0$.

Numerical Analysis. The differential equation (1) can be written in the general form

$$x''(k) + k(t,x,x') = e(t). \tag{9}$$

Now we apply Runge-Kutta method to Eq. (9) using an arbitrary set of initial conditions $(x(0),x'(0)) = (\alpha_0,\beta_0)$. To obtain the desired initial values (α,β), for which a periodic solution is constructed, we proceed as follows. We consider the equations

$$F(\alpha,\beta) = x(2\omega,\alpha,\beta) - \alpha = 0,$$
$$G(\alpha,\beta) = x'(2\omega,\alpha,\beta) - \beta = 0.$$

If (α_0, β_0) satisfy these equations then we have the desired initial condition. If not, we write

$$\begin{aligned} F(\alpha_1, \beta_1) &= x(2\omega, \alpha_0, \beta_0) - \alpha_0, \\ G(\alpha_1, \beta_1) &= x'(2\omega, \alpha_0, \beta_0) - \beta_0. \end{aligned} \tag{10}$$

Using Taylor series expansion for F and G and discarding second and higher order terms we obtain

$$\begin{aligned} F(\alpha_1, \beta_1) &\cong F(\alpha_0, \beta_0) + \frac{\partial F}{\partial \alpha}\triangle\alpha_1 + \frac{\partial F}{\partial \beta}\triangle\beta_1, \\ G(\alpha_1, \beta_1) &\cong G(\alpha_0, \beta_0) + \frac{\partial F}{\partial \alpha}\triangle\alpha_1 + \frac{\partial F}{\partial \beta}\triangle\beta_1, \end{aligned} \tag{11}$$

where all the derivatives of F and G are evaluated at (α_0, β_0). Now we can solve the above equations for $\triangle\alpha_1$ and $\triangle\beta_1$.

Next we choose

$$\begin{aligned} \alpha_1 &= \alpha_0 + \triangle\alpha_1, \\ \beta_1 &= \beta_0 + \triangle\beta_1 \end{aligned}$$

and substitute these values in Eq. (10) and proceed as before. We continue this process until, say, at the k-th step

$$|F(\alpha_k, \beta_k)| + |G(\alpha_k, \beta_k)| < \epsilon,$$

where ϵ is a pressigned tolerance.

To obtain the partial derivatives of F and G in Eq. (11), we note

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= \frac{\partial x}{\partial \alpha} - 1, & \frac{\partial F}{\partial \beta} &= \frac{\partial x}{\partial \beta}, \\ \frac{\partial G}{\partial \alpha} &= \frac{\partial x'}{\partial \alpha}, & \frac{\partial G}{\partial \beta} &= \frac{\partial x'}{\partial \beta} - 1. \end{aligned}$$

Next we use central difference formulas to evaluate the derivatives of x and x' with respect to α and β.

EXAMPLE. We apply the above method of computation to equation

$$x'' + \frac{1}{5}\sin(2\pi t)x' + \frac{1}{10}\sin(2\pi x) = \sin \pi t. \tag{12}$$

Here $c(t) = \frac{1}{5}\sin(2\pi t)$, $f(t,x) = \frac{1}{10}\sin 2\pi x$ and $e(t) = \sin \pi t$. We note $\omega = 1$, $c(t)$ is ω-periodic, and $e(t)$ is 2ω-periodic. It is easily shown that Eq. (9) satisfies all the requirements of Theorem 1 and Corollary 1. Therefore Eq. (12) has a periodic solution of period 2.

A computer program based on the above numerical analysis was written. The suitable initial conditions to give a periodic solution was found to be

$$x(0) = 2.000000 \quad x'(0) = -0.3542160$$

for which

$$x(2) = 2.000000 \quad x'(2) = -0.3542161.$$

ACKNOWLEDGEMENT. We would like to thank the Institute for Studies in Theoretical Physics and Mathematics for the financial support.

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