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GERBER'S INEQUALITY AND SOME RELATED INEQUALITIES

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Abstract. In this paper simple proofs of various extensions of Bernoulli's inequality and then higher order convexity is used to give more precise forms of these generalizations.

1. Introduction

1.1. It is well known that if a real-valued function f has $n+1$ derivatives on $]a,b[$, where we will assume, without loss in generality, that $a<0<b$, then

$$f(x)=T_n(f;0;x)+R_n(f;0;x), \qquad a<x<b, \tag{1}$$

where

$$\begin{aligned} T_n(f;0;x)&=T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k,\\ R_n(f;0;x)&=R_n(x)=\frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,\mathrm{d}t; \end{aligned} \tag{2}$$

see for instance [9, 10].

Further, as remarked in [9, 10] we can write $R_n(x)$ as

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},$$

for some c strictly between 0 and x. Let us call any such c *a mean value point of order* $n+1$ *for* f *on the interval* $[0,x]$, or $[x,0]$ if $x<0$.

1.2. Incidentally the integral form of $R_n(x)$ in (2) gives a very quick and little remarked way of getting the single integral expression for the $(n+1)$-th primitive of F; that is the function Φ where $\Phi(0)=0$, $\Phi^{(k)}(0)=0$, $1\leqslant k\leqslant n$ and $\Phi^{(n+1)}=F$. From (1) and (2)

$$\Phi(x)=\frac{1}{n!}\int_0^x (x-t)^n F(t)\,\mathrm{d}t.$$

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1.3. If $k \geqslant 1$ and $f^{(k)} \geqslant 0$ on $]a,b[$, with $f^{(k)} > 0$ at all but a finite number of points then f is said to be *(strictly) k-convex on* $]a,b[$; if $-f$ is k-convex on $]a,b[$ than f is said to be *(strictly) k-concave on* $]a,b[$. In most standard references the class of k-convex functions is defined more generally and then our class is a sub-class of the strictly k-convex functions; however for the present applications to inequalities this definition will suffice; see [3 & 8; p. 237].

The following lemma collects some properties of k-convex functions that we will need.

LEMMA 1. *If $k \geqslant 1$ and f is k-convex on $]a,b[$, $a < 0 < b$ then:*
(a) *$f^{(k-1)}$ is strictly increasing;*
(b) *$R_{k-1}(f;0,x) > 0$, $a < x < b$;*
(c) *if ℓ is the polynomial of degree at most $k-1$ with $\ell(0) = f(0)$, $\ell^{(j)}(0) = f^{(j)}(0)$, $1 \leqslant j \leqslant k-2$, and $\ell(\beta) = f(\beta)$, $0 < \beta < b$ then $f(x) < \ell(x)$, $0 < x < \beta$.*

Proof. (a) This is an immediate consequence of the mean value theorem of differential calculus.
(b) Immediate from (2).
(c) See [3]. □

2. Gerber's Inequalities

THEOREM 2. *Suppose f has $n+1$ derivatives on $]a,b[$, $a < 0 < b$ and let*

$$\phi(x) = f(x) - T_n(x), \qquad a < x < b,$$

then,
(i) *if $f^{(n+1)} = 0$ then $\phi = 0$;*
(ii) *if f is $(n+1)$-convex then*

$$\phi(x) \begin{cases} > 0 & \text{if } x > 0, \text{ or if } x < 0 \text{ and } n \text{ is odd}, \\ < 0 & \text{if } x < 0 \text{ and } n \text{ is even}; \end{cases} \tag{3}$$

(iii) *if f is $(n+1)$-concave then (~ 3) holds.*

By (~ 3) we mean (3) with the inequality sign reversed; we will use this convention throughout.

Proof. This is an immediate consequence of Lemma 1(b). □

This result has also been proved in [1].

The above can be rewritten as

COROLLARY 3. *With f, ϕ, a, b as in Theorem 2*
(a) *if $x^{n+1}f^{(n+1)}(x) \geqslant 0$, $a < x < b$, being positive except on a finite set of points, then $f(x) > T_n(x)$;*
(b) *if $x^{n+1}f^{(n+1)}(x) = 0$, $a < x < b$, then $f(x) = T_n(x)$;*
(c) *if $x^{n+1}f^{(n+1)}(x) \leqslant 0$, $a < x < b$, being negative except on a finite set of points, then $f(x) < T_n(x)$.*

LEMMA 4. *If* $\alpha \in \mathbb{R}$, $x > -1$ *and* $a(x) = (1+x)^\alpha$ *then*

$$\operatorname{sgn} a^{(k)}(x) = \operatorname{sgn}\binom{\alpha}{k} = \begin{cases} 0 & \text{if } \alpha = j,\ 0 \leqslant j \leqslant k-1, \\ 1 & \text{if } \alpha > k-1, \\ (-1)^k & \text{if } \alpha < 0, \\ (-1)^{k-j-1} & \text{if } j < \alpha < j+1,\ 0 \leqslant j \leqslant k-1. \end{cases}$$

In particular a *is* k*-convex if* $\alpha > k-1$, $\alpha < 0$ *and* k *is even, or if* α *is positive and not an integer and* $j - [\alpha]$ *is odd.*

Proof. Trivial. □

In [4] Gerber proved the following extension of Bernoulli's inequality:

COROLLARY 5. *If* $a \in \mathbb{R}$, $n \in \mathbb{N}$, $x > -1$ *then*

$$\operatorname{sgn}\left((1+x)^\alpha - \sum_{i=0}^{n}\binom{\alpha}{i}x^i\right) = \operatorname{sgn}\left(\binom{\alpha}{n+1}x^{n+1}\right). \tag{4}$$

Proof. Take $f(x) = (1+x)^\alpha$, $a = -1$, $b = \infty$ in Corollary 3, and use Lemma 4. □

Bernoulli's inequality is the case $n = 1$ of (4):

$$(1+x)^\alpha \begin{cases} \geqslant 1 + \alpha x & \text{if } \alpha < 0 \text{ or } \alpha > 1, \\ \leqslant 1 + \alpha x & \text{if } 0 < \alpha < 1, \end{cases} \tag{B}$$

with equality only if $x = 0$; see [4, p. 5 & 5, p. 34].

3. An Inequality of Mitrinović and Pečarić

Now we turn to the interesting note of Mitrinović and Pečarić [7], where the results of Gerber were extended.

THEOREM 6. *If* f, a, b *are as in Theorem* 2 *and if* $a < B < b$, $B \neq 0$ *let*

$$\psi(x) = f(x) - \left[T_{n-1}(x) + \frac{f^{(n)}(B)}{n!}x^n\right],$$

then,

(i) *if* $f^{(n+1)} = 0$ *then* $\psi = 0$

(ii) *if* f *is* $(n+1)$*-convex then*

(a) *if* $B > 0$ *then*

$$\psi(x) \begin{cases} < 0 & \text{if } 0 < x < B, \text{ or if } x < 0 \text{ and } n \text{ is even}, \\ > 0 & \text{if } x < 0 \text{ and } n \text{ is odd}. \end{cases} \tag{5}$$

(*b*) *if* $B < 0$ *then*

$$\psi(x)\begin{cases} > 0 & \text{if } x > 0, \text{ or if } B < x < 0 \text{ and } n \text{ is even}, \\ < 0 & \text{if } B < x < 0 \text{ and } n \text{ is odd}. \end{cases} \tag{6}$$

(*iii*) *If* f *is* $(n+1)$*-concave then* (~ 5) *and* (~ 6) *hold.*

Proof. The reason for $B \neq 0$ is that otherwise $\psi = \phi$, because,

$$\psi(x) = \phi(x) + \frac{f^{(n)}(0) - f^{(n)}(B)}{n!}. \tag{7}$$

Further if $f^{(n+1)} = 0$ then (7) implies that $\psi = \phi$ and so (i) follows from Theorem 1.

Easy calculations give:

$$\psi^{(j)}(x) = f^{(j)}(x) - \left[T_{n-1}^{(j)}(x) + f^{(n)}(B)\frac{n(n-1)\cdots(n-j+1)}{n!}x^{n-j}\right],$$

$$T_{n-1}^{(j)}(x) = \sum_{r=j}^{n-1} f^{(r)}(0)\frac{r(r-1)\cdots(r-j+1)}{r!}x^{r-j}, \quad 1 \leqslant j \leqslant n-1;$$

and

$$\begin{aligned} \psi^{(n)}(x) &= f^{(n)}(x) - f^{(n)}(B); \\ \psi^{(n+1)}(x) &= f^{(n+1)}(x). \end{aligned} \tag{8}$$

Since $\psi^{(j)}(0) = 0$, $1 \leqslant j \leqslant n-1$ it follows from (1), (2) and (8) that

$$\begin{aligned} \psi(x) = R_{n-1}(\psi; 0; x) &= \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\psi^{(n)}(t)\,\mathrm{d}t \\ &= \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\left[f^{(n)}(t) - f^{(n)}(B)\right]\mathrm{d}t. \end{aligned} \tag{9}$$

If f is $(n+1)$-convex then $f^{(n)}$ is strictly increasing, Lemma 1 (a), and so we easily see from (8) that

$$\psi^{(n)}(x)\begin{cases} < 0 & \text{if } x < B, \\ = 0 & \text{if } x = B, \\ > 0 & \text{if } x > B. \end{cases} \tag{10}$$

(*ii*) now follows from (9) and (10).
A similar discussion will give (*iii*). □

The main part of this result can be summarized as follows:

COROLLARY 7. *With* f, ψ, a, b, B *as in Theorem* 6,

(*a*) *if* $B > 0$, $a < B < b$ *then*

(*i*) *if* $x^n f^{(n+1)}(x) \geqslant 0$, $a < x < B$, *and is positive except for at most a finite number of points, then* $\psi(x) < 0$, $a < x < B$;

(*ii*) *if* $x^n f^{(n+1)}(x) = 0$, $a < x < B$, *then* $\psi(x) = 0$, $a < x < B$;

(iii) *if* $x^n f^{(n+1)}(x) \leqslant 0$, $a < x < B$, *and is negative except at at most a finite number of points, then* $\psi(x) > 0$, $a < x < B$;

or more succinctly,

if $f^{(n+1)}$ *has one sign on* $]a, B[$ *then* $x^n f^{(n+1)}(x)\psi(x) \leqslant 0$, $a < x < B$.

(b) *if* $B < 0$, $a < B < b$ *then*

(i) *if* $x^n f^{(n+1)}(x) \geqslant 0$, $B < x < b$, *and is positive except for at most a finite number of points, then* $\psi(x) > 0$, $B < x < b$;

(ii) *if* $x^n f^{(n+1)}(x) = 0$, $B < x < b$, *then* $\psi(x) = 0$, $B < x < b$;

(iii) *if* $x^n f^{(n+1)}(x) \leqslant 0$, $B < x < b$, *and is negative except for at most a finite number of points, then* $\psi(x) < 0$, $B < x < b$;

or more succinctly,

if $f^{(n+1)}$ *has one sign on* $]B, b[$ *then* $x^n f^{(n+1)}(x)\psi(x) \geqslant 0$, $B < x < b$.

COROLLARY 8. *Hypotheses:* $\alpha \in \mathbb{R}$, $B > -1$, $n \in \mathbb{N}$.

Conclusions:

(a) *if* $B \geqslant 0$ *and* $-1 < x < B$ *then*

$$\operatorname{sgn}\left[(1+x)^\alpha - \sum_{i=0}^{n-1}\binom{\alpha}{i}x^i - \binom{\alpha}{n}(1+B)^{\alpha-n}x^n\right] = -\operatorname{sgn}\left[\binom{\alpha}{n+1}x^n\right]; \quad (11)$$

(b) *if* $-1 < B \leqslant 0$ *and* $x > B$ *then*

$$\operatorname{sgn}\left[(1+x)^\alpha - \sum_{i=0}^{n-1}\binom{\alpha}{i}x^i - \binom{\alpha}{n}(1+B)^{\alpha-n}x^n\right] = \operatorname{sgn}\left[\binom{\alpha}{n+1}x^n\right]. \quad (12)$$

Proof. The result is trivial if $n = 0$, when the sum on the left-hand side is empty. The case $B = 0$ is just Corollary 5.

The rest is an application of Corollary 7, using Lemma 4, to the case $a(x) = (1+x)^\alpha$, $a = -1$, $b = \infty$ using Lemma 4. □

The result of Mitrinović & Pečarić [7], is the case $n = 2$ of this last result.

4. Use of Higher Order Convexity

We now use some properties of higher order convexity to make Corollary 8 a little more precise. First suppose that $B > 0$. The above results are deduced from the fact that an integrand in (9) being positive on $[0, B[$ the integral will also be positive; but it is then clear that the integral will be positive on some larger interval, $[0, \beta[$, $\beta > B$, say. Using higher order convexity a value for β will be determined.

If f is $(n+1)$-convex then $f^{(n)}$ is strictly increasing, Lemma 1(a), and so $\psi^{(n)}(0) = f^{(n)}(0) - f^{(n)}(B) < 0$ and this is the first non-zero derivative of ψ at the origin. Hence, by a well known application of Taylor's theorem

(i) if n is even ψ has a local maximum at the origin;

(ii) if n is odd, $n \neq 1$, ψ has a point of inflection of the $-x^3$-type at the origin; (the case of $n = 1$ needs a change in terminology but the deductions will be the same).

If for some $\beta > B$ we have that B is a mean value point for f of order n on $[0,\beta]$ then $\psi(\beta) = 0$; in particular this will occur if $\psi(b) > 0$, or if $\lim_{x\to b}\psi(x) > 0$. Now by Lemma 1(c) the graph of f must lie below that of its Lagrange interpolation polynomial ℓ;

$$\ell(0) = \ell^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant n-1, \qquad \ell(\beta) = 0;$$

that is the function $\ell(x) = 0$; see [3].

Hence β, if it exists, is unique; and either $\psi(x) < 0$, $x > 0$ — the case when there is no β; or $\psi(x) < 0$, $0 < x < \beta$ and $\psi(x) > 0$, $x > \beta$. Let us agree to put $\beta = b$ if no β as defined above exists.

A similar discussion occurs in the case f is $(n+1)$-concave.

Now consider the the case of $B < 0$ and f $(n+1)$-convex; then $\psi^{(n)}(0) > 0$ and so

(i) if n is even ψ has a local minimum at the origin ;
(ii) if n is odd ψ has a point of inflection of the x^3-type.

It is possible that $\psi(\beta) = 0$ for some β, $a < \beta < B$. If so then this β is, as before, unique; again if no such point exists define $\beta = a$

So if n is even: either $\psi(x) > 0$, $x < 0$ — the case when there is no β; or $\psi(x) < 0$, $a < x < \beta$ and $\psi(x) > 0$, $\beta < x < 0$.

While if n is odd: either $\psi(x) < 0$, $x < 0$ — the case when there is no β; or $\psi(x) > 0$, $a < x < \beta$ and $\psi(x) < 0$, $\beta < x < 0$.

Using similar arguments the opposite inequalities are obtained in the cases where f is $(n+1)$-concave.

So Corollary 7(b) has beeen extended to:

COROLLARY 9. *With the above notations*

(a) *if* $0 < B < b$, *and is the mean value of order* n *of* f *on* $[0,\beta]$ *for some* β, $B < \beta < b$, *or if not* $\beta = b$ *then* $x^n\psi(x)f^{(n+1)}(x) \leqslant 0$, $a < x < \beta$*;*

(b) *if* $a < B < 0$ *then and is the mean value of order* n *of* f *on* $[\beta,0]$ *for some* β, $a < \beta < 0$, *or if not* $\beta = a$ *then* $x^n\psi(x)f^{(n+1)}(x) \geqslant 0$, $\beta < x < b$*;* $x^n\psi(x)f^{(n+1)}(x) \geqslant 0$, $B < x < b$.

4.1. An Application.

As an application consider the case $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. It is easy to check that if $0 < x < \pi/2$ then for all n, $f^{(n)}(x) > 0$ and so by Theorem 1 (ii) we have for all n that

$$\tan x > T_n(\tan;0;x) \qquad 0 < x < \pi/2.$$

If then $0 < B < \pi/2$ consider the function ψ in this case. Noting that $\lim_{x\to\pi/2}\psi(x) = \infty$, we see that there is a unique β, $B < \beta < \pi/2$ at which $\psi(\beta) = 0$; that is for which B is the mean value point of order n for f on the interval $[0,\beta]$. Hence by the discussion of the previous section

$$f(x) < T_{n-1}(\tan;0;x) + \frac{f^{(n)}(B)}{n!}x^n, \qquad 0 < x < \beta.$$

The definition of β, $\psi(\beta) = 0$, is equivalent to B being the mean value point of order n of tan on $[0, \beta]$, or

$$\frac{f(\beta) - T_{n-1}(\tan;0;\beta)}{\beta^n} = \frac{f^{(n)}(B)}{n!}.$$

So the above inequality is

$$f(x) < T_{n-1}(\tan;0;x) + \left[f(\beta) - T_{n-1}(\tan;0;\beta)\right]\left(\frac{x}{\beta}\right)^n, \qquad 0 < x < \beta.$$

The case $n = 3$ of this is just 3.4.27 in [6, p. 245].

5. An Extension of the Geometric-Arithmetic Mean Inequality

5.1. It is well known that a simple change of variable in (B) will give the inequality between the arithmetic and geometric means; see [4, p. 6, (3)]

Put $1 + x = u/v$ with both $u, v > 0$ when (B) becomes after multiplying by v

$$u^\alpha v^{1-\alpha} \begin{cases} \geqslant \alpha u + \overline{1-\alpha}\, v & \text{if } \alpha < 0, \text{ or } \alpha > 1, \\ \leqslant \alpha u + \overline{1-\alpha}\, v & \text{if } 0 < \alpha < 1, \end{cases} \tag{GA}$$

with equality only if $u = v$. If $u, v > 0$ then

$$A(u, v : \alpha) = \alpha u + \overline{1-\alpha}\, v, \qquad G(u, v; \alpha) = u^\alpha v^{1-\alpha}$$

are, respectively the *arithmetic and geometric means of* u, v *with weights* α, $1 - \alpha$. It is usual to require that $0 \leqslant \alpha \leqslant 1$ in the classical inequality (GA).

5.2. It is natural then to expect the generalizations of (B) in Corollary 7 should yield generalizations of (GA).

THEOREM 10. *If* $\alpha \in \mathbb{R}$, $n \geqslant 2$, $C \geqslant 1$, *and* $0 < u < Cu$ *then*

$$\operatorname{sgn}\left[v^{n-1}\big(G(u,v;\alpha) - A(u,v;\alpha)\big) - \sum_{i=2}^{n-1}\binom{\alpha}{i}(u-v)^i v^{n-i} - \binom{\alpha}{n}C^{\alpha-n}(u-v)^n\right] = -\operatorname{sgn}\left[\binom{\alpha}{n+1}(u-v)^n\right], \tag{13}$$

and

$$\operatorname{sgn}\left[u^{n-1}\big(G(u,v;\alpha) - A(u,v;\alpha)\big) - \sum_{i=2}^{n-1}\binom{1-\alpha}{i}(v-u)^i u^{n-i} - \binom{1-\alpha}{n}C^{n+\alpha-1}(v-u)^n\right] = \operatorname{sgn}\left[\binom{1-\alpha}{n+1}(v-u)^n\right]. \tag{14}$$

(If $n = 2$ the sum on the left-hand sides of (13) *and* (14) *is taken to be zero.)*

Proof. In Corollary 7 (a) put $1+x=u/v$, $u,v>0$ and $1+B=C$ when that result says: if $C \geqslant 1$ and $0<u<Cv$ then

$$\operatorname{sgn}\Big[\Big(\frac{u}{v}\Big)^{\alpha}-1-\alpha\Big(\frac{u}{v}-1\Big)-\sum_{i=2}^{n-1}\binom{\alpha}{i}\Big(\frac{u}{v}-1\Big)^{i}-\binom{\alpha}{n}C^{\alpha-n}\Big(\frac{u}{v}-1\Big)^{n}\Big] = -\operatorname{sgn}\Big[\binom{\alpha}{n+1}\Big(\frac{u}{v}-1\Big)^{n}\Big]$$

or

$$\operatorname{sgn}\Big[(G(u,v;\alpha)-A(u,v;\alpha)-\sum_{i=2}^{n-1}\binom{\alpha}{i}\frac{(u-v)^{i}}{v^{i-1}}-\binom{\alpha}{n}C^{\alpha-n}\frac{(u-v)^{n}}{v^{n-1}}\Big] = -\operatorname{sgn}\Big[\binom{\alpha}{n+1}\frac{(u-v)^{n}}{v^{n-1}}\Big],$$

which gives (13).

Now similar changes in Corollary 7 (b) give: if $0<C\leqslant 1$ and $0<Cv<u$ then

$$\operatorname{sgn}\Big[\Big(\frac{u}{v}\Big)^{\alpha}-1-\alpha\Big(\frac{u}{v}-1\Big)-\sum_{i=2}^{n-1}\binom{\alpha}{i}\Big(\frac{u}{v}-1\Big)^{i}-\binom{\alpha}{n}C^{\alpha-n}\Big(\frac{u}{v}-1\Big)^{n}\Big] = \operatorname{sgn}\Big[\binom{\alpha}{n+1}\Big(\frac{u}{v}-1\Big)^{n}\Big]. \quad (15)$$

In this last expression substitute $u=1/s$, $v=1/t$, $C=1/D$ and $\alpha=1-\beta$ to get

$$\operatorname{sgn}\Big[\Big(\frac{t}{t}\Big)^{1-\beta}-1-(1-\beta)\Big(\frac{t}{s}-1\Big)-\sum_{i=2}^{n-1}\binom{1-\beta}{i}\Big(\frac{t}{s}-1\Big)^{i} - \binom{1-\beta}{n}D^{n+\beta-1}\Big(\frac{t}{s}-1\Big)^{n}\Big] = -\operatorname{sgn}\Big[\binom{1-\beta}{n+1}\Big(\frac{t}{s}-1\Big)^{n}\Big]$$

or

$$\operatorname{sgn}\Big[s^{n-1}\big((G(u,v;\beta)-A(u,v;\beta)\big)-\sum_{i=2}^{n-1}\binom{1-\beta}{i}(t-s)^{i}s^{n-1} - \binom{1-\beta}{n}D^{n+\beta-1}(t-s)^{n}\Big] = \operatorname{sgn}\Big[\binom{1-\beta}{n+1}(t-s)^{n}\Big].$$

Noting that now $D \geqslant 1$ and $0<s<Dt$ this last expression gives (14). □

A similar pair of results can be obtained by leaving formula (15) alone and applying the s,t,β,D changes in (13).

COROLLARY 11. *If* $0<\alpha<1$ *or if* $\alpha>2$ *and if* $C\geqslant 1$, $0<u<Cv$ *then*

$$\frac{\alpha(1-\alpha)}{2}C^{\alpha-2}\frac{(u-v)^2}{v} < A(u,v;\alpha)-G(u,v;\alpha) < \frac{\alpha(1-\alpha)}{2}C^{\alpha+1}\frac{(u-v)^2}{v},$$

while if $\alpha<0$ *or if* $1<\alpha<2$ *the opposite inequality holds.*

Proof. This is just the case $n=2$ of Theorem 10. □

These results are extensions of some in [1].

5.3. We now use the above results to obtain an inequality for matrices.

THEOREM 13. *Let A, B be two positive definite Hermitian matrices with $A \geqslant cB$ where $0 < c \leqslant 1$, and if $n \geqslant 2$, $\alpha \in \mathbb{R}$ then*

$$\binom{\alpha}{n+1}\left(B^{-1/2}(A-B)B^{-1/2}\right)^n\Big\{B^{1/2}(B^{-1/2}AB^{-1/2})^{\alpha}B^{1/2} - (\alpha A + \overline{1-\alpha}B) - \sum_{i=2}^{n-1}\binom{\alpha}{i}B^{1/2}\left(B^{-1/2}(A-B)B^{-1/2}\right)^i B^{1/2} - \binom{\alpha}{n}c^{\alpha-n}B^{1/2}\left(B^{-1/2}(A-B)B^{-1/2}\right)^n B^{1/2}\Big\} \geqslant 0.$$

Proof. Let M be a positive definite Hermitian matrix with $M \geqslant cI$, $0 < c \leqslant 1$. Then $M = \Gamma B \Gamma^*$ where Γ is unitary, and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, λ_i being a characteristic root of M, $1 \leqslant i \leqslant n$.

Applying (15) with $u/v = \lambda_i$, $i = 1, 2, \ldots, n$ leads to

$$\binom{\alpha}{n+1}(D-I)^n\left\{D^{\alpha} - \sum_{i=0}^{n-1}\binom{\alpha}{i}(D-I)^i - \binom{\alpha}{n}c^{\alpha-n}(D-I)^n\right\} \geqslant 0.$$

So pre-, and post-multiplying by Γ, Γ^* gives

$$\binom{\alpha}{n+1}(M-I)^n\left\{M^{\alpha} - \sum_{i=0}^{n-1}\binom{\alpha}{i}(M-I)^i - \binom{\alpha}{n}c^{\alpha-n}(M-I)^n\right\} \geqslant 0.$$

Now let $M = B^{-1/2}AB^{-1/2}$

$$\binom{\alpha}{n+1}(B^{-1/2}AB^{-1/2}-I)^n\Big\{(B^{-1/2}AB^{-1/2})^{\alpha} - \sum_{i=0}^{n-1}\binom{\alpha}{i}(B^{-1/2}AB^{-1/2}-I)^i - \binom{\alpha}{n}c^{\alpha-n}(B^{-1/2}AB^{-1/2}-I)^n\Big\} \geqslant 0$$

or

$$\binom{\alpha}{n+1}(B^{-1/2}(A-B)B^{-1/2})^n\Big\{(B^{-1/2}AB^{-1/2})^{\alpha} - (I + \alpha B^{-1/2}(A-B)B^{-1/2}) - \sum_{i=2}^{n-1}\binom{\alpha}{i}(B^{-1/2}AB^{-1/2}-I)^i - \binom{\alpha}{n}c^{\alpha-n}(B^{-1/2}AB^{-1/2}-I)^n\Big\} \geqslant 0.$$

This is just the required inequality. □

A reverse inequality can also be obtained if the hypotheses of Theorem 13 are replaced by $c \geqslant 1$ and $A \leqslant cB$.

COROLLARY 14. *If* A, B *are positive definite Hermitian matrices such that* $A \geqslant cB$, $0 < c \leqslant 1$ *then if* $0 < \alpha < 1$, *or* $\alpha < 2$ *then*

$$\alpha A + \overline{1-\alpha} B - B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)B^{1/2} \leqslant \frac{\alpha(1-\alpha)}{2}c^{\alpha-2}B^{1/2}\left(B^{-1/2}(A-B)B^{-1/2}\right)^2 B^{1/2}. \quad (16)$$

If $\alpha < 0$ *or* $1 < \alpha < 2$ *then* (~ 16) *holds.*

Proof. This is just the case $n = 2$ of Theorem 13. □

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