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INEQUALITIES FOR A POLYNOMIAL AND ITS DERIVATIVE

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Abstract. Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative. In this paper we shall obtain certain sharp generalization of some results of Govil, Malik and P. Turān concerning the maximum modulus of $P(z)$ and $P'(z)$.

If $P(z)$ is a polynomial of degree at most n, then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)| \tag{1}$$

and

$$\max_{|z|=r} |P(z)| \geqslant r^n \max_{|z|=1} |P(z)| \quad \text{for } r \leqslant 1. \tag{2}$$

(1) is a famous result known as Bernsteins inequality (for reference see [10]) where as inequality (2) is due to Zarantonelo and Varga [11]. Here in both (1) and (2) equality holds if and only if $P(z)$ has all its zeros at the origin and so it is natural to seek improvements under appropriate assumptions on the zeros of $P(z)$.

If $P(z)$ does not vanish in $|z| < 1$, then the inequalities (1) and (2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)| \tag{3}$$

and

$$\max_{|z|=r} |P(z)| \geqslant \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |P(z)|, \quad \text{for } r \leqslant 1. \tag{4}$$

Inequality (3) was conjectured by Erdös and later proved by Lax [6], whereas, inequality (4) is due to T. J. Rivilin [8]. Inequalities (3) and (4) are respectively much better than the inequalities (1) and (2). As an extension of (3), it was shown by M. A. Malik [7] that if $P(z)$ does not vanish in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{5}$$

Whereas, as a generalization of (4), Govil [4, Theorem 1] proved

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THEOREM A. *If* $P(z) = \sum_{j=0}^{n} c_j z^j$, *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for* $0 \leqslant r \leqslant \rho \leqslant 1$,

$$\max_{|z|=r} |P(z)| \geqslant \left(\frac{1+r}{1+\rho}\right)^n \max_{|z|=\rho} |P(z)|. \tag{6}$$

The result is best possible and equality in (6) *holds for the polynomial* $P(z) = (z+1)^n$.

Recenytly, Bidkham and Devan [2, Theorem 3] generalised inequality (5) and obtained

THEOREM B. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in* $|z| < k$ *where* $k \geqslant 1$, *then*

$$\max_{|z|=r} |P'(z)| \leqslant \frac{n(r+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |P(z)|, \tag{7}$$

for $1 \leqslant r \leqslant k$. *The result is best possible and equality in* (7) *holds for* $P(z) = (z+k)^n$.

In this paper we shall first obtain the following result which is a generalization as well as a refinement of Theorem A.

THEOREM 1. *If* $P(z)$ *is a polynomial of degree at most* n *which does not vanish in* $|z| < k$ *where* $k > 0$, *then for* $0 \leqslant rR \leqslant k^2$ *and* $r \leqslant R$, *we have*

$$\max_{|z|=r} |P(z)| \geqslant \left(\frac{r+k}{R+K}\right)^n \max_{|z|=R} |P(z)| + \left[1 - \left(\frac{r+k}{R+K}\right)^n\right] \min_{|z|=k} |P(z)|. \tag{8}$$

Here the result is best possible and equality (8) *holds for* $P(z) = (z+k)^n$.

The following result which is a refinement of Theorem A, is obtained by taking $k = 1$ in Theorem 1.

COROLLARY 1. *If* $P(z)$ *is a polynomial of degree at most* n *which does not vanish in* $|z| < 1$, *then for* $0 \leqslant r\rho \leqslant 1$ *and* $r \leqslant \rho$, *we have*

$$\max_{|z|=r} |P(z)| \geqslant \left(\frac{r+1}{\rho+1}\right)^n \max_{|z|=\rho} |P(z)| + \left[1 - \left(\frac{r+1}{\rho+1}\right)^n\right] \min_{|z|=1} |P(z)|. \tag{9}$$

The result is best possible and equality in (9) *holds for* $P(z) = (z+1)^n$.

REMARK 1. Theorem A is a special case of corollary 1, since $0 \leqslant r \leqslant \rho \leqslant 1$ implies $0 \leqslant r\rho \leqslant 1$.

Next we extend Theorem B for the class of polynomials $P(z) = a_0 + \sum_{j=m}^{n} a_j z^j$ not vanishes in $|z| < k$ where $k \geqslant 1$. In fact, we prove

THEOREM 2. *If* $P(z) = a_0 + \sum_{j=m}^{n} a_j z^j$ *has no zeros in* $|z| < k$ *where* $k \geqslant 1$ *then for* $0 \leqslant r \leqslant R \leqslant k$, *we have*

$$\max_{|z|=R} |P'(z)| \leqslant \frac{nR^{m-1}\big(R^m + k^m)\big)^{\frac{n}{m}-1}}{\big(r^m + k^m\big)^{\frac{n}{m}}} \max_{|z|=r} |P(z)|. \tag{10}$$

The result is best possible and equality in (10) *holds for* $P(z) = (z^m + k^m)^{\frac{n}{m}}$, *where* n *is a multiple of* m.

REMARK 2. If we take $m = 1$ and $r = 1$ in Theorem 2 we get Theorem B.

If $P(z)$ is a polynomial of degree n which has all its zeros in the unit disk $|z| \leqslant 1$, then according to a result of Turān [9], we have

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{11}$$

As a refinement of (11) it was shown by Aziz and Dawood [1] that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2}\left\{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\right\}. \tag{12}$$

Here, we finally present the following generalization of (12).

THEOREM 3. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then for* $rR \geqslant k^2$ *and* $r \leqslant R$, *we have*

$$\max_{|z|=R} |P'(z)| \geqslant \frac{n(R+k)^{n-1}}{(r+k)^n} \left\{\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)|\right\}. \tag{13}$$

Equality in (13) holds for the polynomial $P(z) = (z+k)^n$.

If we take $r = 1$ and $k = 1$ in Theorem 3 we immediately get the following result.

COROLLARY 2. *If* $P(z)$ *is a polynomial of degree* n *having all the zeros in* $|z| \leqslant 1$, *then for* $R \geqslant 1$ *we have*

$$\max_{|z|=R} |P'(z)| \geqslant \frac{n}{2}\left(\frac{1+R}{2}\right)^{n-1} \left\{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\right\}. \tag{14}$$

The result is best possible and equality in (14) holds for $P(z) = \left(\frac{z+1}{2}\right)^n$.

REMARK 3. For $R = 1$ Corollary 2 reduced to (12).

Lemmas

For the proofs of these theorems we require the following Lemmas. The first result is due to Chan and Malik [3, Theorem 1].

LEMMA 1. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *has no zeros in* $|z| < k$, $k \geqslant 1$, *then*

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^{\mu}} \max_{|z|=1} |P(z)|. \tag{15}$$

Equality in (15) *holds for* $P(z) = (z^{\mu} + k^{\mu})^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

LEMMA 2. *If* $P(z) = a_0 + \sum_{j=m}^{n} a_j z^j$ *is apolynomial of degree* n *having no zeros in* $|z| < k$, $k > 0$, *then for* $0 \leqslant r \leqslant R \leqslant k$,

$$\max_{|z|=r} |P(z)| \geqslant \left(\frac{r^m + k^m}{R^m + k^m}\right)^{\frac{n}{m}} \max_{|z|=R} |P(z)|. \tag{16}$$

There is equality in (16) *for* $P(z) = (z^m + k^m)^{\frac{n}{m}}$, *where* n *is a multiple of* m.

Lemma 2 is due to V. K. Jain [5].

LEMMA 3. *If* $P(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *such that* $P(z)$ *has all its zeros in* $|z| < k$, $k > 0$, *then* $rR \geqslant k^2$ *and* $r \leqslant R$, *we have for* $|z| = 1$,

$$|P(Rz)| \geqslant \left(\frac{R+k}{r+k}\right)^n |P(rz)|. \tag{17}$$

Equality in (17) *holds for the polynomial* $P(z) = (z+k)^n$.

Proof of Lemma 3. Since all the zeros of $P(z)$ lie in $|z| \leqslant k$, $k > 0$, we write

$$P(z) = C \prod_{j=1}^{n} \left(z - R_j e^{i\theta_j}\right),$$

where $R_j \leqslant k$, $j = 1, 2, \ldots, n$, therefore, for $0 \leqslant \theta < 2\pi$, we have

$$\left|\frac{P(re^{i\theta})}{P(Re^{i\theta})}\right| = \prod_{j=1}^{n} \left|\frac{re^{i\theta} - R_j e^{i\theta_j}}{Re^{i\theta} - R_j e^{i\theta_j}}\right| = \prod_{j=1}^{n} \left|\frac{re^{i(\theta-\theta_j)} - R_j}{Re^{i(\theta-\theta_j)} - R_j}\right|. \tag{18}$$

Now for $R \geqslant r$, $Rr \geqslant R_j^2$ and for each θ, $0 \leqslant \theta < 2\pi$, it can be easily seen that

$$\begin{aligned} \left|\frac{re^{i(\theta-\theta_j)} - R_j}{Re^{i(\theta-\theta_j)} - R_j}\right|^2 &= \frac{r^2 + R_j^2 - 2rR_j\cos(\theta-\theta_j)}{R^2 + R_j^2 - 2RR_j\cos(\theta-\theta_j)} \\ &\leqslant \left(\frac{r+R_j}{R+R_j}\right)^2. \end{aligned}$$

Since $R_j \leqslant k$, for all $j = 1, 2, \ldots, n$ therefore it follows from (18) that if $r \leqslant R$ and $rR \geqslant k^2$, then

$$\left|\frac{P(re^{i\theta})}{P(Re^{i\theta})}\right| \leqslant \prod_{j=1}^{n} \left(\frac{r+R_j}{R+R_j}\right) \leqslant \left(\frac{r+k}{R+k}\right)^n.$$

Hence for $r \leqslant R$, $rR \geqslant k^2$ and for each θ, $0 \leqslant \theta < 2\pi$, we have

$$|P(re^{i\theta})| \leqslant \left(\frac{r+k}{R+k}\right)^n |P(Re^{i\theta})|$$

and this complete the proof of Lemma 3.

Proofs of the theorems

Proof of Theorem 1. By hypothesis the polynomial $P(z)$ has all its zeros in $|z| \geqslant k$, $k > 0$ and $m = \min_{|z|=k} |P(z)|$, therefore $m \leqslant |P(z)|$ for $|z| \leqslant k$. We show for any given complex number α with $|\alpha| \leqslant 1$, the polynomial $F(z) = P(z) + \alpha m$ has all its zeros in $|z| \geqslant k$. This is obvious if $m = 0$, that is if $P(z)$ has a zero on $|z| = k$. We now suppose all the zeros of $P(z)$ lie in $|z| > k$ so that $m = \min_{|z|=k} |P(z)| > 0$. Hence $\frac{m}{P(z)}$ is analytic for $|z| \leqslant k$ and $\left|\frac{m}{P(z)}\right| \leqslant 1$ for $z = k$. Moreover, $\frac{m}{P(z)}$ is is not a constant and therefore it follows by Maximum modulus principle that

$$m < |P(z)| \quad \text{for} \ |z| < k. \tag{19}$$

Now assume that $F(z) = P(z) + \alpha m$ has a zero in $|z| < k$, say at $z = z_0$ with $|z_0| < k$, then

$$P(z_0) + \alpha m = F(z_0) = 0.$$

This implies

$$|P(z_0)| = |\alpha m| \leqslant m,$$

which is a contradiction to (19). Hence we conclude that in any case $F(z) = P(z) + \alpha m$ has all its zeros in $|z| \geqslant k$. If

$$R_1 e^{i\theta_1} \cdot R_2 e^{i\theta_2} \cdot \ldots \cdot R_n e^{i\theta_n}$$

are the zeros of $F(z)$ then $R_j \geqslant k$, $j = 1, 2, \ldots, n$, and we have

$$F(z) = C \prod_{j=1}^{n} (z - R_j e^{i\theta_j}).$$

This gives for each θ, $0 \leqslant \theta < 2\pi$

$$\left|\frac{F(re^{i\theta})}{F(Re^{i\theta})}\right| = \prod_{j=1}^{n} \left|\frac{re^{i\theta} - R_j e^{i\theta_j}}{Re^{i\theta} - R_j e^{i\theta_j}}\right|. \tag{20}$$

Now it can be easily verified that if $r \leqslant R$ and $0 \leqslant rR \leqslant R_j^2$, then

$$\begin{aligned}
\left|\frac{re^{i(\theta-\theta_j)} - R_j}{Re^{i(\theta-\theta_j)} - R_j}\right|^2 &= \frac{r^2 + R_j^2 - 2rR_j \cos(\theta - \theta_j)}{R^2 + R_j^2 - 2RR_j \cos(\theta - \theta_j)} \\
&\geqslant \frac{r^2 + R_j^2 - 2rR_j}{r^2 + R_j^2 - 2RR_j} \\
&= \left(\frac{r + R_j}{R + R_J}\right)^2.
\end{aligned}$$

Since $R_j \geqslant k$ for all $j = 1, 2, \dots, n$, therefore it follows from (20) that if $r \leqslant R$ and $0 \leqslant rR \leqslant k^2$, then

$$\Big|\frac{F(re^{i\theta})}{F(Re^{i\theta})}\Big| \geqslant \prod_{j=1}^{n}\Big(\frac{r+R_j}{R+R_j}\Big) \geqslant \Big(\frac{r+k}{R+k}\Big)^n \tag{21}$$

for each θ, $0 \leqslant \theta \leqslant 2\pi$. Replacing $F(z)$ by $P(z) + \alpha m$, from (21) we obtain for each θ, $0 \leqslant \theta < 2\pi$,

$$|P(re^{i\theta}) + \alpha m| \geqslant \Big(\frac{r+k}{R+k}\Big)^n |P(Re^{i\theta}) + \alpha m|,$$

which gives

$$|P(rz + \alpha m| \geqslant \Big(\frac{r+k}{R+k}\Big)^n \Big\{|P(Rz)| - |\alpha m|\Big\} \quad \text{for } |z| = 1.$$

Now choosing argument of α with $|\alpha| = 1$ such that for $|z| = 1$

$$|P(rz)| - m \geqslant \Big(\frac{r+k}{R+k}\Big)^n \Big\{P(Rz) - m\Big\} \quad \text{for } |z| = 1.$$

This implies for $0 \leqslant rR \leqslant k^2$ and $r \leqslant R$, that

$$\max_{|z|=r} |P(z)| \geqslant \Big(\frac{r+k}{R+k}\Big)^n \max_{|z|=R} |P(z)| + \Big\{1 - \Big(\frac{r+k}{R+k}\Big)^n\Big\} \min_{|z|=k} |P(z)|$$

which is (8) and this completes the proof of Theorem 1.

Proof of Theorem 2. By hypothesis the polynomial $P(z) = a_0 + \sum_{j=m}^{n} a_j z^j$ has no zero in $|z| < k$ where $k \geqslant 1$, therefore it follows that $F(z) = P(Rz)$ has no zeros in $|z| < \frac{k}{R}$ where $\frac{k}{R} \geqslant 1$. Applaying Lemma 1 to the polynomial $F(z)$, we get

$$\max_{|z|=1} |F'(z)| \leqslant \frac{n}{1 + \frac{k^m}{R^m}} \max_{|z|=1} |F(z)|$$

which gives

$$\max_{|z|=1} |P'(z)| \leqslant \frac{nR^{m-1}}{R^m + k^m} \max_{|z|=R} |P(z)|. \tag{22}$$

Now if $0 \leqslant r \leqslant R \leqslant k$, then by Lemma 2, we have

$$\max_{|z|=R} |P(z)| \leqslant \Big(\frac{R^m + k^m}{r^m + k^m}\Big)^{\frac{n}{m}} \max_{|z|=r} |P(z)|. \tag{23}$$

From (22) and (23) it follows that

$$\max_{|z|=R} |P'(z)| \leqslant \frac{nR^{m-1}(R^m + k^m)^{\frac{n}{m}-1}}{(r^m + k^m)^{\frac{n}{m}}} \max_{|z|=r} |P(z)|,$$

which is (10) and Theorem 2 is proved.

Proof of Theorem 3. Let $m = \min_{|z|=k} |P(z)|$, then $m \leqslant |P(z)|$ for $|z| = k$. Since all the zeros of $P(z)$ lie in $|z| \leqslant k \leqslant 1$ therefore for every complex number α such that $|\alpha| < 1$, it follows (by Rouches theorem for $m > 0$) that the polynomial $G(z) = P(z) + \alpha m$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$ and therefore $H(z) = G(Rz)$ has all its zeros in $|z| \leqslant \frac{k}{R} < 1$. Hence if $z_1, z_2, \dots, z_n$ are the zeros of $H(z)$ then $|z_j| \leqslant \frac{k}{R} \leqslant 1$, for all $j = 1, 2, \dots, n$ and we have

$$\frac{zH'(z)}{H(z)} = \sum_{j=1}^{n} \frac{z}{z - z_j}.$$

This gives

$$\begin{aligned} \operatorname{Re} \frac{e^{i\theta} H'(e^{i\theta})}{H(e^{i\theta})} &= \sum_{j=1}^{n} \operatorname{Re} \frac{e^{i\theta}}{e^{i\theta} - z_j} \geqslant \sum_{j=1}^{n} \frac{1}{1 + \frac{k}{R}} \\ &= \sum_{j=1}^{n} \frac{R}{R+k} = \frac{nR}{R+k}, \end{aligned}$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which is not a zero of $H(z)$. Hence

$$\left| \frac{H'(e^{i\theta})}{H(e^{i\theta})} \right| \geqslant \operatorname{Re} \left| \frac{e^{i\theta} H'(e^{i\theta})}{H(e^{i\theta)})} \right| \geqslant \frac{nR}{R+k},$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which is not a zero of $H(z)$. Therefore, it follows that

$$|H'(e^{i\theta})| \geqslant \frac{nR}{R+k} |H(e^{i\theta})|, \tag{24}$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which is not a zero of $H(z)$. Since (24) is trivially true for points $e^{i\theta}$, which are the zeros of $H(z)$. Hence it follows that

$$|H'(z)| \geqslant \frac{nR}{R+k} |H(z)|, \quad \text{for } |z| = 1.$$

Replacing $H(z)$ by $G(Rz)$, we get

$$|G'(Rz)| \geqslant \frac{n}{R+k} |G(Rz), | \quad \text{for } |z| = 1.$$

Now, applaying Lemma 3 to the polynomial $G(z)$, we obtain

$$|G'(Rz) \geqslant \frac{n}{R+k} \left(\frac{R+k}{r+k} \right)^n |G(rz)|, \quad \text{for } |z| = 1, \tag{25}$$

where $r \leqslant R$ and $rR \geqslant k^2$ (we note that $R \geqslant r$ and $rR \geqslant k^2$ implies that $R \geqslant k$). Since $G(z) = P(z) + \alpha m$, from (25) it follows that

$$|P'(Rz)| \geqslant \frac{n(R+k)^{n-1}}{(r+k)^n} |P(rz) + \alpha m|, \quad \text{for } |z| = 1, \tag{26}$$

choosing argument of α suitably on the R. H. S. of (26) we get for $|z| = 1$ and for every α with $|\alpha| < 1$,

$$|P'(Rz)| \geqslant \frac{n(R+k)^{n-1}}{(r+k)^n}\left\{|P(rz)| + |\alpha|m\right\}$$

where $r \leqslant R$ and $rR \geqslant k^2$. Letting $|\alpha| \to 1$ we finally obtain

$$\max_{|z|=R} |P'(z)| \geqslant \frac{n(R+k)^{n-1}}{(r+k)^n}\left\{\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)|\right\}$$

where $r \leqslant R$ and $rR \geqslant k^2$, which is (13) and this completes the proof of Theorem 3.

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