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MULTI–DIMENSIONAL INTEGRAL INEQUALITIES OF THE WIRTINGER–TYPE

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(*communicated by B. G. Pachpatte*)

Abstract. In this paper some new Wirtinger-type integral inequalities involving many functions of many variables are established. These on the one hand improve existing results in the subject concerned and on the other hand can serve as generators of other integral inequalities of such type.

1. Introduction

Wirtinger's inequality is one of the most inspiring and fundamental integral inequalities in the analysis of finite elements and the study of qualitative as well as quantitative properties of solutions of differential and integral equations. It states that if $\varphi : \mathbb{R} \to \mathbb{R}$ is periodic with period 2π, $\varphi' \in \mathcal{L}^2[0, 2\pi]$, and if $\int_0^{2\pi} \varphi(x)dx = 0$, then

$$\int_0^{2\pi} \varphi(x)^2 dx \leqslant \int_0^{2\pi} \varphi'(x)^2 dx \tag{1}$$

with equality holds if and only if $\varphi(x) = A\cos x + B\sin x$ for some $A, B \in \mathbb{R}$. As pointed out by Mitrinović [15], (1) was first proved by Blaschke [8] in 1916 but a stronger version had already been established well before that. In fact, back in 1905 Almansi [3] showed that under the weaker conditions that φ, $\varphi' \in C(a,b)$, $\varphi(a) = \varphi(b)$, and $\int_a^b \varphi(x)dx = 0$, the inequality

$$\int_a^b \varphi(x)^2 dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b \varphi'(x)^2 dx \tag{2}$$

holds. Since (1) and (2) together with many of their variations have proved to be extremely useful in the study of differential and integral equations, a vast stock of important generalizations of them have been established. These include the works of Beesack [5, 6], Bellman [7], Opial [17], Schmidt [23], Sz.-Nagy [16], and recently Agarwal-Pang [1], Agarwal-Pečarić-Brnetić [2], Agarwal-Sheng [3], Cheung [9, 10, 11, 12], Milovanović-Mitrinović-Rassias [14], Pachpatte [18, 19, 20], and Rassias [21, 22].

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It is the purpose of this paper to establish some generalizations of Wirtinger's inequality (1), (2) to the case of several independent variables, which improve existing results in [1, 3]. A noteworthy point is that in contrast to the usual techniques of using divergence theorems and considering certain eigenvalue problems, the methods used in this paper are rather elementary. More importantly, the methods used here are kind of systematic and furthermore, the same techniques can be used to establish other types of integral inequalities in a rather efficient and easy way.

Throughout this paper, $m,n \geqslant 1$ will always denote two fixed integers. For uniformity, let $\alpha, \beta, \ldots$ be indices running from 1 to m, and $i, j, \ldots$ from 1 to n. Let $\Omega = \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n$ be a fixed rectangular region. As usual, a general point in $\mathbb{R}^n$ will be denoted by $x = (x_1, \ldots, x_n)$ and the volume form of $\mathbb{R}^n$ by $dx = dx_1 \cdots dx_n$. If $x = (x_1, \ldots, x_n) \in \Omega$, we shall write $\Omega_x := \prod_{i=1}^{n} [a_i, x_i]$ and $V(\Omega_x)$:= the volume of Ω_x. For the sake of simplicity we write $V = V(\Omega_b) = V(\Omega)$. Let $C(\Omega)$ be, as usual, the set of all continuous functions on Ω. If $f \in C(\Omega)$, the partial derivatives of f will be denoted by f_i, f_{ij}, etc. The function $f_{1\cdots n}$ will be abbreviated by $\dot{f}$.

Since all summations and products appearing in this paper have clear terminals, for the sake of simplicity we shall drop the references to these and simply use the symbols $\sum_{\alpha}$, $\prod_{i}$ etc. without terminals.

The following elementary inequality will be needed in the sequel.

LEMMA 0. *For any p_α, q_α, $c_\alpha > 0$ with $\sum q_\alpha / p_\alpha = 1$,*

$$\prod c_\alpha^{q_\alpha} \leqslant \sum \frac{q_\alpha}{p_\alpha} c_\alpha^{p_\alpha},$$

where the equality holds if and only if $c_1 = \cdots = c_m$.

The proof of Lemma 0 depends on the arithmetic-geometric mean inequality and is rather elementary, so it is not included here. Interested readers may consult, e.g., [13, 15].

2. Main Results

Let

$$\begin{aligned}
\mathcal{F} = \{f \in C(\Omega) : \; & f_1, f_{12}, \ldots, f_{1\cdots n} \text{ exist and are continuous on } \Omega \\
& \text{such that } \; f(a_1, x_2, \ldots, x_n) = f_1(x_1, a_2, x_3, \ldots, x_n) = \cdots \\
& = f_{1\cdots n-1}(x_1, \ldots, x_{n-1}, a_n) = 0 \text{ for all } x \in \Omega\}, \\
\mathcal{G} = \{f \in \mathcal{F} : \; & f(b_1, x_2, \ldots, x_n) = f_2(x_1, b_2, x_3, \ldots, x_n) = \cdots \\
& = f_{1\cdots n-1}(x_1, \ldots, x_{n-1}, b_n) = 0 \text{ for all } x \in \Omega\}.
\end{aligned}$$

THEOREM 1. *For any* $f^\alpha \in \mathcal{F}$ *and any real numbers* $p_\alpha \geqslant 1$, $q_\alpha > 0$ *with* $\sum_\alpha q_\alpha/p_\alpha = 1$, *if we write* $q := \sum_\alpha q_\alpha$, *then*

$$\int_\Omega w \prod_\alpha |f^\alpha|^{q_\alpha} \leqslant \frac{1}{C^q} K(w,q) \sum_\alpha \frac{q_\alpha}{p_\alpha} C^{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} , \tag{3}$$

where $w(t)$ *is any non-negative continuous weight function on* Ω, $C > 0$ *is any constant, and*

$$K(w,q) := \int_\Omega w(t) V(\Omega_t)^{q-1} dt .$$

In particular, if $w(t) \equiv 1$, *we have*

$$\int_\Omega \prod_\alpha |f^\alpha|^{q_\alpha} \leqslant \frac{1}{q^n}\left(\frac{V}{C}\right)^q \sum_\alpha \frac{q_\alpha}{p_\alpha} C^{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} . \tag{4}$$

To prove Theorem 1 we need the following lemma.

LEMMA 1. *For any* $f \in \mathcal{F}$, $f(t) = \int_{\Omega_t} \dot{f}(u)du$ *for all* $t \in \Omega$.

Proof. Since $f \in \mathcal{F}$, $f(t) = \int_{a_1}^{t_1} f_1(u_1, t_2, \ldots, t_n) du_1$, and so the lemma follows by induction. □

Proof of Theorem 1. By Lemma 1 and Hölder's inequality, we have

$$\begin{aligned} |f^\alpha(t)| &\leqslant \int_{\Omega_t} |\dot{f}^\alpha(u)| du \\ &\leqslant V(\Omega_t)^{(p_\alpha - 1)/p_\alpha} \left[\int_\Omega |\dot{f}^\alpha(u)|^{p_\alpha} du \right]^{1/p_\alpha} \end{aligned}$$

and so

$$\prod_\alpha |f^\alpha(t)|^{q_\alpha} \leqslant V(\Omega_t)^{q-1} \prod_\alpha \left[\int_\Omega |\dot{f}^\alpha(u)|^{p_\alpha} du \right]^{q_\alpha/p_\alpha} .$$

Therefore,

$$\begin{aligned} \int_\Omega w(t) \prod_\alpha |f^\alpha(t)|^{q_\alpha} dt &\leqslant \int_\Omega w(t) V(\Omega_t)^{q-1} dt \cdot \prod_\alpha \left[\int_\Omega |\dot{f}^\alpha(u)|^{p_\alpha} du \right]^{q_\alpha/p_\alpha} \\ &= \frac{1}{C^q} K(w,q) \prod_\alpha \left[C^{p_\alpha} \int_\Omega |\dot{f}^\alpha(u)|^{p_\alpha} du \right]^{q_\alpha/p_\alpha} \\ &\leqslant \frac{1}{C^q} K(w,q) \sum_\alpha \frac{q_\alpha}{p_\alpha} C^{p_\alpha} \int_\Omega |\dot{f}^\alpha(u)|^{p_\alpha} du \end{aligned}$$

by Lemma 0. Finally, if $w(t) \equiv 1$, then

$$\begin{aligned} K(w,q) &= \int_\Omega V(\Omega_t)^{q-1} dt \\ &= \frac{V^q}{q^n} \end{aligned}$$

and so (4) follows immediately from (3). □

COROLLARY 1. *Under the hypotheses of Theorem 1, we have*

$$\int_\Omega \prod_\alpha |f^\alpha|^{q_\alpha} \leqslant \frac{1}{q^n} \cdot \min\left\{ V^q \sum_\alpha \frac{q_\alpha}{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} , \quad \sum_\alpha \frac{q_\alpha}{p_\alpha} V^{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} \right\} . \tag{5}$$

In particular,

$$\int_\Omega \prod_\alpha |f^\alpha| \leqslant \frac{1}{m^n} \cdot \min\left\{ V^m \sum_\alpha \frac{1}{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} , \quad \sum_\alpha \frac{1}{p_\alpha} V^{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} \right\} \tag{6}$$

for all $p_\alpha \geqslant 1$ *with* $\sum_\alpha 1/p_\alpha = 1$.

Proof. (5) is immediate by putting $C = 1$ and $C = V$ respectively in (4), and (6) follows from (5) by letting $q_\alpha = 1$ for all α. □

REMARK 1. Instead of requiring $f^\alpha \in \mathcal{F}$ for all α, a set of seemingly more "natural" conditions on the f^α's would be

$$f^\alpha \Big|_{x_i = a_i} = 0 \quad \text{for all} \quad \alpha, i.$$

Observe that these are stronger than $f^\alpha \in \mathcal{F}$ and so Theorem 1 and hence Corollary 1 are automatically true under such conditions.

For any $t \in \Omega$, let Ω_t^k, $k = 1, \dots, 2^n$, be the sub-regions of Ω determined by the hyperplanes $x_i = t_i$, $i = 1, \dots, n$. Let

$$\Phi(t) := \prod_i [(t_i - a_i)(b_i - t_i)] .$$

THEOREM 2. *For any* $f^\alpha \in \mathcal{G}$ *and any real numbers* $p_\alpha \geqslant 1$, $q_\alpha > 0$ *with* $\sum_\alpha q_\alpha/p_\alpha = 1$, *if we write* $q := \sum_\alpha q_\alpha$, *then*

$$\int_\Omega w \prod_\alpha |f^\alpha|^{q_\alpha} \leqslant \frac{1}{C^q} L(w,q) \sum_\alpha \frac{q_\alpha}{p_\alpha} C^{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} , \tag{7}$$

where $w(t)$ *is any non-negative continuous weight function on* Ω, $C > 0$ *is any constant, and*

$$L(w,q) := \frac{1}{2^n} \int_\Omega w(t) \Phi(t)^{(q-1)/2} dt .$$

In particular, if $w(t) \equiv 1$, *we have*

$$\int_\Omega \prod_\alpha |f^\alpha|^{q_\alpha} \leqslant \left[\frac{1}{2} B\left(\frac{q+1}{2}, \frac{q+1}{2} \right) \right]^n \left(\frac{V}{C} \right)^q \sum_\alpha \frac{q_\alpha}{p_\alpha} C^{p_\alpha} \int_\Omega |\dot{f}^\alpha|^{p_\alpha} , \tag{8}$$

where B *is the Beta function.*

To prove Theorem 2 we need a couple of lemmas.

LEMMA 2. *For any* $t \in \Omega$,

$$\prod_{k=1}^{2^n} V(\Omega_t^k) = \Phi(t)^{2^{n-1}} .$$

Proof. Observe that $\prod_k V(\Omega_t^k)$ is a product of powers of $(t_i - a_i)$ and $(b_i - t_i)$, $i = 1, \dots, n$. Now for each i, the hyperplane $x_i = t_i$ seperates Ω into two components each of which consists of exactly 2^{n-1} sub-regions Ω_t^k's. Note that each Ω_k in one of these two components has a side of length $t_i - a_i$, while none of the Ω_k's in the other component has such. Thus $(t_i - a_i)$ appears exactly 2^{n-1} times in $\prod_k V(\Omega_t^k)$. Similarly, $(b_i - t_i)$ appears exactly 2^{n-1} times in $\prod_k V(\Omega_t^k)$. Hence the lemma. □

LEMMA 3. *For any* $f \in \mathcal{G}$ *and any constant* $p \geqslant 1$,

$$|f(t)|^p \leqslant \frac{1}{2^n}\Phi(t)^{(p-1)/2} \int_\Omega |\dot{f}|^p.$$

Proof. Observe that one of the Ω_t^k's, say Ω_t^1, equals Ω_t. Thus by Lemma 1,

$$f(t) = \int_{\Omega_t^1} \dot{f}(u)du.$$

Since $f \in \mathcal{G}$, by similar arguments we have

$$f(t) = \pm \int_{\Omega_t^k} \dot{f}(u)du , \quad k = 1, \dots, 2^n,$$

and so by Hölder's inequality,

$$|f(t)| \leqslant \int_{\Omega_t^k} |\dot{f}(u)|du \leqslant V(\Omega_t^k)^{(p-1)/p}\Big(\int_{\Omega_t^k} |\dot{f}(u)|^p du\Big)^{1/p} , \quad k = 1, \dots, 2^n.$$

Multiplying these 2^n inequalities together and using Lemma 2, we have

$$\begin{aligned} |f(t)|^{2^n} &\leqslant \Big[\prod_k V(\Omega_t^k)\Big]^{(p-1)/p}\Big[\prod_k \int_{\Omega_t^k} |\dot{f}|^p\Big]^{1/p} \\ &= \Phi(t)^{(2^{n-1})(p-1)/p}\Big[\prod_k \int_{\Omega_t^k} |\dot{f}|^p\Big]^{1/p}, \end{aligned}$$

thus

$$|f(t)| \leqslant \Big\{\Phi(t)^{(p-1)/2}\Big[\prod_k \int_{\Omega_t^k} |\dot{f}|^p\Big]^{1/2^n}\Big\}^{1/p}$$

and so by the arithmetic-geometric mean inequality,

$$\begin{aligned} |f(t)| &\leqslant \Big\{\Phi(t)^{(p-1)/2} \cdot \frac{1}{2^n}\sum_k \int_{\Omega_t^k} |\dot{f}|^p\Big\}^{1/p} \\ &= \Big\{\frac{1}{2^n}\Phi(t)^{(p-1)/2} \int_\Omega |\dot{f}|^p\Big\}^{1/p} . \end{aligned}$$

□

Proof of Theorem 2. By Lemma 3, we have, for each α,

$$|f^{\alpha}(t)|^{q_\alpha} \leqslant \Big[\frac{1}{2^n}\Phi(t)^{(p_\alpha-1)/2}\int_\Omega |\dot f^{\alpha}|^{p_\alpha}\Big]^{q_\alpha/p_\alpha}.$$

Thus

$$\begin{aligned}\prod_\alpha |f^{\alpha}(t)|^{q_\alpha} &\leqslant \prod_\alpha \Big[\frac{1}{2^n}\Phi(t)^{(p_\alpha-1)/2}\Big]^{q_\alpha/p_\alpha}\prod_\alpha\Big[\int_\Omega |\dot f^{\alpha}|^{p_\alpha}\Big]^{q_\alpha/p_\alpha}\\ &= \frac{1}{2^n}\Phi(t)^{(q-1)/2}\prod_\alpha\Big[\int_\Omega |\dot f^{\alpha}|^{p_\alpha}\Big]^{q_\alpha/p_\alpha}\end{aligned}$$

and so

$$\begin{aligned}\int_\Omega w(t)\prod_\alpha |f^{\alpha}(t)|^{q_\alpha}dt &\leqslant \frac{1}{2^n}\int_\Omega w(t)\Phi(t)^{(q-1)/2}dt\cdot\prod_\alpha\Big[\int_\Omega |\dot f^{\alpha}|^{p_\alpha}\Big]^{q_\alpha/p_\alpha}\\ &= L(w,q)\cdot\frac{1}{C^q}\cdot\prod_\alpha\Big[C^{p_\alpha}\int_\Omega |\dot f^{\alpha}|^{p_\alpha}\Big]^{q_\alpha/p_\alpha}\\ &\leqslant \frac{1}{C^q}L(w,q)\sum_\alpha\frac{q_\alpha}{p_\alpha}C^{p_\alpha}\int_\Omega |\dot f^{\alpha}|^{p_\alpha}\end{aligned}$$

by Lemma 0. Finally, if $w(t)\equiv 1$, then

$$\begin{aligned}L(w,q) &= \frac{1}{2^n}\int_\Omega \Phi(t)^{(q-1)/2}dt\\ &= \frac{1}{2^n}\int_\Omega\prod_i[(t_i-a_i)(b_i-t_i)]^{(q-1)/2}dt\\ &= \frac{1}{2^n}\prod_i\Big[(b_i-a_i)^q B\Big(\frac{q+1}{2},\frac{q+1}{2}\Big)\Big]\\ &= \Big[\frac{1}{2}B\Big(\frac{q+1}{2},\frac{q+1}{2}\Big)\Big]^n V^q,\end{aligned}$$

where

$$B(r,s) := \int_0^1 u^{r-1}(1-u)^{s-1}du, \quad r>0, \quad s>0,$$

is the Beta function. Thus we have

$$\int\prod_\alpha |f^{\alpha}|^{q_\alpha} \leqslant \Big[\frac{1}{2}B\Big(\frac{q+1}{2},\frac{q+1}{2}\Big)\Big]^n\Big(\frac{V}{C}\Big)^q\sum_\alpha\frac{q_\alpha}{p_\alpha}C^{p_\alpha}\int_\Omega |\dot f^{\alpha}|^{p_\alpha}.$$

□

COROLLARY 2. *Under the hypotheses of Theorem 2, we have*

$$\int_\Omega\prod_\alpha |f^{\alpha}|^{q_\alpha} \leqslant \Big[\frac{1}{2}B\Big(\frac{q+1}{2},\frac{q+1}{2}\Big)\Big]^n\cdot \min\Big\{V^q\sum_\alpha\frac{q_\alpha}{p_\alpha}\int_\Omega |\dot f^{\alpha}|^{p_\alpha},\ \sum_\alpha\frac{q_\alpha}{p_\alpha}V^{p_\alpha}\int_\Omega |\dot f^{\alpha}|^{p_\alpha}\Big\}. \tag{9}$$

In particular,

$$\int_\Omega \prod_\alpha |f^\alpha| \leqslant \Big[\frac{1}{2}B\Big(\frac{m+1}{2},\frac{m+1}{2}\Big)\Big]^n \cdot$$
$$\min\Big\{V^m \sum_\alpha \frac{1}{p_\alpha}\int_\Omega |\dot f^\alpha|^{p_\alpha}, \sum_\alpha \frac{1}{p_\alpha} V^{p_\alpha}\int_\Omega |\dot f^\alpha|^{p_\alpha}\Big\} \tag{10}$$

for all $p_\alpha \geqslant 1$ *with* $\sum_\alpha 1/p_\alpha = 1$.

Proof. (9) is immediate by putting $C = 1$ and $C = V$ respectively in (8), and (10) follows from (9) by letting $q_\alpha = 1$ for all α. □

REMARK 2. Similar to Remark 1, instead of requiring $f^\alpha \in \mathcal{G}$ for all α, a set of seemingly more "natural" conditions on the f^α's are that they vanish on the boundary of Ω, that is,

$$f^\alpha\Big|_{x_i=a_i} = f^\alpha\Big|_{x_i=b_i} = 0 \quad \text{for all} \quad \alpha\,,\ i\,.$$

Again since these are indeed stronger than $f^\alpha \in \mathcal{G}$, Theorem 2 and hence Corollary 2 are automatically true under such stronger conditions.

REMARK 3. Theorem 2 is a significant improvement of the results of Agarwal and Sheng in [3] in two senses. Firstly, putting $C = 1$ in inequalities (7) and (8) respectively gives

$$\int_\Omega w \prod_\alpha |f^\alpha|^{q_\alpha} \leqslant L(w,q) \sum_\alpha \frac{q_\alpha}{p_\alpha}\int_\Omega |\dot f^\alpha|^{p_\alpha}$$

and

$$\int_\Omega \prod_\alpha |f^\alpha|^{q_\alpha} \leqslant \Big[\frac{1}{2}B\Big(\frac{q+1}{2},\frac{q+1}{2}\Big)\Big]^n V^q \sum_\alpha \frac{q_\alpha}{p_\alpha}\int_\Omega |\dot f^\alpha|^{p_\alpha}\,,$$

which are precisely those obtained in [3]. Secondly, and perhaps more importantly, the basic conditions imposed on the f^α's in [3] are

$$f^\alpha\Big|_{x_i=a_i} = f^\alpha\Big|_{x_i=b_i} = 0\,, \quad \text{for all} \quad \alpha, i\,,$$

which are, as discussed in REMARK 2 above, significantly stronger than our basic assumptions that $f^\alpha \in \mathcal{G}$ for all α.

3. The Case $m = 1$ (the case of 1 function in many variables)

The most frequently encountered situation among all is clearly the case where $m = 1$, that is, the case where there is only one dependent function. Obviously in this case the following are immediate from Corollaries 1 and 2 by setting $m = 1$ and $p = q = k \geqslant 1$:

COROLLARY 3. *For any* $f \in \mathcal{F}$,

$$\int_\Omega |f|^k \leqslant \frac{V^k}{k^n} \int_\Omega |\dot{f}|^k \quad \text{for all real} \quad k \geqslant 1 \, .$$

COROLLARY 4. *For any* $f \in \mathcal{G}$,

$$\int_\Omega |f|^k \leqslant \left[\frac{1}{2} B\Big(\frac{k+1}{2}, \frac{k+1}{2}\Big)\right]^n V^k \int_\Omega |\dot{f}|^k \quad \text{for all real} \quad k \geqslant 1 \, .$$

More importantly, the general results obtained in §2 by no means exhaust their power of generating new Wirtinger-type inequalities involving one dependent function of several variables in these corollaries. For instance, by choosing any integer $N > 1$ and letting $f^1 = \cdots = f^N = f$, we may obtain, for different combinations of p_α's and q_α's, new Wirtinger-type inequalities involving one dependent function of several variables from the general results in §2.

4. Remark

The method used in §2 and §3 above in establishing new multi-dimensional integral inequalities of the Wirtinger-type is, comparing to the usual techniques of considering certain eigenvalue problems, rather elementary and easy to apply. More importantly, this method does not exhaust itself in establishing integral inequalities of such type, in fact, the same techniques can also be used to arrive at other types of integral inequalities in several independent variables. These include the Poincaré-type [10], the Opial-type [12], the Sobolev-type [9], the Gronwall-Wendroff type [11], etc. Although this method is not as sophisticated, in many cases the results obtained are better than those found by using other more complicated techniques with heavy machinery. It is believed that many more important multi-dimensional integral inequalities for our disposal in the study of both qualitative and quantitative properties of solutions of differential and integral equations can be established by using the techniques used here.

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