AN APPLICATION OF THE HAUSDORFF–YOUNG INEQUALITY

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Abstract. In this paper a generalization theorem of Zahid of a [8] has been obtained, by considering the condition $S_p(\delta)$, $p > 1$ instead of $S(\delta)$.

1. Introduction

A sequence $\{a_k\}$ of positive numbers is said to be quasi-monotone if $k^{-\beta}a_k \downarrow 0$ for some $\beta$, or equivalently if $\Delta a_k \geq -\beta \frac{a_k}{k}$.

A sequence $\{a_k\}$ is said to be $\delta$–quasi-monotone if $a_k \to 0$, $a_k > 0$ ultimately and $\Delta a_k \geq -\delta_k$, where $\{\delta_k\}$ is a sequence of positive numbers.

A sequence $\{a_k\}$ is said to satisfy condition $S'$ if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is quasi-monotone, $\sum_{k=1}^{\infty} A_k < \infty, |\Delta a_k| \leq A_k$, for all $k$.

On the other hand, a sequence $\{a_k\}$ is said to satisfy condition $S(\delta)$, if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is $\delta$–quasi-monotone, $\sum_{k=1}^{\infty} k\delta_k < \infty, \sum_{k=1}^{\infty} A_k < \infty$, and $|\Delta a_k| \leq A_k$, for all $k$.

Now, we say that a sequence $\{a_k\}$ of numbers satisfies conditions $S_p(\delta)$ or $a_k \in S_p(\delta)$, if $a_k \to 0$ as $k \to \infty$ and there exists a sequence of numbers $\{A_k\}$ such that:

(a) $\{A_k\}$ is $\delta$–quasi-monotone and $\sum_{k=1}^{\infty} k\delta_k < \infty$,

(b) $\sum_{k=1}^{\infty} A_k < \infty$,

(c) $\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$.

Thus, in view of the above definitions it is obvious that $S' \subset S(\delta) \subset S_p(\delta)$.


Key words and phrases: $\delta$–quasi-monotone sequence, cosine trigonometric series, Fourier series, Dirichlet kernel, Abel’s transformation, Holder inequality, Hausdorff–Young inequality.
2. Preliminaries

Let \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \) be the cosine trigonometric series.

Quite recently, S. Zahid Ali Zenei [8] proved the following theorem.

**THEOREM A.** [8] Let the coefficients of the series \( f(x) \) satisfy the condition \( S(\delta) \). Then the series is a Fourier series and the following relation holds:

\[
\pi \int_{0}^{\pi} |f(x)| \, dx \leq C \sum_{n=0}^{\infty} A_n,
\]

where \( C \) is an absolute constant.

**LEMMA 1.** [3] (Hausdorff–Young). Let the sequence of complex numbers \( \{c_n\} \in l^p \). Then \( \{c_n\} \) is the sequence of Fourier coefficients of some \( \varphi \in L^q \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \), and

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |\varphi(x)|^q \, dx \right)^{\frac{1}{q}} \leq \left( \sum_{n=-\infty}^{\infty} |c_n|^p \right)^{\frac{1}{p}}.
\]

**LEMMA 2.** ([8] case \( \nu = 1 \) ) If \( \{a_n\} \) is a \( \delta \)--quasi-monotone sequence with \( \sum_{n=1}^{\infty} n\delta_n < \infty \), then the convergence of \( \sum_{n=1}^{\infty} a_n \) implies that \( na_n = o(1), \, n \to \infty \).

**LEMMA 3.** [8] Let \( \{a_n\} \) be a \( \delta \)--quasi-monotone sequence with \( \sum_{n=1}^{\infty} n\delta_n < \infty \). If

\( \sum_{n=1}^{\infty} a_n < \infty \), then \( \sum_{n=1}^{\infty} (n+1)|\Delta a_n| < \infty \).

3. Main results

**THEOREM.** Let the coefficients of the series \( f(x) \) satisfy the condition \( S_p(\delta) \). Then the series is a Fourier series and the following relation holds:

\[
\pi \int_{0}^{\pi} |f(x)| \, dx \leq C \sum_{n=0}^{\infty} A_n,
\]

where \( C \) is an absolute constant.

**Proof.** By virtue of hypothesis \( \Delta A_n \geq -\delta_n \), we have

\[
|\Delta A_n| \leq \Delta A_n + 2\delta_n.
\]

We suppose that

\( A_0 = \max(a_1, \delta_1, \delta_1 + 2\delta_2, \delta_1 + 2\delta_2 + 3\delta_3, \ldots, \delta_1 + 2\delta_2 + \ldots + n_0\delta_{n_0}) \), \( n \geq n_0 \).
We see that \( \sum_{k=1}^{n} k\delta_k \leq A_0 \leq \sum_{k=0}^{n} A_k, \ n \in \mathbb{N} \).

By summation by parts, we have:

\[
\sum_{k=1}^{n} |\Delta a_k| = \sum_{k=1}^{n} A_k \frac{|\Delta a_k|}{A_k}
\]

\[
= \sum_{k=1}^{n-1} |\Delta A_k| \cdot \sum_{j=1}^{k} |\Delta a_j| \cdot A_n \cdot \sum_{j=1}^{n} |\Delta a_j| \cdot A_j
\]

\[
\leq \sum_{k=1}^{n-1} k|\Delta A_k| \left( \frac{1}{k} \sum_{j=1}^{k} |\Delta a_j|^p \right)^{\frac{1}{p}} + nA_n \left( \frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right)^{\frac{1}{p}}
\]

\[
= O(1) \left[ \sum_{k=1}^{n-1} k|\Delta A_k| + nA_n \right]
\]

\[
\leq O(1) \left[ \sum_{k=1}^{n-1} k(|\Delta A_k| + 2\delta_k + nA_n) \right]
\]

\[
= O(1) \left[ \sum_{k=1}^{n-1} k|\Delta A_k| + 2 \sum_{k=1}^{n-1} k\delta_k + nA_n \right]
\]

\[
= O(1) \left( \sum_{k=1}^{n} A_k - nA_n + 2 \sum_{k=1}^{n-1} k\delta_k + nA_n \right)
\]

\[
= O(1) \left( \sum_{k=1}^{n} A_k + 2 \sum_{k=1}^{n-1} k\delta_k \right).
\]

Letting \( n \to \infty \) we get \( \sum_{n=1}^{\infty} |\Delta a_n| < \infty \).

Thus \( \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \) converges to \( f(x) \) for all \( x \) except possibly \( x = 0 \). By summation by parts, we have:

\[
f(x) = \lim_{n \to \infty} \left[ \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{a_0}{2} + \sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_nD_n(x) - \frac{a_0}{2} \right]
\]

\[
= \lim_{n \to \infty} \left[ \sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_nD_n(x) \right]
\]

\[
= \sum_{k=1}^{\infty} \Delta a_k D_k(x),
\]
by the fact that \( \lim_{n \to \infty} a_n D_n(x) = 0 \), if \( x \neq 0 \) where \( D_n(x) \) is the Dirichlet kernel. Now applications of Abel’s transformation yield,

\[
\int_{0}^{\pi} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, dx = \int_{0}^{\pi} \sum_{k=0}^{\infty} \Delta a_k D_k(x) \, dx
\]

\[
= \int_{0}^{\pi} \sum_{k=0}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \, dx
\]

\[
\leq \sum_{k=0}^{\infty} |\Delta A_k| \int_{0}^{\pi} \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \, dx + A_n \int_{0}^{\pi} \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \, dx.
\]

Then

\[
\int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| \, dx = \int_{0}^{\frac{\pi}{k}} + \int_{\frac{\pi}{k}}^{\pi} = I_k + J_k.
\]

Recalling the uniform estimate of the Dirichlet kernel we have:

\[
I_k \leq A \sum_{j=0}^{k} \frac{|\Delta a_j|}{A_j} \leq A k \left( \frac{1}{k} \sum_{j=0}^{k} \frac{|\Delta a_k|^p}{A_j^p} \right)^{\frac{1}{p}},
\]

where \( A \) is an absolute constant.

Let us estimate the second integral:

\[
J_k = \int_{\frac{\pi}{k}}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| \, dx = \int_{\frac{\pi}{k}}^{\pi} \frac{1}{\sin \frac{x}{2}} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} \sin \left( j + \frac{1}{2} \right) x \right| \, dx.
\]

We shall first apply the Holder inequality, where \( \frac{1}{p} + \frac{1}{q} = 1 \),

\[
J_k \leq \left[ \int_{\frac{\pi}{k}}^{\pi} \left( \frac{1}{\sin \frac{x}{2}} \right)^p \, dx \right]^\frac{1}{p} \left[ \int_{0}^{\pi} \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} \sin \left( j + \frac{1}{2} \right) x \right| \, dx \right]^\frac{1}{q}.
\]

Since \( \int_{\frac{\pi}{k}}^{\pi} \frac{dx}{\left( \sin \frac{x}{2} \right)^p} \leq \frac{\pi}{k} \int_{\frac{\pi}{k}}^{\pi} \frac{dx}{x^p} \leq \frac{\pi}{p-1} k^{p-1} \), it follows that

\[
J_k \leq \left( \frac{\pi}{p-1} \right)^\frac{1}{q} k^{p-1} \left[ \int_{0}^{\pi} \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} \sin \left( j + \frac{1}{2} \right) x \right| \, dx \right]^\frac{1}{q}.
\]
Then using the Hausdorff-Young inequality we get:

\[
\left[ \int_0^\pi \left| \sum_{j=0}^k \frac{\Delta a_j}{A_j} \sin \left( j + \frac{1}{2} \right) x \right|^q dx \right]^\frac{1}{q} \lesssim \left[ \int_0^\pi \left| \sum_{j=0}^k \frac{\Delta a_j e^{ijx}}{A_j} \right|^q dx \right]^\frac{1}{q} \lesssim \left( \sum_{j=0}^k \frac{|\Delta a_j|^p}{A_j^p} \right)^\frac{1}{p}.
\]

Finally, \( J_k \leq Bk \left( \frac{1}{k} \sum_{j=0}^k \frac{|\Delta a_j|^p}{A_j^p} \right)^\frac{1}{p} \), where \( B \) is an absolute constant.

Thus,

\[
\int_0^\pi \left| \sum_{j=0}^k \frac{\Delta a_j}{A_j} D_j(x) \right| dx = O(k + 1).
\]

Then,

\[
\int_0^\pi \left| \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx \right| dx \leq M \left[ \sum_{k=0}^\infty (k + 1) |\Delta A_k| + (n + 1) A_n \right]
\]

\[
\leq M \left[ \sum_{k=0}^\infty (k + 1) |\Delta A_k| + (n + 1) \sum_{k=n}^\infty |\Delta A_k| \right]
\]

\[
\leq M \left[ \sum_{k=0}^\infty (k + 1) |\Delta A_k| + \sum_{k=n}^\infty (k + 1) |\Delta A_k| \right].
\]

Application of Lemma 3 yields

\[
\int_0^\pi \left| \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx \right| dx < \infty.
\]

On the other hand,

\[
\sum_{k=0}^n (k + 1) |\Delta A_k| \leq \sum_{k=0}^n (k + 1) \Delta A_k + 2 \sum_{k=0}^n (k + 1) \delta_k
\]

\[
= \sum_{k=0}^n A_k - (n + 1) A_{n+1} + 2 \sum_{k=0}^n (k + 1) \delta_k
\]

\[
\leq \sum_{k=0}^n A_k - (n + 1) A_{n+1} + 4 \sum_{k=0}^n k \delta_k
\]

\[
\leq \sum_{k=0}^n A_k - (n + 1) A_{n+1} + 4 \sum_{k=0}^\infty A_k.
\]

From Lemma 2, we have: \((n + 1)A_{n+1} = o(1), \ n \to \infty\).

Thus, \( \sum_{k=0}^n (k + 1) |\Delta A_k| = O(\sum_{k=0}^\infty A_k) \).
Finally, the following inequality is satisfied:

\[
\int_{0}^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \leq C \sum_{k=0}^{\infty} A_k.
\]

REFERENCES


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