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INEQUALITIES FOR THE MINIMAL EIGENVALUE OF THE LAPLACIAN IN AN ANNULUS

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Abstract. We discuss the behavior of the minimal eigenvalue λ of the Dirichlet Laplacian in the domain $D_1 \backslash D_2 := D$ (an annulus) where D_1 is a circular disc and $D_2 \subset D_1$ is a smaller circular disc. It is conjectured that the minimal eigenvalue λ has a maximum value when D_2 is a concentric disc. If h is a displacement of the center of the disc D_2 and $\lambda(h)$ is the corresponding minimal eigenvalue, then $\frac{d\lambda(h)}{dh} < 0$ so that $\lambda(h)$ is minimal when ∂D_2 touches ∂D_1, where ∂D is the boundary of D. Numerical results are given to back the conjecture. Upper and lower bounds are given for $\lambda(h)$.

1. Introduction

Let D_1 be a disc on $\mathbf{R}^2$, centered at the origin, of radius 1, $D_2 \subset D_1$ be a disc of radius $a < 1$, the center $(h, 0)$ of which is at the distance h from the origin. Denote by $\lambda(h)$ the minimal Dirichlet eigenvalue of the Laplacian in the annulus $D := D_h := D_1 \backslash D_2$.

In this paper the following conjecture is formulated and partly justified:

CONJECTURE C.. *The minimal eigenvalue $\lambda(h)$ is a monotonically decreasing function of h on the interval $0 \leqslant h \leqslant 1 - a$. In particular*

$$\lambda(0) > \lambda(h), \qquad h > 0. \tag{1.1}$$

Let $\dot{\lambda} := \frac{d\lambda}{dh}$ and let S denote ∂D_2, the boundary of D_2.

The following results are given to back this conjecture:

LEMMA 1. *One has*

$$\dot{\lambda} = \int_S u_N^2 N_1 ds, \tag{1.2}$$

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where N is the unit normal to $S = S_h$ pointing into the annulus D_h, N_1 is the projection of N onto x_1-axis, u_N is the normal derivative of u, and $u(x) = u(x_1, x_2)$ is the normalized in $L^2(D)$ eigenfunction corresponding to the first eigenvalue λ:

$$\Delta u + \lambda u = 0 \quad \textit{in} \quad D,\ u = 0 \quad \textit{on} \quad \partial D_1 \cup \partial D_2 := \partial D, \tag{1.3}$$

$$\|u\|_{L^2(D)} = 1. \tag{1.4}$$

It is argued at the end of Section 2 that

$$\dot{\lambda} < 0 \quad \textit{if} \quad 0 < h < 1 - a. \tag{1.5}$$

In Lemma 2 below we give upper and lower bounds (1.6) for $\lambda(h)$. These bounds are practically convenient, especially for small h.

Let $D(r)$ be the disc $|x| \leqslant r$, $\mu(r)$ be the first Dirichlet eigenvalue of the Laplacian in $D_1 \backslash D(r)$, and in Section 3 (1.5) is illustrated by numerical results.

LEMMA 2. *One has*

$$\mu(a-h) < \lambda(h) < \mu(a+h), \qquad 0 < h < 1 - a. \tag{1.6}$$

In section 2 proofs are given.

2. Proofs

Proof of Lemma 2. Lemma 2 is an immediate consequence of the variational principle for λ since $D(a+h) \subset D_h \subset D(a-h)$. Note that $\mu(b)$, $a \leqslant b < 1$, can be calculated efficiently. Indeed, by symmetry the first eigenfunction ϕ of the Dirichlet Laplacian in $D_1 \backslash D(b)$ depends on the radial variable $r = |x|$ only, and solves the problem

$$\phi'' + \frac{1}{r}\phi' + \mu\phi = 0, \qquad b \leqslant r \leqslant 1; \qquad \phi(b) = \phi(1) = 0. \tag{2.1}$$

Thus

$$\phi = c_1 J_0(\sqrt{\mu} r) + c_2 N_0(\sqrt{\mu} r), \tag{2.2}$$

where J_0 and N_0 are the Bessel functions, and c_1, c_2 are constants. The boundary conditions (2.1) are satisfied if $\mu = \mu(b) > 0$ is a positive root of the equation:

$$J_0(\sqrt{\mu} b) N_0(\sqrt{\mu}) - J_0(\sqrt{\mu}) N_0(\sqrt{\mu} b) = 0. \tag{2.3}$$

The smallest positive root $\mu = \mu(b)$ of (2.3) is the desired first eigenvalue of the Dirichlet Laplacian in $D_1 \backslash D(b)$. Equation (2.3) can be solved numerically. This makes (1.6) an efficient estimate of $\lambda(h)$, especially for small $h > 0$. □

Proof of Lemma 1. We use the known technique based on the domain derivative [1].

It is known that $\lambda(h)$ is continuously differentiable with respect to h [2]. Let $\dot{u} = \frac{du}{dh}$, where u solves (1.3)-(1.4). Differentiate the equation and the boundary condition (1.3) with respect to h and get

$$\Delta\dot{u} + \lambda\dot{u} = -\dot{\lambda}u \qquad in \qquad D = D_h, \tag{2.4}$$

$$\dot{u} + u_N N_1 = 0 \; on \; S = S_h. \tag{2.5}$$

Multiply (2.4) by u, (1.3) by $\dot{u}$, subtract, integrate over $D = D_h$, use Green's formula, and (2.5) and get:

$$\dot{\lambda}\int_D u^2 dx = \int_S (u\dot{u}_N - \dot{u}u_N)ds = \int_S u_N^2 N_1 ds. \tag{2.6}$$

From (2.6) and (1.4) one gets (1.2). Lemma 1 is proved. $\square$

It follows from (1.2) by symmetry that $\dot{\lambda}(0) = 0$. Indeed, if $h = 0$, then $u_N^2|_{S_0}$ $=$ const by symmetry, and $\int_{S_0} N_1 ds = 0$.

If $h > 0$, then u_N^2 on the half circle S_h^+, the part of the boundary of S_h which is closer to ∂D_1, is likely to be less than on the other half S_h^- of S_h, while $N_1 > 0$ on S_h^+ and $N_1 < 0$ on S_h^-. Moreover, $|N_1|$ is the same at the symmetric points of S_h^+ and S_h^-, where the axis of symmetry is the vertical diameter of D_2. Therefore one expects $\dot{\lambda}(h) < 0$ for $h > 0$, which is the conjecture C.

3. Numerical Results

We use a finite element method to calculate u_N^2 at a number of nodal points φ on ∂D_2, where φ is the angle between the radial line at the positive x_1-axis. Due to symmetry, it is sufficient to consider $0 \leqslant \varphi \leqslant \pi$. The following tables give values for u_N^2 for various values of h and φ. The last row gives $\lambda(h)$ for different values of h.

Table 1
Values for u_N^2
$a = 0.1$ $\lambda(0) = 10.98324859$

φ	$h = 0.1$	$h = 0.3$	$h = 0.6$	$h = 0.8$
0°	0.18340156	0.08997194	0.03502936	0.00538128
15°	0.18586555	0.09354750	0.03875921	0.00792279
30°	0.19312533	0.10408993	0.04977909	0.01615017
45°	0.20478642	0.12105508	0.06745736	0.03118122
60°	0.22019869	0.14357017	0.09052611	0.05294918
75°	0.23846941	0.17048691	0.11728631	0.07901455
90°	0.25848609	0.20042583	0.14624640	0.10706183
105°	0.27895498	0.23176494	0.17645804	0.13678539
120°	0.29846292	0.26256868	0.20707716	0.16793719
135°	0.31556947	0.29053976	0.23653390	0.19964213
150°	0.32892971	0.31313057	0.26197403	0.22879649
165°	0.33743644	0.32789921	0.27954557	0.24990529
180°	0.34035750	0.33304454	0.28585725	0.25766770
$\lambda(h)$	10.51624800	8.76956649	6.91928150	6.21431318

Table 2
Values for u_N^2
$a = 0.3$ $\lambda(0) = 19.46950428$

φ	$h = 0.1$	$h = 0.3$	$h = 0.6$
0°	0.04651448	0.00601084	0.00006665
15°	0.05078040	0.00792264	0.00029224
30°	0.06389146	0.01432651	0.00162487
45°	0.08665951	0.02711431	0.00616138
60°	0.12001996	0.04901522	0.01734345
75°	0.16444947	0.08285892	0.03916871
90°	0.21927390	0.13049149	0.07481155
105°	0.28204163	0.19150347	0.12521694
120°	0.34820007	0.26211532	0.18784387
135°	0.41130766	0.33475001	0.25580537
150°	0.46389778	0.39885669	0.31827254
165°	0.49888764	0.44319924	0.36272535
180°	0.51117180	0.45907590	0.37887932
$\lambda(h)$	17.00607073	12.31240018	8.54494014

Table 3
Values for u_N^2
$a = 0.6$ $\lambda(0) = 61.2854372$

φ	$h = 0.1$	$h = 0.3$
0°	0.00010994	0.00000018
15°	0.00025775	0.00000144
30°	0.00101252	0.00002268
45°	0.00370221	0.00026580
60°	0.01190759	0.00195778
75°	0.03332159	0.00947178
90°	0.08086609	0.03287792
105°	0.17026477	0.08782665
120°	0.32267905	0.18896048
135°	0.49728793	0.33653240
150°	0.69311417	0.50402714
165°	0.84533543	0.64040281
180°	0.90307061	0.69330938
$\lambda(h)$	42.71463081	23.79696055

In all the cases above, u_N^2 increases in value as φ increases from zero to π, thereby confirming that $\dot{\lambda} < 0$ (see formula (2.6)). From the above tables we also note that for fixed a, $\lambda(h)$ is a decreasing function of h, and that $\lambda(h) < \lambda(0)$ for $h > 0$ thus confirming the Conjecture C.

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