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DECOMPOSITION OF HOMOGENEOUS MEANS AND CONSTRUCTION OF SOME METRIC SPACES

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Abstract. Any (positively) homogeneous mean on $(0,\infty)^2$ can be decomposed multiplicatively into the arithmetic mean A and a one–place function, called A*–index function*. Index functions characterize a homogeneous mean in many respects, and their graphs are suitable for geometrical comparisons of several properties of homogeneous means. Moreover, index functions can facilitate proofs of inequalities between different types of homogeneous means. With the aid of A–index functions, some metrics are introduced in the set of homogeneous means.

0. Introduction

Let $m:(0,\infty)^2\to(0,\infty)$ be a fixed positively homogeneous mean. Then any positively homogeneous mean defined on $(0,\infty)^2$ can be decomposed (multiplicatively) into m and a one–place function, called m*-index function*. In fact, there is a one–to–one correspondence between the family of all homogeneous means and the set of index functions. The main part of the paper is devoted to the case $m=A$, where A is the arithmetic mean. The A–index function of a mean characterizes the mean in many respects, e.g. symmetry of a mean is equivalent to the evenness of its A–index function, and subadditivity of a mean is equivalent to the convexity of its A–index function.

Index functions can be useful tools in proving inequalities between different types of positively homogeneous means (as an application, in section 5, we give the best estimation of the contra–harmonic mean by power means). Graphs of index functions are suitable for geometrical interpretations and visual comparisons of several properties of positively homogeneous means. In section 6, we show that index functions allow to introduce metrics in the set of positively homogeneous means. In section 7, the M-convexity of power functions is treated via index functions.

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1. Preliminaries and motivation

Let $I \subset \mathbf{R}$ be an interval. By a *mean* we understand a two–place function $M : I^2 \to \mathbf{R}$ such that

$$\min(x,y) \leqslant M(x,y) \leqslant \max(x,y), \qquad x,\ y \in I,$$

(in particular, we have $M(x,x) = x$ for all $x \in I$). A mean M is called *strict* if for all x, $y \in I$, $x \neq y$, these inequalities are sharp; and it is *symmetric*, if $M(x,y) = M(y,x)$ for all x, $y \in I$ (for a more exhaustive theory of means cf. Bullen, Mitrinović and Vasić [2], also Aczél and Dhombres [1]).

In the present paper we are mainly interested in positively homogeneous means. Therefore the interval I has to be of one of the following forms: $(0,\infty)$, $[0,\infty)$, $(-\infty,0)$, $(-\infty,0]$ and $\mathbf{R}$. Since the cases $(-\infty,0)$ and $(-\infty,0]$ easily reduce to $(0,\infty)$, $[0,\infty)$, respectively, we omit them. Recall that a function $M : I^2 \to \mathbf{R}$ is *positively homogeneous of order* p $(p \in \mathbf{R})$, if

$$M(tx,ty) = t^p M(x,y), \qquad t > 0,\ \ x,\ y \in I.$$

It is easy to see that if M in this homogeneity condition is a mean then $p = 1$, i.e. M is *positively homogeneous* (for short: M is *homogeneous*),

$$M(tx,ty) = tM(x,y), \qquad t > 0,\ \ x,\ y \in I.$$

To present a general concept of this paper let us fix a mean $N : (0,\infty)^2 \to (0,\infty)$. Then for every mean $M : (0,\infty)^2 \to (0,\infty)$ there is a trivial decomposition

$$M(x,y) = N(x,y) f_{M,N}(x,y), \qquad x,\ y > 0,$$

where, of course, $f_{M,N} = M/N$. For example, the exponential mean F on $(0,\infty)$, $F(x,y) = \log\left(\left(\exp(x) + \exp(y)\right)/2\right)$, “decomposed” by the arithmetic mean A (restricted to $(0,\infty)$), $A(x,y) = (x+y)/2$, gives, trivially, $F = Af_{F,A}$ where

$$f_{F,A}(x,y) := \frac{2}{x+y} \log\left(\frac{\exp(x)+\exp(y)}{2}\right)$$

has a rather complicated form (without the possibility of any simplifying reduction). However, if N and M are homogeneous means, the two–place function $f_{M,N}$ can be reduced to a one–place function, which is useful in representing every homogeneous mean M nontrivially as a product $Nf_{M,N}$.

To characterize a positively homogeneous function $M : (0,\infty)^2 \to \mathbf{R}$ by a one–place function $h : (0,\infty) \to (0,\infty)$, it is often useful to take

$$M(x,y) = xh(y/x), \qquad x,\ y > 0,$$

where $h(s) := M(1,s)$, $s > 0$. When the mean M is symmetric, the corresponding symmetry property of h (which reduces to the functional equation $h(s) = sh\left(\frac{1}{s}\right)$,

$s > 0$) is not easily recognizable. However, an expedient symmetry may be achieved by a transformation of the variable s ($s = y/x$) to a new variable t; taking the homographic involution $s = \frac{1-t}{1+t}$ (implying $t = \frac{1-s}{1+s} = \frac{x-y}{x+y}$), we obtain

$$M(x,y) = A(x,y)f\left(\frac{x-y}{x+y}\right), \qquad x,\, y > 0,$$

where A stands for the arithmetic mean and $f(t) = M(1+t, 1-t)$, $t \in (-1,1)$. In the next section we discuss such a decomposition in more detail.

2. Decomposition by the arithmetic mean

Denote by A the arithmetic mean $A(x,y) = \frac{x+y}{2}$, $(x,\, y > 0)$.

DEFINITION 1. For an arbitrary homogeneous mean $M : (0,\infty)^2 \to (0,\infty)$ the function $f_{M,A} : (-1,1) \to \mathbf{R}$, given by

$$f_{M,A}(t) := M(1+t, 1-t), \qquad t \in (-1,1),$$

is called *index function of M with respect to A* (for short: *A–index function of M*).

The following decomposition result justifies this definition.

THEOREM 1. *If $M : (0,\infty)^2 \to (0,\infty)$ is a homogeneous mean, then*

$$M(x,y) = A(x,y)f_{M,A}\left(\frac{x-y}{x+y}\right), \qquad x,\, y > 0, \tag{1}$$

and, moreover,

1° $f_{M,A}((-1,1)) \subseteq (0,2)$*;*

2° $f_{M,A}(0) = 1$*;*

3° *M is symmetric iff $f_{M,A}$ is even, i.e.*

$$f_{M,A}(-t) = f_{M,A}(t), \qquad t \in (-1,1);$$

4° *for all $t \in (-1,1)$,*

$$1 - |\,t\,| \leqslant f_{M,A}(t) \leqslant 1 + |\,t\,|;$$

5° *for all $t \in (-1,1)$,*

$$f_{A,A}(t) = 1;$$

6° *for the extremal means,* min *and* max*, we have, respectively,*

$$f_{\min,A}(t) = 1 - |\,t\,|, \qquad f_{\max,A}(t) = 1 + |\,t\,|, \qquad t \in (-1,1);$$

7° *if M is one of the projective means, i.e, if $M = P_1$ or $M = P_2$, where*

$$P_1(x,y) := x, \qquad P_2(x,y) := y,$$

then

$$f_{P_1,A}(t) = 1 + t, \qquad f_{P_2,A}(t) = 1 - t, \qquad t \in (-1,1);$$

8° *for all homogeneous means* M, $N:(0,\infty)^2\to(0,\infty)$,

$$M(x,y)\leqslant N(x,y),\quad (x,\ y>0)\quad \textit{iff}\quad f_{M,A}(t)\leqslant f_{N,A}(t),\quad t\in(-1,1);$$

9° M *is subadditive, i.e.*

$$M\left(x_1+x_2,y_1+y_2\right)\leqslant M\left(x_1,y_1\right)+M\left(x_2,y_2\right),\qquad x_1,\ x_2,\ y_1,\ y_2>0,$$

iff the function $f_{M,A}$ *is convex.*

Proof. Let M be a homogeneous mean on $(0,\infty)^2$. Then, making use of the definition of the A–index function of M, we have

$$\begin{aligned}M(x,y)&=\frac{x+y}{2}\frac{2}{x+y}M(x,y)=\frac{x+y}{2}M\left(\frac{2x}{x+y},\frac{2y}{x+y}\right)\\&=\frac{x+y}{2}M\left(1+\frac{x-y}{x+y},1-\frac{x-y}{x+y}\right)=A(x,y)f_{M,A}\left(\frac{x-y}{x+y}\right)\end{aligned}$$

for all x, $y>0$, which proves the decomposition formula (1).

By the definition of a mean, we have

$$0\leqslant\min(1+t,1-t)\leqslant M(1+t,1-t)\leqslant\max(1+t,1-t),$$

for all $t\in(-1,1)$. The definition of $f_{M,A}$ proves $1°$. Setting $t=0$ in the definition of the A–index function gives $f_{M,A}(0)=M(1,1)=1$ and proves $2°$. If M is symmetric then, for all $t\in(-1,1)$,

$$f_{M,A}(-t)=M(1-t,1+t)=M(1+t,1-t)=f_{M,A}(t).$$

Conversely, $f_{M,A}(-t)=f_{M,A}(t)$, for all $t\in(-1,1)$, implies that

$$f_{M,A}\left(\frac{x-y}{x+y}\right)=f_{M,A}\left(\frac{y-x}{x+y}\right),\qquad x,\ y>0.$$

Now the symmetry of A and decomposition formula (1) imply that

$$M(x,y)=A(x,y)f_{M,A}\left(\frac{x-y}{x+y}\right)=A(y,x)f_{M,A}\left(\frac{y-x}{x+y}\right)=M(y,x)$$

for all x, $y>0$, which proves $3°$. We omit easy proofs of properties $4°$ – $8°$. It is known that a positively homogeneous function $M:(0,\infty)^2\to\mathbf{R}$ is subadditive iff the function $\phi:(0,\infty)\to\mathbf{R}$, $\phi(t):=M(t,1)$, $t>0$, is convex (cf. Matkowski [5]). Thus to prove $9°$ it is enough to show that $f_{M,A}$ is convex iff the function ϕ is convex (cf. also Remark 1). We have

$$\begin{aligned}f_{M,A}(t)&=M(1+t,1-t)=(1+t)M\left(1,\frac{1-t}{1+t}\right)=(1+t)\phi\left(\frac{1-t}{1+t}\right)\\&=2\frac{1+t}{2}\phi\left(\frac{2}{1+t}-1\right)=2\frac{1+t}{2}\psi\left(\frac{2}{1+t}\right),\end{aligned}$$

where $\psi(u) := \phi(u-1)$. But the function ψ is convex iff the function $u \to u\psi(1/u)$ is convex (cf. Matkowski [4]). This proves 9°, and the proof is completed.

REMARK 1. If $f_{M,A}$ is twice differentiable in $(-1,1)$, the proof of 9° can be simplified. Then, of course, ϕ is twice differentiable in $(0,\infty)$ and we get

$$f''_{M,A}(t) = \frac{4}{(1+t)^3}\phi''\left(\frac{1-t}{1+t}\right), \qquad t \in (-1,1),$$

which shows that $f_{M,A}$ is convex iff ϕ is convex. – To give a complete argument for 9°, other than that presented above, it is enough to observe that every convex function is a limit of a sequence of convex and twice differentiable functions.

It turns out that property 4° characterizes the family of all homogeneous means on $(0,\infty)$. In a sense, the following result is the converse of Theorem 1.4°.

THEOREM 2. *For every function $f : (-1,1) \to \mathbf{R}$ such that*

$$1-|t| \leqslant f(t) \leqslant 1+|t|, \qquad t \in (-1,1), \tag{2}$$

the function $M : (0,\infty)^2 \to \mathbf{R}$ defined by

$$M(x,y) := A(x,y)f\left(\frac{x-y}{x+y}\right), \qquad x,\, y > 0, \tag{3}$$

is a homogeneous mean such that $f = f_{M,A}$.

Proof. Suppose that f satisfies condition (2) and let M be defined by (3). It is obvious that M is homogeneous. Take arbitrary x, $y > 0$, and assume, for simplicity of notation, that $x \leqslant y$. Then, applying (2) and (3), gives

$$\begin{aligned}
\min(x,y) &= x = \frac{x+y}{2}\left(1 - \frac{y-x}{x+y}\right) \\
&= \frac{x+y}{2}\left(1 - \left|\frac{x-y}{x+y}\right|\right) \leqslant \frac{x+y}{2} f\left(\frac{x-y}{x+y}\right) \\
&= M(x,y) \leqslant \frac{x+y}{2}\left(1 + \left|\frac{x-y}{x+y}\right|\right) \\
&= \frac{x+y}{2}\left(1 + \frac{y-x}{x+y}\right) = y = \max(x,y),
\end{aligned}$$

which shows that M is a mean. Setting $x = 1+t$ and $y = 1-t$ for $t \in (-1,1)$ gives $M(1+t, 1-t) = f(t)$, which means that $f = f_{M,A}$, and the proof is completed.

REMARK 2. Theorems 1 and 2 establish a bijection of the class of all homogeneous means $M : (0,\infty)^2 \to (0,\infty)$ onto the class of all functions $f : (-1,1) \to (0,2)$ satisfying condition (2). Formula (1) gives a general construction of homogeneous means.

REMARK 3. The set

$$\Delta := \left\{(t,s) \in \mathbf{R}^2 : \ t \in (-1,1); \quad 1-|t| \leqslant s \leqslant 1+|t|\right\}$$

is of butterfly shape. Theorem 2 can be interpreted geometrically in the following way: every function $f : (-1,1) \to \mathbf{R}$ such that the graph of f is contained in Δ, is an A–index function of a mean. Note that Δ is not a convex set. In this context it is interesting that the set of all A–index functions is convex. Namely, for all homogeneous means M, N and $\lambda \in (0,1)$ the function $\lambda f_{M,A} + (1-\lambda) f_{N,A}$ is an A–index function of a mean $\lambda M + (1-\lambda)N$.

This fact can be generalized. For homogeneous means V, M, N on $(0,\infty)$ define their composition $U : (0,\infty)^2 \to (0,\infty)$ by

$$U(x,y) := V\left(M(x,y), N(x,y)\right), \qquad x,\ y > 0.$$

Then, of course, U is a homogeneous mean, and

$$f_{U,A}(t) = V\left(f_{M,A}(t), f_{N,A}(t)\right), \qquad t \in (-1,1).$$

In particular, the graph of a homogeneous mean of any two A–index functions, the graphs of which are of course in Δ, is also located in the region Δ.

Note also that the A–index functions $f_{M,A}$ of homogeneous means M need not be continuous. To show this it is enough to apply Theorem 2 where f is an arbitrary discontinuous function satisfying condition (2).

The next result gives conditions under which a homogeneous mean defined on $(0,\infty)^2$ can be, in a natural way, extended to a homogeneous mean defined on the closed quadrant $[0,\infty)^2$.

THEOREM 3. *Let $M : (0,\infty)^2 \to (0,\infty)$ be a homogeneous mean. If the limits*

$$f_{M,A}(1-) := \lim_{t \to 1-} f_{M,A}(t),$$

$$f_{M,A}(-1+) := \lim_{t \to -1+} f_{M,A}(t),$$

exist, then they are finite, and $\overline{M} : [0,\infty)^2 \to [0,\infty)$ defined by

$$\overline{M}(x,y) := \begin{cases} M(x,y) & x,\ y > 0, \\ A(x,0) f_{M,A}(1-) & x > 0,\ y = 0, \\ A(0,y) f_{M,A}(-1+) & x = 0,\ y > 0, \\ 0 & x = y = 0. \end{cases}$$

is a homogeneous mean defined on $[0,\infty)^2$.

Proof. In view of Theorem 1.1° the limit $f_{M,A}(1-)$ is finite and, making use of (1), we infer that, for every $x > 0$, the limit

$$M(x,0+) := \lim_{y \to 0+} M(x,y) = \lim_{y \to 0+} A(x,y) f_{M,A}\left(\frac{x-y}{x+y}\right) = A(x,0) f_{M,A}(1-)$$

exists, and is finite. Similarly, the limit $f_{M,A}(-1+)$ is finite, and, for every $y > 0$, the limit

$$M(0+,y) := \lim_{x\to 0+} M(x,y) = \lim_{x\to 0+} A(x,y) f_{M,A}\left(\frac{x-y}{x+y}\right) = A(0,y) f_{M,A}(-1+)$$

exists, and it is finite. As the homogeneity of $\overline{M}$ is obvious the proof is completed.

REMARK 4. Applying this theorem, it is easy to verify that harmonic and logarithmic means (whose natural domain is $(0,\infty)^2$) can be extended onto the closed quadrant $[0,\infty)^2$. Note also that for every homogeneous mean M defined on $[0,\infty)^2$ we can define the A–index function $f_{M,A} : [-1,1] \to [0,2]$, and that the counterparts of Theorems 1 and 2 remain true.

Taking in Theorem 2 a function $f : (-1,1) \to \mathbf{R}$ satisfying condition (2) and such that at least one of the limits $f(1-)$ or $f(-1+)$ does not exist, we obtain a homogeneous mean defined on $(0,\infty)^2$ that is not extendable to a continuous homogeneous mean defined on the closed quadrant $[0,\infty)^2$.

EXAMPLE 1. (Decomposition of power mean $M^{[p]}$ by arithmetic mean A.) The power means $M^{[p]} : (0,\infty)^2 \to (0,\infty)$, $p \in \mathbf{R}$, are defined by the formula

$$M^{[p]}(x,y) := \left(\frac{x^p + y^p}{2}\right)^{1/p}, \quad p \neq 0; \qquad M^{[0]}(x,y) := G(x,y), \quad x,\, y > 0,$$

where $G : (0,\infty)^2 \to (0,\infty)$ stands for the geometric mean. We have

$$f_{M^{[p]},A}(t) = \left(\frac{(1+t)^p + (1-t)^p}{2}\right)^{1/p}, \quad f_{G,A}(t) = \left(1-t^2\right)^{1/2}, \qquad t \in (-1,1).$$

PARTICULAR CASES.

(i) $p = -1$ (decomposition of harmonic mean $M^{[-1]} = H$ by arithmetic mean):

$$H(x,y) = A(x,y) f_{H,A}\left(\frac{x-y}{x+y}\right), \qquad x,\, y > 0;$$

$$f_{H,A}(t) = H(1+t, 1-t) = 1 - t^2, \qquad t \in (-1,1).$$

Note that here the limits $f_{H,A}(1-)$, $f_{H,A}(-1+)$ exist and equal zero. Therefore, in view of Theorem 3, the harmonic mean H can be (uniquely) extended onto the closed quadrant $[0,\infty)^2$, and $\overline{H}$, the homogeneous extension of H, vanishes on the boundary of its domain.

(ii) $p = 2$ (decomposition of RMS mean $M^{[2]} = R$ by arithmetic mean):

$$R(x,y) = A(x,y) f_{R,A}\left(\frac{x-y}{x+y}\right), \qquad x,\ y > 0;$$

$$f_{R,A}(t) = R(1+t, 1-t) = \left(1+t^2\right)^{1/2}, \qquad t \in (-1,1).$$

Let us note the following easy to verify

REMARK 5. For any mean $M:(0,\infty)^2\to(0,\infty)$, the function $M^*:(0,\infty)^2\to\mathbf{R}$ defined by

$$M^*(x,y):=x+y-M(x,y),\qquad x,\ y>0,$$

is a mean. In the sequel it is called the *contra–mean of* M. Note that $(M^*)^*=M$.

Moreover,

$$A(M,M^*)=A\quad\text{and}\qquad f_{M^*,A}+f_{M,A}=2.$$

In connection with this remark let us note another property of A–index functions.

THEOREM 4. *If* $M:(0,\infty)^2\to(0,\infty)$ *is a homogeneous mean and* $f_{M,A}$ *is its* A *–index function, then the function*

$$f:=2-f_{M,A}$$

is also an A*–index function, namely of a mean which is the contra–mean of* M.

Proof. In view of Theorem 1.4°,

$$1-|\,t\,|\leqslant f_{M,A}(t)\leqslant 1+|\,t\,|,\qquad t\in(-1,1).$$

Hence, by the definition of f, we get

$$1-|\,t\,|\leqslant f(t)\leqslant 1+|\,t\,|,\qquad t\in(-1,1),$$

and, according to Theorem 2, the function f is an A–index function of a certain homogeneous mean $m:(0,\infty)^2\to(0,\infty)$, and for all $x,\ y>0$,

$$m(x,y)=A(x,y)f\left(\frac{x-y}{x+y}\right)=2A(x,y)-A(x,y)f_{M,A}\left(\frac{x-y}{x+y}\right)=x+y-M(x,y),$$

which completes the proof.

EXAMPLE 2. (Decomposition of contra–harmonic mean $K=H^*$ by arithmetic mean):

$$K(x,y)=\frac{x^2+y^2}{x+y}=A(x,y)f_{K,A}\left(\frac{x-y}{x+y}\right),\qquad x,\ y>0;$$

$$f_{K,A}(t)=K(1+t,1-t)=1+t^2,\qquad t\in(-1,1).$$

EXAMPLE 3. (Decomposition of Heronic mean E by arithmetic mean):

$$E(x,y)=\frac{1}{3}\left(x+y+(xy)^{1/2}\right)=A(x,y)f_{E,A}\left(\frac{x-y}{x+y}\right),\qquad x,\ y>0;$$

$$f_{E,A}(t)=E(1+t,1-t)=\frac{1}{3}\left(2+(1-t^2)^{1/2}\right),\qquad t\in(-1,1).$$

EXAMPLE 4. (Decomposition of logarithmic mean L by arithmetic mean):

$$L(x,y) = \frac{x-y}{\log(x)-\log(y)} = A(x,y)f_{L,A}\left(\frac{x-y}{x+y}\right), \qquad x,\, y > 0, \quad x \neq y;$$

$$f_{L,A}(t) = L(1+t, 1-t) = \frac{2t}{\log\frac{1+t}{1-t}}, \qquad t \in (-1,1).$$

Here $f_{L,A}(1-)$, $f_{L,A}(-1+)$ exist and equal zero. In view of Theorem 3, the mean L is extendable onto the quadrant $[0,\infty)^2$. According to Theorem 3, the extension $\overline{L}$ vanishes on the boundary of its domain.

3. Decomposition by any homogeneous mean

Here we show that upon replacing the arithmetic mean by another homogeneous reference mean, the counterparts of Theorem $1.1^\circ - 8^\circ$ remain true.

DEFINITION 2. Let $m : (0,\infty)^2 \to (0,\infty)$ be a fixed homogeneous mean. For an arbitrary homogeneous mean $M : (0,\infty)^2 \to (0,\infty)$ the function $f_{M,m} : (-1,1) \to \mathbf{R}$, given by

$$f_{M,m}(t) := \frac{M(1+t,1-t)}{m(1+t,1-t)}, \qquad t \in (-1,1),$$

is said to be the *index function of* M *with respect to* m (for short: m*–index function of* M).

REMARK 6. Note that, under the assumption of the definition,

$$f_{M,m}(t) := \frac{f_{M,A}(t)}{f_{m,A}(t)} = \frac{M(1+t,1-t)}{f_{m,A}(t)} = M\left(\frac{1+t}{f_{m,A}(t)}, \frac{1-t}{f_{m,A}(t)}\right), \qquad t \in (-1,1).$$

THEOREM 5. *Let* $m : (0,\infty)^2 \to (0,\infty)$ *be a fixed homogeneous mean. If* $M : (0,\infty)^2 \to (0,\infty)$ *is a homogeneous mean, then*

$$M(x,y) = m(x,y)f_{M,m}\left(\frac{x-y}{x+y}\right), \qquad x,\, y > 0,$$

and, moreover,

1° $f_{M,m}(-1,1) \subseteq (0,\alpha)$, *where* $\alpha := \sup\left\{\frac{1+|t|}{f_{m,A}(t)} : t \in (-1,1)\right\}$;

2° $f_{m,m}(0) = 1$;

3° *if* m *is symmetric, then* M *is symmetric iff* $f_{M,m}$ *is even, i.e.*

$$f_{M,m}(-t) = f_{M,m}(t), \qquad t \in (-1,1);$$

4° *for every* $t \in (-1,1)$,

$$\frac{1-|t|}{f_{m,A}(t)} \leqslant f_{M,m}(t) \leqslant \frac{1+|t|}{f_{m,A}(t)};$$

5° *for all* $t \in (-1,1)$, $f_{m,m}(t) = 1$;

6° *for the extremal means,* min *and* max, *we have, respectively,*

$$f_{\min,m}(t) = \frac{1-|t|}{f_{m,A}(t)}, \qquad f_{\max,m}(t) = \frac{1+|t|}{f_{m,A}(t)}, \qquad t \in (-1,1);$$

7° *if* M *is one of the projective means, i.e. if* $M = P_1$ *or* $M = P_2$, *where* $P_1(x,y) := x$, *and* $P_2(x,y) := y$, *then*

$$f_{P_1,m}(t) = \frac{1+t}{f_{m,A}(t)}, \qquad f_{P_2,A}(t) = \frac{1-t}{f_{m,A}(t)}, \qquad t \in (-1,1);$$

8° *for all homogeneous means* M, $N : (0,\infty)^2 \to (0,\infty)$,

$$M(x,y) \leqslant N(x,y), \ \ (x,\ y>0) \quad \textit{iff} \quad f_{M,m}(t) \leqslant f_{N,m}(t), \quad t \in (-1,1).$$

We omit easy arguments (analogous to the suitable parts of Theorem 1).

REMARK 7. If m, $M : (0,\infty)^2 \to (0,\infty)$ are homogeneous means then

$$f_{M,m}(t) = \frac{1}{f_{m,M}(t)}, \qquad t \in (-1,1).$$

The next result, a counterpart of Theorem 2, is, in a sense, the converse of Theorem 5.4°. It gives a construction of a homogeneous mean from a given suitable function in a single variable and a given homogeneous reference mean.

THEOREM 6. *Let* $m : (0,\infty)^2 \to (0,\infty)$ *be a homogeneous mean. For every function* $f : (-1,1) \to \mathbf{R}$ *satisfying the condition*

$$\frac{1-|t|}{f_{m,A}(t)} \leqslant f(t) \leqslant \frac{1+|t|}{f_{m,A}(t)}, \qquad t \in (-1,1), \tag{4}$$

the function $M : (0,\infty)^2 \to \mathbf{R}$ *defined by*

$$M(x,y) := m(x,y) f\left(\frac{x-y}{x+y}\right), \qquad x,\ y > 0,$$

is a homogeneous mean such that $f = f_{M,m}$.

As the proof is similar to that of Theorem 2, we omit it.

EXAMPLE 5. (Decomposition of a power mean $M^{[p]}$ by another power mean $M^{[q]}$.) By the definition of power means (Example 1) we get the docomposition

$$M^{[p]}(x,y) = M^{[q]}(x,y) f_{M^{[p]},M^{[q]}}\left(\frac{x-y}{x+y}\right), \qquad p,\ q \in \mathbf{R}, \quad (x,\ y > 0);$$

$$f_{M^{[p]},M^{[q]}}(t) = \frac{M^{[p]}(1+t,1-t)}{M^{[q]}(1+t,1-t)}, \qquad t \in (-1,1).$$

PARTICULAR CASES.

(i) $p=-1$, $q=0$ (decomposition of harmonic mean $M^{[-1]} = H$ by geometric mean $M^{[0]} = G$):

$$H(x,y) = G(x,y) f_{H,G}\left(\frac{x-y}{x+y}\right), \qquad f_{H,G}(t) = \left(1-t^2\right)^{1/2}, \quad t \in (-1,1).$$

(ii) $p=1$, $q=0$ (decomposition of arithmetic mean $M^{[1]} = A$ by geometric mean $M^{[0]} = G$):

$$A(x,y) = G(x,y) f_{A,G}\left(\frac{x-y}{x+y}\right), \qquad f_{A,G}(t) = (1-t^2)^{-1/2}, \qquad t \in (-1,1),$$

conforming to $f_{A,G}(t) = 1/f_{G,A}(t)$ for all $t \in (-1,1)$ (cf. Example 1) by Remark 7.

EXAMPLE 6. Choose for reference the harmonic mean, $m = H$, and consider the function $f : (-1,1) \to \mathbf{R}$, $f(t) = 1 + \frac{t}{2}$. From the decomposition $H = A f_{H,A}$ (cf. Example 1) we know $f_{H,A}(t) = 1 - t^2$. The given f fulfills condition $4°$ of Theorem 5, therefore, f is an H–index function for a certain homogeneous mean $M : (0,\infty)^2 \to (0,\infty)$. By Theorem 6 this mean has the form

$$M(x,y) = \frac{2xy}{x+y} f\left(\frac{x-y}{x+y}\right) = \frac{3x^2y + xy^2}{(x+y)^2}, \qquad x,\, y > 0.$$

Although the reference mean $m = H$ is symmetric, the resulting mean M is not symmetric since the function f is not even.

4. Graphs of index functions

Let $m : (0,\infty)^2 \to (0,\infty)$ be a fixed homogeneous mean. Then the graphs of all m–index functions $f_{M,m}$, where M is a homogeneous mean on $(0,\infty)$, are contained in a butterfly–shaped region; they are suitable for geometrical interpretations and visual comparisons of several properties of homogeneous means.

4.1. Graphs of index functions with respect to the arithmetic mean.

The A–index function $f_{M,A}$ is, according to Theorem 1.4°, bounded by the functions

$$f_{\min,A}(t) = 1 - |\, t\,|, \qquad f_{\max,A}(t) = 1 + |\, t\,|, \qquad t \in (-1,1);$$

their graphs constitute a region of butterfly shape.

EXAMPLE 7. The graphs of the A–index functions $f_{M,A}$ of some power means (cf. Example 1) are given in Figure 1. The arithmetic mean A (as reference mean) appears as a horizontal straight line; geometric mean G and harmonic mean H are represented by a semicircle and a parabola, respectively. All graphs must pass through the point $(0,1)$ (cf. Theorem 1.2°). The present means are symmetric, implying that the graphs of the corresponding A–index functions are symmetric with respect to the second coordinate axis. The well-known relation between harmonic, geometric, logarithmic, arithmetic and root-mean-square mean, expressed by the inequality $\min \leqslant H \leqslant G \leqslant L \leqslant A \leqslant R \leqslant \max$, is mirrored in any graph of index functions (according to Theorem 1.8°).

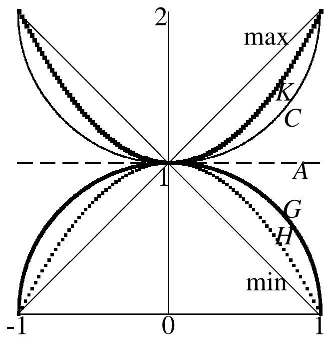

Fig. 1. Graphs of A–index functions.
The labels refer to the following means: A arithmetic mean, G geometric mean, H harmonic mean, C contra-geometric mean, K contra-harmonic mean, max maximum, min minimum

EXAMPLE 8. The graphs of the A–index functions $f_{M,A}$ of some contra–means (cf. Example 2) are also given in Figure 1. — The Heronic mean E from Example 3 can be written as a weighted arithmetic mean of A and G, namely

$$E = \frac{1}{3}(2A+G) = A\left(\frac{4}{3}A, \frac{2}{3}G\right),$$

which is mirrored in the corresponding A–index function as

$$f_{E,A} = \frac{1}{3}\,(2+f_{G,A}) = A\left(\frac{4}{3}, \frac{2}{3}f_{G,A}\right).$$

(It is easy to read off the inequality $G \leqslant E \leqslant A$ from any graph of index functions.)

4.2. Graphs of index functions with respect to any homogeneous mean.

The index function $f_{M,m}$ (of a homogeneous mean M with respect to another homogeneous mean m) is, according to Theorem 5.4°, bounded by the functions

$$f_{\min,m}(t) = \frac{1-\mid t \mid}{f_{m,A}(t)}, \qquad f_{\max,m}(t) = \frac{1+\mid t \mid}{f_{m,A}(t)}, \qquad t \in (-1,1),$$

which form a region of butterfly shape in graphical representations.

EXAMPLE 9. The graphs of the G–index functions $f_{M,G}$ of some power means (cf. Example 5) are given in Figure 2. The geometric mean G (as reference mean) appears as a horizontal straight line; the harmonic mean H is now represented by a semicircle, and (the graph of) the G–index function $f_{A,G}$ is not bounded above. Thus, in Theorem 5.1°, we have $\alpha = +\infty$.

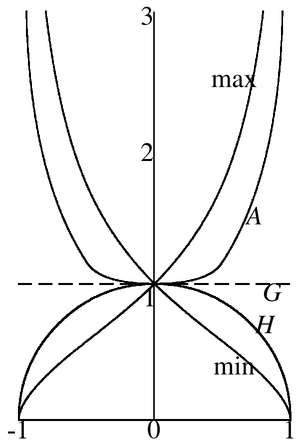

Fig. 2. Graphs of G–index functions.
The labels refer to the following means: A arithmetic mean, G geometric mean, H harmonic mean, max maximum, min minimum

5. Use of A–index functions in proving inequalities

To get the best estimation of the contra–harmonic mean

$$K(x,y) = \frac{x^2+y^2}{x+y}, \qquad x,y > 0,$$

by power means $M^{[p]}$,

$$M^{[p]}(x,y) := \left(\frac{x^p+y^p}{2}\right)^{1/p}, \quad p \neq 0; \qquad M^{[0]}(x,y) := \sqrt{xy}; \quad (x,y>0),$$

we now apply Theorem 1.8°. (Cf. also Zs. Páles [7] where, by a different method, a more general result is proved.)

THEOREM 7. *For every* $p \in \mathbf{R}$, $M^{[p]} \leqslant K$ *iff* $p \leqslant 3$. *Moreover, there is no* $p \in \mathbf{R}$ *such that* $K \leqslant M^{[p]}$.

Proof. Suppose that $M^{[p]} \leqslant K$ for some $p > 0$. By Theorem 1.8^0 this inequality is equivalent to

$$f_{M^{[p]},A}(t) \leqslant f_{K,A}(t), \qquad t \in (-1,1).$$

Since

$$f_{K,A}(t) = 1 + t^2, \qquad f_{M^{[p]},A}(t) = 2^{-1/p}\left[(1+t)^p + (1-t)^p\right]^{1/p}, \quad t \in (-1,1),$$

we get

$$(1+t)^p + (1-t)^p \leqslant 2(1+t^2)^p, \qquad t \in (-1,1).$$

As, by Taylor's theorem,

$$(1+t)^p + (1-t)^p = 2 + \binom{p}{2} t^2 + o(t^2), \qquad t \in (-1,1),$$

$$2(1+t^2)^p = 2 + \binom{p}{1} t^2 + o(t^2), \qquad t \in (-1,1),$$

the last inequality implies that $\binom{p}{2} \leqslant \binom{p}{1}$, i.e. $p \leqslant 3$.

Since, obviously,

$$f_{M^{[3]},A}(t) = (1+3t^2)^{1/3} \leqslant 1 + t^2 = f_{K,A}(t), \qquad t \in (-1,1),$$

Theorem 1.8^0 implies that $M^{[3]} \leqslant K$. Since the function $\mathbf{R} \ni p \to M^{[p]}$ is increasing, we infer that $M^{[p]} \leqslant K$ for all $p \leqslant 3$.

Suppose now that there exists a $p \in \mathbf{R}$ such that

$$f_{K,A}(t) \leqslant f_{M^{[p]},A}(t), \qquad t \in (-1,1).$$

In view of the previous part of the proof we have $p \geqslant 3$. Letting here $t \to 1$ gives $2 \leqslant 2^{(p-1)/p}$, which is a contradiction, and the proof is completed.

REMARK 8. A similar reasoning allows to give a simple proof of the inequality of Lin [3]:

$$G \leqslant L \leqslant M^{[1/3]},$$

which is the best estimation of the logarithmic mean by power means.

6. Some metrics in the family of homogeneous means

We begin this section with the following

THEOREM 8. *Let* $\mathscr{M}$ *denote the set of all homogeneous means* $M:(0,\infty)^2\to(0,\infty)$, *and let* $d_A:\mathscr{M}^2\to\mathbf{R}$ *be defined by*

$$d_A(M,N):=\sup\{|f_{M,A}(t)-f_{N,A}(t)|:t\in(-1,1)\}.$$

Then $(\mathscr{M},d_A)$ *is a bounded complete metric space. Moreover,*

1° *for every sequence* $M_k\in\mathscr{M}$, $k\in\mathbf{N}$, *and* $M\in\mathscr{M}$,

$$\lim_{k\to\infty}d_A(M_k,M)=0$$

iff $M_k\to M$ *uniformly on compact subsets of* $(0,\infty)^2$;

2°

$$\sup\{d_A(M,N):M,\,N\in\mathscr{M}\}=d_A(\min,\max)=2;$$

3° *the set* $\mathscr{M}$ *is convex, i.e. for all* M, $N\in\mathscr{M}$ *and* $\lambda\in(0,1)$,

$$W:=\lambda M+(1-\lambda)N\in\mathscr{M},$$

and the metric space $(\mathscr{M},d_A)$ *is convex in the sense of Menger (it is metrically convex), i.e.*

$$d_A(M,W)+d_A(W,N)=d_A(M,N).$$

Proof. From Theorem 1.4° we infer that

$$|f_{M,A}(t)-f_{N,A}(t)|\leqslant 2\mid t\mid,\qquad t\in(-1,1),$$

which shows that the function d_A has finite values on $\mathscr{M}$, and $d_A(M,N)\leqslant 2$. If $d_A(M,N)=0$ then $f_{M,A}=f_{N,A}$, and by (1), we have $M=N$. Conversely, if $M=N$ for some M, $N\in\mathscr{M}$ then $f_{M,A}=f_{N,A}$ and, consequently, $d_A(M,N)=0$. Since the symmetry and the triangle inequality are obvious, d_A is a metric in $\mathscr{M}$. Let $(M_k)_{k=1}^{\infty}$ be a Cauchy sequence in the metric space $(\mathscr{M},d_A)$ and $\varepsilon>0$. Thus there is a $k_0\in\mathbf{N}$ such that $d_A(M_k,M_l)\leqslant\varepsilon$ for all k, $l\geqslant k_0$, k, $l\in\mathbf{N}$. By the definition of d_A we have

$$\left|f_{M_k,A}(t)-f_{M_l,A}(t)\right|\leqslant\varepsilon,\qquad k,\,l\geqslant 0,\quad t\in(-1,1).\tag{5}$$

It follows that there exists an $f:(-1,1)\to\mathbf{R}$ such that for every $t\in(-1,1)$,

$$\lim_{k\to\infty}f_{M_k,A}(t)=f(t).$$

Since, in view of Theorem 1.4°, we have $1-\mid t\mid\leqslant f_{M_k,A}(t)\leqslant 1+\mid t\mid$, for all $t\in(-1,1)$, we hence get $1-\mid t\mid\leqslant f(t)\leqslant 1+\mid t\mid$ for all $t\in(-1,1)$. By Theorem 2 there exists an $M\in\mathscr{M}$ such that $f=f_{M,A}$. Letting $l\to\infty$ in (5) gives

$$\left|f_{M_k,A}(t)-f_{M,A}(t)\right|\leqslant\varepsilon,\qquad k\geqslant 0,\quad t\in(-1,1),$$

i.e. $d_A(M_k,M)\leqslant\varepsilon$ for all $k\geqslant k_0$. Thus the sequence (M_k) converges to an element of $\mathscr{M}$ in the sense of the metric d_A, and the completeness of the metric space is proved. Part 1° is an easy consequence of decomposition formula (1). Part 2° and 3^0 are obvious.

REMARK 9. Let $\mathscr{M}$ denote the set of all homogeneous means $M:(0,\infty)^2\to(0,\infty)$, and let $\rho_A:\mathscr{M}^2\to\mathbf{R}$ be defined by

$$\rho_A(M,N):=\sup\left\{\frac{|f_{M,A}(t)-f_{N,A}(t)|}{|t|}:t\in(-1,1)\right\}.$$

Then, similarly as in Theorem 8, one can show that $(\mathscr{M},\rho_A)$ is a bounded complete metric space; the statements 2^0 – 3^0 remain valid. It is obvious that the metric ρ_A is stronger than d_A. To show that ρ_A is essentially stronger, consider the following

EXAMPLE 10. Let $M_n\in\mathscr{M}$, $n\in\mathbf{N}$, be a sequence of means such that

$$f_{M_n,A}(t)=\begin{cases}1, & |t|\in[1/n,1];\\ |t|, & |t|<1/n.\end{cases}$$

Then $d_A(M_n,A)=\frac{1}{n}, n\in\mathbf{N}$, and (M_n) converges to A, as $n\to\infty$, in the sense of the metric d_A. On the other hand, we have

$$\rho_A(M_n,A)=1,\quad n\in\mathbf{N};$$

obviously, the sequence (M_n) is not convergent in the sense of the metric ρ_A.

The metric in Remark 9 seems to be a most proper one, and the last example shows that this (apparently most proper) metric is independently interesting.

Choosing special subsets of $\mathscr{M}$ one can define some other metric spaces; cf., for instance, the following remarks.

REMARK 10. Let $\mathscr{L}$ be the set of all homogeneous means $M:(0,\infty)^2\to(0,\infty)$ such that the A–index function $f_{M,A}$ is Lebesgue measurable, and let $p\geqslant 1$ be fixed. Then, by Theorem 1.1°, for every $M\in\mathscr{L}$, the function $f_{M,A}$ is Lebesgue integrable, and $l_A^p:\mathscr{L}\times\mathscr{L}\to\mathbf{R}$ defined by

$$l_A^p(M,N):=\left(\int_{-1}^{1}|f_{M,A}(t)-f_{N,A}(t)|^p\,dt\right)^{\frac{1}{p}},\qquad M,\,N\in\mathscr{L},$$

is a metric in $\mathscr{L}$. Similarly as in Theorem 8, it can be shown that the metric space $(\mathscr{L},l_A^p)$ is complete, and metrically convex (and the set $(\mathscr{L})$ is convex, cf. Remark 3).

REMARK 11. Denote by $\mathscr{C}_n$ the set of all homogeneous means $M:(0,\infty)^2\to(0,\infty)$ such that the A–index function $f_{M,A}$ is n times continuously differentiable. Then $\varrho_A:\mathscr{C}_n\times\mathscr{C}_n\to\mathbf{R}$ defined by

$$\varrho_A(M,N):=\sum_{k=1}^{n-1}\left|f_{M,A}^{(k)}(0)-f_{N,A}^{(k)}(0)\right|+\sup\left\{\left|f_{M,A}^{(n)}(t)-f_{N,A}^{(n)}(t)\right|:|t|<1\right\}$$

is a metric in $\mathscr{C}_n$. The metric space $(\mathscr{C}_n,\varrho_A)$ is complete, and (metrically) convex.

7. M–convexity of power functions

Let $M:(0,\infty)^2\to(0,\infty)$ be a homogeneous mean. A function $\phi:(0,\infty)\to(0,\infty)$ is called *M–convex* if, for all $x,\ y>0$,

$$\phi(M(x,y))\leqslant M(\phi(x),\phi(y)).$$

For every $p\in\mathbf{R}$ define $\phi_p:(0,\infty)\to(0,\infty)$ by $\phi_p(x)=x^p\quad(x>0)$. The following criterion of M–convexity for the power functions is proved in [Matkowski and Rätz, 6].

All functions ϕ_p, $p>1$, are M–convex iff the following function is increasing:

$$\tau_M:(0,\infty)\to(0,\infty),\qquad \tau_M(x):=\left(M\left(e^x,1\right)\right)^{1/x}\qquad(x>0).$$

REMARK 12. Due to the homogeneity of M, the test function τ_M can be written in a more symmetric form,

$$\tau_M(x)=c\left(M\left(e^{x/2},e^{-x/2}\right)\right)^{1/x}\qquad(x>0),\ \text{where } c:=e^{1/2},$$

and we get the following:

1. *Let $r\in\mathbf{R}$, $r\neq 0$, be fixed. All functions ϕ_p, $p>1$, are $M^{[r]}$–convex iff the function*

$$\tau_{M^{[r]}}(x)=c\left(\cosh\left(\frac{rx}{2}\right)\right)^{1/(rx)}\quad(x>0)\quad \textit{is increasing}.$$

2. *All functions ϕ_p, $p>1$, are L–convex iff the function*

$$\tau_L(x)=c\left(\frac{\sinh(\frac{x}{2})}{\frac{x}{2}}\right)^{1/x}\quad(x>0)\quad \textit{is increasing}.$$

Let us note that, via the hyperbolic functions, there is a strict connection of the test function τ_M with our index function. Namely, we have the following

REMARK 13. *For every positively homogeneous mean $M:(0,\infty)^2\to(0,\infty)$,*

$$\tau_M(x)=c\left[\cosh\left(\frac{x}{2}\right)f_{M,A}\left(\tanh(\frac{x}{2})\right)\right]^{1/x}\qquad(x>0),\quad c=e^{1/2}.$$

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