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AN ENERGY—TYPE INEQUALITY

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Abstract. In this paper we will prove an Energy-Type inequality for the mild solution of the linear evolution equation. And for showing the strength and power of this inequality we will use this inequality to show continuity of the solution with respect to a parameter of the semilinear evolution equation with monotone nonlinearity.

1. Introduction

Let H be a real separable Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$. Let $T > 0$ and let $S = [0, T]$. suppose f be an H–valued uniformly bounded function on S define $\|f\|_\infty = \sup_{t\in S}\|f(t)\|$. Consider on H the linear evolution equation formally written as

$$\begin{cases} \dot{X}(t) &= A(t)X(t) + a(t) \\ X(0) &= X_0, \end{cases} \tag{1}$$

where $\{A(t), \quad t \in S\}$ is a family of closed linear operators on H whose domain D is independent of $t \in S$ and is dense in H.

Suppose that $\{A(t) \ : \ t \in S\}$ generates a unique *evolution operator* $\{U(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$, i.e, the $U(t,s)$ are bounded linear operators on H such that

$$U(t,t) = I, \quad U(t,s)\,U(s,r) = U(t,r) \quad \text{for} \quad 0 \leqslant r \leqslant s \leqslant t \leqslant T,$$

and $(t,s) \to U(t,s)$ is strongly continuous for $0 \leqslant s \leqslant t \leqslant T$, and certain relationships between A and U hold, which we will introduce later on.

DEFINITION 1. An H-valued process X is a *mild solution* of (1) if and only if
(i) $X. \in L^1(S,H)$;
(ii) $X(t) = U(t,0)X(0) + \int_0^t U(t,s)a(s)ds$ for each $t \in S$.

In this paper we will prove an Energy Inequality for the mild solution of (1). And for showing the strength and power of this inequality we will use this inequality to show continuity of the solution with respect to a parameter of the semilinear evolution equation with monotone nonlinearity.

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DEFINITION 2. We say the evolution operator $U(t,s)$ is an *almost strong evolution operator* with *generator* $A(t)$ if it satisfies the following:

(a) For almost all $s \leqslant t$ and for each $x \in D$

$$U(t,s)x - x = \int_s^t U(t,r)A(r)xdr. \tag{2}$$

(b) Let $x \in D$ and $s \in S$. For almost all $t > s$

$$U(t,s)D \subseteq D; \tag{3}$$

$$\int_s^t A(r)U(r,s)xdr = (U(t,s) - I)x. \tag{4}$$

If U and A satisfy (2), (3), and (4) for every $s \in S$, u is called a *strong evolution operator.*

REMARK 1. *(i)* If $\{A(t) : t \in S\}$ is the generator of an almost strong evolution operator $U(t,s)$, then (1) (with $a = 0$ and $X_0 \in D$) has a unique solution $X(t) = U(t,0)X_0$ which is differentiable almost everywhere.

(ii) For a.e. $0 \leqslant s \leqslant t \leqslant T$ and each $x \in D$ we have

$$\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x, \tag{5}$$

$$\frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x. \tag{6}$$

We say $U(t,s)$ is an *exponentially bounded with parameter* λ *on* S if there is $\lambda \in \mathbf{R}$ such that

$$\|U(t,s)\|_L \leqslant e^{\lambda(t-s)} \quad \text{for a.e. } 0 \leqslant s \leqslant t \leqslant T. \tag{7}$$

Note that if an almost strong evolution operator $U(t,s)$ is exponentially bounded on S with parameter λ, we have

$$\langle A(t)x, x\rangle \leqslant \lambda \|x\|^2, \quad \forall x \in D. \tag{8}$$

This can be seen because if $x \in D$ and $t > s$,

$$\frac{\|U(t,s)x\|^2 - \|x\|^2}{t-s} \leqslant \frac{(e^{2\lambda(t-s)} - 1)\|x\|^2}{t-s}$$

or

$$\lim_{t \to s^+} \frac{\|U(t,s)x\|^2 - \|x\|^2}{t-s} \leqslant \lim_{t \to s^+} \frac{(e^{2\lambda(t-s)} - 1)\|x\|^2}{t-s} = 2\lambda \|x\|^2;$$

but

$$\lim_{t \to s} \frac{\|U(t,s)x\|^2 - \|x\|^2}{t-s} = \frac{d}{dt^+}\|U(t,s)x\|^2|_{t=s} = 2\langle A(s)x, x\rangle \text{ a.e..}$$

The following are the relevant hypotheses concerning A and U:

HYPOTHESIS 1. *(a) The domain* $\mathscr{D}(A(t)) := D$ *is independent of* t *for* $t \in S$ *and is dense in* H*;*

(b) $\{A(t) : t \in S\}$ *generates a unique almost strong evolution operator* $U(t,s)$*;*

(c) $U(t,s)$ *is exponentially bounded on* S *with parameter* λ*;*

We refer to Pazy [6] and Tanabe [9] for sufficient conditions for the existence of an evolution operator with the properties 1(a)–(c).

These conditions apply to a large class of delay equations, and to parabolic and hyperbolic equations [see for example [2]].

Let A be an unbounded operator on H with dense domain D. Let $\| \quad \|_D$ be the norm defined on D by

$$\|x\|_D^2 = \|Ax\|^2 + \|x\|^2, \quad x \in D.$$

This norm is called *the graph norm* on D. Note that it generated by the inner product

$$\langle x, y \rangle_D = \langle Ax, Ay \rangle + \langle x, y \rangle$$

REMARK 2. *(i)* An operator A is closed if and only if its domain D is complete under the graph norm [see [7],problem 15(a), p314.]

(ii) Suppose A is a closed linear operator with dense domain D. Then D is a Hilbert space with graph norm $\| \quad \|_D$.

2. An Energy–type Inequality

THEOREM 1. (Energy's inequality) *Let* $a(.)$ *be an* H*-valued integrable function on* S*. Suppose* U *and* A *satisfy Hypothesis 1(a)–(c). If*

$$X(t) = U(t,0)X_0 + \int_0^t U(t,s)a(s)ds, \tag{9}$$

then

$$\|X(t)\|^2 \leqslant e^{2\lambda t}\|X_0\|^2 + 2\int_0^t e^{2\lambda(t-s)}\langle X(s), a(s)ds \rangle, \quad t \in S. \tag{10}$$

Before proving Theorem 1 we are going to prove two lemmas. Suppose $U(t,s)$ satisfies Hypothesis 1c for some $\lambda \in \mathbf{R}$. Define

$$U_1(t,s) = e^{-\lambda(t-s)}U(t,s), \quad A_1(t) = A(t) - \lambda I, \textit{and} \quad a_1(t) = e^{-\lambda t}a(t).$$

and $X_1(t) = e^{-\lambda t}X_t$.

LEMMA 1. *If* U *and* A *satisfy Hypothesis 1, then* U_1 *and* A_1 *satisfy Hypothesis 1 with* $\lambda \equiv 0$*. Moreover,* $X(t)$ *satisfies (9) if and only if* $X_1(t)$ *satisfies*

$$X_1(t) = U_1(t,0)X_0 + \int_0^t U_1(t,s)a_1(s)ds. \tag{11}$$

Proof. $\|U_1(t,s)\|_L = e^{-\lambda(t-s)}\|U(t,s)\|_L \leqslant 1$ a.e. By the definition of U_1 we can rewrite (9) as $X(t) = e^{\lambda t}U_1(t,0)X_0 + e^{\lambda t}\int_0^t e^{-\lambda s}U_1(t,s)a_1(s)ds$.

Using the definition of X_t^1 and $a_1(t)$ we can rewrite the above as (11). $\square$

LEMMA 2. *If $a(.)$ is an H-valued integrable function on S and if $X(t) := X_0 + \int_0^t a(s)ds$, then*

$$\|X(t)\|^2 = \|X_0\|^2 + 2\int_0^t \langle X(s), a(s)\rangle\, ds.$$

Proof. Since $a(s)$ is integrable, then $X(t)$ is absolutely continuous and $X'(t) = a(t)$ a.e. on S. Then $\|X(t)\|$ is also absolutely continuous and

$$\frac{d}{dt}\|X(t)\|^2 = 2\langle \frac{dX(t)}{dt}, X(t)\rangle = 2\langle a(t), X(t)\rangle \text{ a.e.}$$

so that

$$\int_0^t \frac{d}{ds}\|X(s)\|^2 ds = \|X(t)\|^2 - \|X_0\|^2.$$

Thus

$$\|X(t)\|^2 - \|X_0\|^2 = 2\int_0^t \langle X(s), a(s)\rangle\, ds.$$

$\square$

Proof of Theorem 1. By Lemma 1 we can assume $\lambda = 0$ in Hypothesis 1c. Then for all $x \in D$, $\langle A(t)x, x\rangle \leqslant 0$ for a.e. t.

Define a map $R_n(t) : H \to D$ by $R_n(t) = n(nI - A(t))^{-1}$. Then $R_n(t)$ is defined on all of H. Since $\langle A(t)x, x\rangle \leqslant 0$ for a.e. t, then $\langle \frac{1}{n}(nI - A(t))x, x\rangle \geqslant \|x\|^2$ for a.e. $t \in S$, $\forall x \in D$. By the Schwarz inequality we have for all $x \in D$ that

$$\|\frac{1}{n}(nI - A(t))x\| \geqslant \|x\|,$$

so $\|R_n(t)\|_L \leqslant 1$ for a.e. t .

We approximate X_t by Yosida's method. Define $a_n(t)$ by $a_n(t) := R_n(s)a(s)$

Note that since $R_n(t) : H \to D$,then $a_n(t) \in D$. Let $\{X_0^n\}$ be a sequence in D which converges to X_0 such that $\|X_0^n\| \leqslant \|X_0\|$ for all n.

Define

$$X_n(t) := U(t,0)X_0^n + \int_0^t U(t,s)a_n(s)ds. \tag{12}$$

We are going to prove that $\|X_n - X\|_\infty \to 0$.

Since

$$\|U(.,0)(X_0^{(n)} - X_0)\|_\infty \leqslant \|X_0^n - X_0\| \to 0 \quad \text{boundedly,}$$

it is enough to show that

$$\sup_{0\leqslant t\leqslant T}\|\int_0^t U(t,s)(a_n(s)\ -\ a(s))ds\|\ \to\ 0. \tag{13}$$

Now $\|U(t,s)\|_L \leqslant 1$ so we have

$$\|\int_0^{\cdot} U(.,s)(a_n(s)\ -\ a(s))ds\|_\infty \leqslant \int_0^T \|(R_n(s)-I)a(s)\|ds. \tag{14}$$

Since $R_n(s)\ \to\ I$ strongly then $\|(R_n(s)-I)a(s)\|\ \to\ 0$ for a.e. $s\in S$, and since $\|R_n(s)-I\|_L\ \leqslant\ 2$ then the integrand is $\leqslant\ 2$ a.e. Then by the dominated convergence theorem, the right hand side of (14) approaches zero so we get (13).

Hence $\|X_n\ -\ X\|_\infty\ \to\ 0$.

Let us first prove Energy's inequality (10) for

$$\bar{X}_t = U(t,0)\bar{X}_0 + \int_0^t U(t,s)\bar{a}(s)ds, \tag{15}$$

where $\bar{a}$ satisfies the following.

HYPOTHESIS 2. *(a) $\bar{a}$ is a D-valued integrable function*
(b) $\bar{X}_0$ is a D-valued

LEMMA 3. *If $\bar{X}_0$ and $\bar{a}$ satisfy Hypothesis 2 and if $\bar{X}$ is a solution of (15), then*

$$\|\bar{X}_t\|^2 \leqslant \|\bar{X}_0\|^2 + \int_0^t \langle\, \bar{X}(s),\bar{a}(s)\rangle\, ds. \tag{16}$$

Proof. Since $\bar{a}$ and $\bar{X}_0$ satisfy Hypothesis 2, so by [Theorem 2.38, page 45 [2]], $\bar{X}_t$ satisfies

$$\bar{X}(t) = \bar{X}_0 + \int_0^t A(s)\bar{X}(s)ds + \int_0^t \bar{a}(s)\, ds. \tag{17}$$

Since $A(.)\bar{X}_k(.)\ \in\ L^1(S,H)$, we can apply Lemma 2 to see that

$$\begin{aligned}\|\bar{X}(t)\|^2\ \ =\ \ \|\bar{X}_0\|^2\ +\ 2\int_0^t \langle A(s)\bar{X}(s),\bar{X}(s)\rangle\, ds \\ +\, 2\int_0^t \langle\, \bar{X}(s),\bar{a}(s)\rangle\, ds.\end{aligned} \tag{18}$$

But $\langle A(s)\bar{X}_k(s),\bar{X}_k(s)\rangle\ \ \leqslant 0$. a.e., so (18) implies that

$$\|\bar{X}(t)\|^2\ \leqslant\ \|\bar{X}_0\|^2 + 2\int_0^t \langle\ \bar{X}(s),d\bar{a}(s)\ \rangle\ ds. \tag{19}$$

This proves the Lemma.

To complete the proof of the theorem, we only need to show that

$$\int_0^t \langle X_n(s), a_n(s) \rangle \, ds \to \int_0^t \langle X(s), a(s) \rangle \, ds . \tag{20}$$

Now

$$\begin{aligned} &| \int_0^t \langle X_n(s), a_n(s) \rangle \, ds - \int_0^t \langle X(s), a(s) \rangle \, ds| \\ &\leqslant | \int_0^t \langle X_n(s) - X(s), a_n(s) \rangle \, ds| + | \int_0^t \langle X(s), (a(s) - a_n(s)) \rangle \, ds| \\ &:= |I_n^1(t)| + |I_n^2(t)|. \end{aligned}$$

- Since $a_n(s) = R_n(s)\, a(s)$ for a.e. s and $\|R_n(s)\, a(s)\| \leqslant 1$ for a.e. s, then

$$\sup_{0 \leqslant t \leqslant T} |I_n^1(t)| \leqslant \|X_n - X\|_\infty \int_0^T \|a(s)\| ds.$$

Since $\|X_n - X\|_\infty \to 0$, $\sup_{0 \leqslant t \leqslant T} |I_n^1(t)| \to 0$.

- Since $(a(s) - a_n(s)) = (I - R_n(s))a(s)$ a.e. s, then

$$\sup_{0 \leqslant t \leqslant T} |I_n^2(t)| \leqslant \|X\|_\infty \int_0^T \|(A_n(s) - I)a(s)\| ds.$$

But $(A_n(s) - I)a(s)$ converges to zero a.e. and its norm is bounded by 2, so by the bounded convergence theorem $\sup_{0 \leqslant t \leqslant T} |I_n^2(t)|$ tends to zero. □

3. Application

Let g be an H-valued function defined on a set $D(G) \subset H$. Recall that g is *monotone* if for each pair $x, y \in D(g)$,

$$\langle g(x) - g(y), x - y \rangle \geqslant 0,$$

and g is *semi-monotone* with *parameter* M if, for each pair $x, y \in D(g)$,

$$\langle g(x) - g(y), x - y \rangle \geqslant -M\|x - y\|^2.$$

We say g is *bounded* if there exists an increasing continuous function ψ on $[0, \infty)$ such that $\|g(x)\| \leqslant \psi(\|x\|), \forall x \in D(g)$. g is *demi-continuous* if, whenever (x_n) is a sequence in $D(g)$ which converges strongly to a point $x \in D(g)$, then $g(x_n)$ converges weakly to $g(x)$.

Consider the integral equation

$$X(t) = U(t, 0)X_0 + \int_0^t U(t, s) f(s, X(s)) ds + V(t), \quad t \in S. \tag{21}$$

When A and U satisfy Hypothesis 1 , f and V satisfy the following

HYPOTHESIS 3. *For each $x \in H$ $t \to f(t,x)$ is continuous. For each $t \in S$, $x \to f(t,x)$ is demicontinuous and uniformly bounded in t. (That is, there is a function $\varphi = \varphi(x)$ on $\mathbf{R}_+$ which is continuous and increasing in x and such that for all $t \in S$, $x \in H$, $\|f(t,x)\| \leqslant \varphi(\|x\|)$.)*
(d) There exists a non-negative number M such that for each $t \in S$, $x \to -f(t,x)$ is semimonotone with parameter M.
(e) $t \to V(t)$ is cadlag.

Faris and Jona-Lasinio (1982) have proved that the solution X of (21) is a continuous function of V in the special case when the generator of U is $\frac{d^2}{dx^2}$ and $f(x) = -\lambda x^3 - \mu x$. Da Prato and Zabczyk (1988) generalized this result to the case where U is a general analytic semigroup and f is a locally Lipschitz function on a Banach space.

As an application of the Energy-Type Inequality we will prove a generalization of Faris and Jona-Lasino's theorem for monotone $-f$ and more general U; this was open after Faris and Jona-Lasinio (1982) [see for example Smolenski et al (1986), page 230].

The existence and uniqueness of the above integral equation is a well-known theorem of Browder(1964) and Kato(1964). That is

PROPOSITION 1. *Suppose that X_0, f and V satisfy Hypothesis 3. Suppose A and U satisfy Hypothesis 1. Then (21) has a unique cadlag (continuous, if V_t is continuous) solution. Furthermore*

$$\|X\|_\infty \leqslant \|X_0\| + \|V\|_\infty + C_T\varphi(\|X_0\| + \|V\|_\infty), \tag{22}$$

where

$$C_T = \begin{cases} \frac{1}{M+\lambda}e^{(M+\lambda)T} & \text{if } M+\lambda \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. See [1] or [5].

In [10], the measurability of the solution of (21) is proved. In [12], the measurability of the solution of (21) is used to prove the existence of the solution of the stochastic semilinear integral equation

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)f_s(X_s)ds + \int_0^t U(t,s)g_s(X)dW_s + V_t,$$

where

- $g_s(.)$ is a uniformly-Lipschitz predictable functional with values in the space of Hilbert-Schmidt operators on H;
- $\{W_t, t\in\mathbf{R}\}$ is an H-valued cylindrical Brownian motion with respect to $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$.

THEOREM 2. *Let f, V^1 and V^2 satisfy Hypothesis 3. Suppose A and U satisfy Hypotheses 1. Let $X^i(t),\ i = 1,2$ be solutions of the integral equations:*

$$X^i(t) = \int_0^t U(t,s)f(s,X^i(s))ds + V^i(t). \tag{23}$$

Then there is a constant C such that

$$\|X^2 - X^1\|_\infty \leqslant C\|V^2 - V^1\|_\infty^{\frac{1}{2}} \tag{24}$$

Proof. Define $Y^i(t) = X^i(t) - V^i(t),\ i = 1,2$. Then we can write (23) in the form

$$Y^i(t) = \int_0^t U(t,s)f(s,X^i(s))ds, \quad i = 1,2,$$

so that

$$Y^2(t) - Y^1(t) = \int_0^t U(t,s)[f(s,X^2(s)) - f(s,X^1(s))]ds.$$

Since U satisfies Hypothesis 1(a)–(c), then by Theorem 1 we have

$$\|Y^2(t) - Y^1(t)\|^2 \leqslant 2\int_0^t e^{2\lambda(t-s)}\langle Y^2(s) - Y^1(s), f(X^2(s)) - f(X^1(s))\rangle\, ds. \tag{25}$$

Note that because Y^i and X^i are cadlag and the f^i are bounded by φ_i, then the integrands are dominated by cadlag functions and hence are integrable. Since $Y^i = X^i - V^i$ and $-f^2$ is monotone. By the Schwartz inequality this is

$$\leqslant\ 2\int_0^t e^{2\lambda s}\|V^2(s) - V^1(s)\|\ \|f(X^2(s)) - f(X^1(s))\|ds$$

Since X^1 and X^2 are bounded and f is bounded then there is a constant K such that

$$\|Y^2 - Y^1\|_\infty \leqslant K\|V^2 - V^1\|_\infty^{\frac{1}{2}} \tag{26}$$

since $Y^i = X^i - V^i$ the proof of theorem is complete. □

REMARK 3. Let $D(S,H)$ be the set of H-valued cadlag functions on S with norm

$$\|f\|_\infty = \sup_{t\in S}\|f(t)\|.$$

By Theorem 2 there is a continuous mapping $\psi : S\times D(S,H) \to D(S,H)$ such that if $X(t)$ is a solution of

$$X(t) = \int_0^t U(t,s)f(X(s))ds + V(t),$$

then $X(t) = \psi(t,V)(t)$. Moreover there is a constant C such that

$$\|\psi(.,V^2) - \psi(.,V^1)\|_\infty \leqslant C\|V^2 - V^1\|_\infty^{\frac{1}{2}},$$

so ψ is Hölder continuous with exponent $1/2$.

REFERENCES

[1] F. E. BROWDER, *Non-linear equations of evolution*, Ann. of Math., **80** (1964), 485–523.

[2] R. F. CURTAIN AND A. J. PRITCHARD, *Infinite dimensional linear system theory*, LN in control and information sciences, **8** (1978), Springer-Verlag, Berlin-Heidelberg, New York.

[3] G. DA PRATO AND J. ZABCZYK, *A note on semilinear stochastic equations*, Differential and Integral Equations, **1** (2) (1988), 1–13.

[4] W. G. FARIS AND G. JONA-LASINIO, *Large fluctuations for a non-linear heat equation with noise*, J. Phys A: Math. Gen., **15** (1982), 3025–3055.

[5] T. KATO, *Nonlinear evolution equations in Banach spaces*, Proc. Symp. Appl. Math., **17** (1964), 50–67.

[6] A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, **44** (1983), Springer-Verlag, Berlin.

[7] M. REED, AND B. SIMON, *Methods of modern mathematical physics I: Functional analysis*, New York, London: Academic Press 1972.

[8] W. SMOLENSKI, R. SZTENCEL, AND ZABCZYK,. *Large deviations estimates for semilinear stochastic equations*, Proceeding of the 5th IFIP Conference on Stochastic Differential Systems, Eisenach, 1986.

[9] H. TANABE,. *Equations of evolution*. Pitman, London 1979.

[10] B. Z. ZANGENEH, *Measurability of the Solution of a Semilinear Evolution Equation*, Seminar on Stochastic Processes, Birkhãuser, Boston, 1990.

[11] ———, *Existence and uniqueness of the solution of a semilinear stochastic evolution equation on the whole real line*, Seminar on Stochastic Processes, Birkhãuser, Boston, 1992.

[12] ———, *Semilinear stochastic evolution equations with monotone nonlinearities*, Stochastics and Stochastics Reports, **53** (1995), 129–174.

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