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AN APPLICATION OF THE HAUSDORFF–YOUNG INEQUALITY

ŽIVORAD TOMOVSKI

(*communicated by H. M. Srivastava*)

Abstract. In this paper a generalization theorem of Zahid of a [8] has been obtained, by considering the condition $S_p(\delta)$, $p>1$ instead of $S(\delta)$.

1. Introduction

A sequence $\{a_k\}$ of positive numbers is said to be quasi-monotone if $k^{-\beta}a_k \downarrow 0$ for some β, or equivalently if $\Delta a_k \geqslant -\beta \dfrac{a_k}{k}$.

A sequence $\{a_k\}$ is said to be δ–quasi-monotone if $a_k \to 0$, $a_k > 0$ ultimately and $\Delta a_k \geqslant -\delta_k$, where $\{\delta_k\}$ is a sequence of positive numbers.

A sequence $\{a_k\}$ is said to satisfy condition S' if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is quasi-monotone, $\sum_{k=1}^{\infty} A_k < \infty$, $|\Delta a_k| \leqslant A_k$, for all k.

On the other hand, a sequence $\{a_k\}$ is said to satisfy condition $S(\delta)$, if $a_k \to 0$ as $k\to\infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is δ–quasi-monotone, $\sum_{k=1}^{\infty} k\delta_k < \infty$, $\sum_{k=1}^{\infty} A_k < \infty$, and $|\Delta a_k| \leqslant A_k$, for all k.

Now, we say that a sequence $\{a_k\}$ of numbers satisfies conditions $S_p(\delta)$ or $a_k \in S_p(\delta)$, if $a_k \to 0$ as $k \to \infty$ and there exists a sequence of numbers $\{A_k\}$ such that:

(a) $\{A_k\}$ is δ–qusi-monotone and $\sum_{k=1}^{\infty} k\delta_k < \infty$,

(b) $\sum_{k=1}^{\infty} A_k < \infty$,

(c) $\dfrac{1}{n}\sum_{k=1}^{n} \dfrac{|\Delta a_k|^p}{A_k^p} = O(1)$.

Thus, in view of the above definitions it is obvious that $S' \subset S(\delta) \subset S_p(\delta)$.

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2. Preliminaries

Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$ be the cosine trigonometric series.

Quite recently, S. Zahid Ali Zenei [8] proved the following theorem.

THEOREM A. [8] *Let the coefficients of the series $f(x)$ satisfy the condition $S(\delta)$. Then the series is a Fourier series and the following relation holds:*

$$\int_0^{\pi} |f(x)|\,dx \leqslant C \sum_{n=0}^{\infty} A_n,$$

where C is an absolute constant.

LEMMA 1. [3] (Hausdorff–Young). *Let the sequence of complex numbers $\{c_n\} \in l^p$. Then $\{c_n\}$ is the sequence of Fourier coefficients of some $\varphi \in L^q \left(\dfrac{1}{p} + \dfrac{1}{q} = 1\right)$, and*

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |\varphi(x)|^q\,dx\right)^{\frac{1}{q}} \leqslant \left(\sum_{n=-\infty}^{\infty} |c_n|^p\right)^{\frac{1}{p}}.$$

LEMMA 2. ([8] case $\nu = 1$) *If $\{a_n\}$ is a δ–quasi-monotone sequence with $\sum\limits_{n=1}^{\infty} n\delta_n < \infty$, then the convergence of $\sum\limits_{n=1}^{\infty} a_n$ implies that $na_n = o(1)$, $n \to \infty$.*

LEMMA 3. [8] *Let $\{a_n\}$ be a δ–quasi-monotone sequence with $\sum\limits_{n=1}^{\infty} n\delta_n < \infty$. If $\sum\limits_{n=1}^{\infty} a_n < \infty$, then $\sum\limits_{n=1}^{\infty} (n+1)|\Delta a_n| < \infty$.*

3. Main results

THEOREM. *Let the coefficients of the series $f(x)$ satisfy the condition $S_p(\delta)$. Then the series is a Fourier series and the following relation holds:*

$$\int_0^{\pi} |f(x)|\,dx \leqslant C \sum_{n=0}^{\infty} A_n,$$

where C is an absolute constant.

Proof. By virtue of hypothesis $\Delta A_n \geqslant -\delta_n$, we have

$$|\Delta A_n| \leqslant \Delta A_n + 2\delta_n.$$

We suppose that

$$A_0 = \max\bigl(a_1, \delta_1, \delta_1 + 2\delta_2, \delta_1 + 2\delta_2 + 3\delta_3, \ldots, \delta_1 + 2\delta_2 + \ldots + n_0\delta_{n_0}\bigr), \quad n \geqslant n_0.$$

We see that $\sum_{k=1}^{n} k\delta_k \leqslant A_0 \leqslant \sum_{k=0}^{n} A_k$, $n \in \mathbf{N}$.
By summation by parts, we have:

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta a_k| &= \sum_{k=1}^{n} A_k \frac{|\Delta a_k|}{A_k} \\
&= \sum_{k=1}^{n-1} |\Delta A_k| \cdot \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} + A_n \cdot \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} \\
&\leqslant \sum_{k=1}^{n-1} k|\Delta A_k| \left(\frac{1}{k}\sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right)^{\frac{1}{p}} + nA_n\left(\frac{1}{n}\sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p}\right)^{\frac{1}{p}} \\
&= O(1)\left[\sum_{k=1}^{n-1} k|\Delta A_k| + nA_n\right] \\
&\leqslant O(1)\left[\sum_{k=1}^{n-1} k((\Delta A_k) + 2\delta_k) + nA_n\right] \\
&= O(1)\left[\sum_{k=1}^{n-1} k(\Delta A_k) + 2\sum_{k=1}^{n-1} k\delta_k + nA_n\right] \\
&= O(1)\left(\sum_{k=1}^{n} A_k - nA_n + 2\sum_{k=1}^{n-1} k\delta_k + nA_n\right) \\
&= O(1)\left(\sum_{k=1}^{n} A_k + 2\sum_{k=1}^{n-1} k\delta_k\right).
\end{aligned}$$

Letting $n \to \infty$ we get $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$.

Thus $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ converges to $f(x)$ for all x except possibly $x = 0$. By summation by parts, we have:

$$\begin{aligned}
f(x) &= \lim_{n\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx\right] \\
&= \lim_{n\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_n D_n(x) - \frac{a_0}{2}\right] \\
&= \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_n D_n(x)\right] \\
&= \sum_{k=1}^{\infty} \Delta a_k D_k(x),
\end{aligned}$$

by the fact that $\lim_{n\to\infty} a_n D_n(x) = 0$, if $x \neq 0$ where $D_n(x)$ is the Dirichlet kernel. Now applications of Abel's transformation yield,

$$\begin{aligned}\int_0^\pi \Big|\frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx\Big|\, dx &= \int_0^\pi \Big|\sum_{k=0}^\infty \Delta a_k D_k(x)\Big|\, dx \\ &= \int_0^\pi \Big|\sum_{k=0}^\infty A_k \frac{\Delta a_k}{A_k} D_k(x)\Big|\, dx \\ &\leqslant \sum_{k=0}^\infty |\Delta A_k| \int_0^\pi \Big|\sum_{j=0}^k \frac{\Delta a_j}{A_j} D_j(x)\Big|\, dx + A_n \int_0^\pi \Big|\sum_{j=0}^k \frac{\Delta a_j}{A_j} D_j(x)\Big|\, dx.\end{aligned}$$

Then

$$\int_0^\pi \Big|\sum_{j=0}^k \frac{\Delta a_j}{A_j} D_j(x)\Big|\, dx = \int_0^{\frac{\pi}{k}} + \int_{\frac{\pi}{k}}^\pi = I_k + J_k.$$

Recalling the uniform estimate of the Dirichlet kernel we have:

$$I_k \leqslant A \sum_{j=0}^k \frac{|\Delta a_j|}{A_j} \leqslant Ak\Big(\frac{1}{k}\sum_{j=0}^k \frac{|\Delta a_k|^p}{A_j^p}\Big)^{\frac{1}{p}},$$

where A is an absolute constant.

Let us estimate the second integral:

$$J_k = \int_{\frac{\pi}{k}}^\pi \Big|\sum_{j=0}^k \frac{\Delta a_j}{A_j} D_j(x)\Big|\, dx = \int_{\frac{\pi}{k}}^\pi \frac{1}{\sin\frac{x}{2}} \Big|\sum_{j=0}^k \frac{\Delta a_j}{A_j} \sin\Big(j+\frac{1}{2}\Big)x\Big|\, dx.$$

We shall first apply the Holder inequality, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$J_k \leqslant \Bigg[\int_{\frac{\pi}{k}}^\pi \Big(\frac{1}{\sin\frac{x}{2}}\Big)^p dx\Bigg]^{\frac{1}{p}} \Bigg[\int_0^\pi \Big|\sum_{j=0}^k \frac{\Delta a_j}{A_j} \sin\Big(j+\frac{1}{2}\Big)x\Big|^q dx\Bigg]^{\frac{1}{q}}.$$

Since $\int_{\frac{\pi}{k}}^\pi \frac{dx}{\left(\sin\frac{x}{2}\right)^p} \leqslant \pi^p \int_{\frac{\pi}{k}}^\pi \frac{dx}{x^p} \leqslant \frac{\pi}{p-1} k^{p-1}$, it follows that

$$J_k \leqslant \Big(\frac{\pi}{p-1}\Big)^{\frac{1}{p}} k^{\frac{p-1}{p}} \Bigg[\int_0^\pi \sum_{j=0}^k \Big|\frac{\Delta a_j}{A_j} \sin\Big(j+\frac{1}{2}\Big)x\Big|^q dx\Bigg]^{\frac{1}{q}}.$$

Then using the Hausdorff-Young inequality we get:

$$\left[\int_0^\pi \left|\sum_{j=0}^k \frac{\Delta a_j}{A_j}\sin\left(j+\frac{1}{2}\right)x\right|^q dx\right]^{\frac{1}{q}} \leqslant \left[\int_0^\pi \sum_{j=0}^k \left|\frac{\Delta a_j}{A_j}e^{ijx}\right|^q dx\right]^{\frac{1}{q}} \leqslant \left(\sum_{j=0}^k \frac{|\Delta a_j|^p}{A_j^p}\right)^{\frac{1}{p}}.$$

Finally, $J_k \leqslant Bk\left(\frac{1}{k}\sum_{j=0}^k \frac{|\Delta a_j|^p}{A_j^p}\right)^{\frac{1}{p}}$, where B is an absolute constant.

Thus,

$$\int_0^\pi \left|\sum_{j=0}^k \frac{\Delta a_j}{A_j}D_j(x)\right| dx = O(k+1).$$

Then,

$$\begin{aligned}\int_0^\pi \left|\frac{a_0}{2}+\sum_{n=1}^\infty a_n\cos nx\right| dx &\leqslant M\left[\sum_{k=0}^\infty (k+1)|\Delta A_k| + (n+1)A_n\right]\\ &\leqslant M\left[\sum_{k=0}^\infty (k+1)|\Delta A_k| + (n+1)\sum_{k=n}^\infty \Delta A_k\right]\\ &\leqslant M\left[\sum_{k=0}^\infty (k+1)|\Delta A_k| + \sum_{k=n}^\infty (k+1)|\Delta A_k|\right].\end{aligned}$$

Application of Lemma 3 yields

$$\int_0^\pi \left|\frac{a_0}{2}+\sum_{n=1}^\infty a_n\cos nx\right| dx < \infty.$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^n (k+1)\,|\Delta A_k| &\leqslant \sum_{k=0}^n (k+1)\,\Delta A_k + 2\sum_{k=0}^n (k+1)\delta_k\\ &= \sum_{k=0}^n A_k - (n+1)\,A_{n+1} + 2\sum_{k=0}^n (k+1)\delta_k\\ &\leqslant \sum_{k=0}^n A_k - (n+1)\,A_{n+1} + 4\sum_{k=0}^n k\delta_k\\ &\leqslant \sum_{k=0}^n A_k - (n+1)\,A_{n+1} + 4\sum_{k=0}^\infty A_k.\end{aligned}$$

From Lemma 2, we have: $(n+1)A_{n+1} = o(1)$, $n \to \infty$.

Thus, $\sum_{k=0}^\infty (k+1)\,|\Delta A_k| = O\left(\sum_{k=0}^\infty A_k\right)$.

Finally, the following inequality is satisfied:

$$\int_{0}^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \leqslant C \sum_{k=0}^{\infty} A_k .$$

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Faculty of Mathematical and Natural Sciences
91 000 Skopje
Macedonia
e-mail: tomovski@iunona.pmf.ukim.edu.mk