Mathematical
Inequalities
& Applications
Volume 1, Number 4 (1998), 565–584

GLOBAL LIMITING EMBEDDINGS OF LOGARITHMIC BESSEL POTENTIAL SPACES

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(*communicated by V. Burenkov*)

Abstract. The paper is a continuation of [EGO II-IV], where it was shown that Bessel potential spaces $H^{\sigma}Y(\mathbb{R}^n)$, modelled upon appropriate generalized Lorentz-Zygmund spaces $Y(\mathbb{R}^n)$ may be embedded into Orlicz spaces $L_{\Phi}(\Omega)$, where $\Phi(t) = \exp(\exp(\ldots\exp t^{\nu})\ldots))$ for large t, $\nu > 0$, and Ω is a subset of $\mathbb{R}^n$ with finite volume. Using weighted Hardy inequalities, we modify the Young function Φ near the origin so that the above embedding holds with Ω replaced by $\mathbb{R}^n$. The resulting Young function dominates globally the Young function $\Psi(t) = t^q$, $t > 0$, for q sufficiently large and consequently, $H^{\sigma}Y(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$. We also obtain an estimate of the norms of the last embeddings which is sharp in their dependence upon q provided that q is large enough.

1. Introduction

In the recent paper [EGO IV] an embedding theory for certain logarithmic Bessel potential spaces $H^{\sigma}Y(\mathbb{R}^n)$ modelled upon generalized Lorentz-Zygmund spaces $Y(\mathbb{R}^n)$ was established and the role of the logarithmic terms involved in the norms of spaces $H^{\sigma}Y(\mathbb{R}^n)$ was clarified. Since generalized Lorentz-Zygmund spaces include many familiar objects including Lebesgue, Lorentz, Lorentz-Zygmund, and Zygmund spaces (see Section 2), in [EGO IV] we got the refinements of the Sobolev embedding theorems, Trudinger's limiting embedding as well as embeddings of Sobolev spaces into space of $\lambda(\cdot)$-Hölder-continuous functions including the result of Brézis and Wainger. In these embedding theorems all the target spaces are spaces of functions defined on $\mathbb{R}^n$ with the exception of the embedding which generalizes Trudinger's limiting embedding. In this limiting case the result has *a local character*: For any bounded subset Ω of $\mathbb{R}^n$ with finite volume we have

$$H^{\sigma}Y(\mathbb{R}^n) \hookrightarrow L_{\Phi}(\Omega); \tag{1.1}$$

the target space in (1.1) is the Orlicz space with the Young function Φ given for large t by

$$\Phi(t) = \exp(\exp(\ldots\exp t^{\nu})\ldots)), \tag{1.2}$$

Mathematics subject classification (1991): 46E35, 46E30, 26D15.

Key words and phrases: Generalized Lorentz-Zygmund spaces, logarithmic Bessel potential spaces, Orlicz spaces, embeddings.

The research was partially supported by grant no. 201/97/0744 of the Grant Agency of the Czech Republic.

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Paper MIA-01-55

ν is a positive number. (Note that the number of exponential functions appearing in (1.2) is determined by the number of parameters of the generalized Lorentz-Zygmund space $Y(\mathbb{R}^n)$ which are in the limiting state.)

It is well known (cf. [A, Section 8.26] or [EE]) that the Young function Φ from Trudinger's result

$$W^{1,n}(\Omega) \hookrightarrow L_\Phi(\Omega) \tag{1.3}$$

(here Ω is a bounded domain in $\mathbb{R}^n$ whose boundary is sufficiently regular) can be modified near the origin so that the embedding (1.3) holds even if the volume of Ω is infinite. This leads us to the idea that also the Young function Φ from (1.2) can be modified near the origin so that the resulting Young function Φ_0 (equivalent to Φ near infinity) is such that

$$H^\sigma Y(\mathbb{R}^n) \hookrightarrow L_{\Phi_0}(\mathbb{R}^n), \tag{1.4}$$

that is, the local embedding (1.1) can be replaced by a *global* one. In distinction to [A, Section 8.26] or [EE], this modification is done by making use of convenient Hardy inequalities with power-logarithmic weights. (Note that (1.4) with a particular choice of the generalized Lorentz-Zygmund space Y extends to $\mathbb{R}^n$ the results of [EK] and [FLS] (cf. [EGO IV, page 133]).) Since the Young function Φ_0 from (1.4) dominates globally the Young function Ψ given by $\Psi(t) = t^q$, $t \geqslant 0$, for large q, (1.4) implies that

$$H^\sigma Y(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \tag{1.5}$$

provided that q is large enough.

We also obtain estimates of the norms of the embeddings (1.5) which are sharp in their dependence on q (for large q). This extends the result from [EGO V], where such estimates were established for the embedding

$$H^\sigma Y(\mathbb{R}^n) \hookrightarrow L^q(\Omega) \tag{1.6}$$

with $\Omega \subset \mathbb{R}^n$ having non-empty interior and finite volume.

The paper is organized as follows. Section 2 contains the basic notation and auxiliary assertions. The main results (Theorems 3.1 and 3.4) are given in Section 3 which also contains examples. The proofs of main results can be found in Sections 4 and 5.

2. Notation and preliminaries

Let $(\mathscr{R}, \mu)$ be a totally σ-finite measure space. When $\mathscr{R} \subseteq \mathbb{R}^n$, we shall always take μ to be n-dimensional Lebesgue measure μ_n, and we shall put $|G| = |G|_n = \mu_n(G)$ for any measurable subset G of $\mathbb{R}^n$. The family of all extended scalar-valued (real or complex) μ-measurable functions on $\mathscr{R}$ will be denoted by $\mathscr{M}(\mathscr{R}, \mu)$; $\mathscr{M}^+(\mathscr{R}, \mu)$ will represent the subset of $\mathscr{M}(\mathscr{R}, \mu)$ of all those functions which are non-negative μ-a.e. The symbol $\mathscr{M}^+(a,b)$ with $(a,b) \subseteq \mathbb{R}$ will stand for $\mathscr{M}^+((a,b), \mu_1)$.

For $f \in \mathscr{M}(\mathscr{R}, \mu)$, the *distribution function* μ_f of f is given by

$$\mu_f(\lambda) = \mu_{f,\mathscr{R}}(\lambda) = \mu(\{x \in \mathscr{R};\ |f(x)| > \lambda\}), \quad \lambda \geqslant 0,$$

and the *non-increasing rearrangement* f^* of f is defined by

$$f^*(t) = f^*_{(\mathscr{R},\mu)}(t) = \inf\{\lambda; \mu_f(\lambda) \leqslant t\}, \quad t \geqslant 0.$$

Recall that if $f \in \mathscr{M}(\mathscr{R},\mu)$, then $\operatorname{supp} f^* \subset [0,\mu(\mathscr{R})]$, f^* is non-increasing on $(0,\infty)$, and f and f^* are equimeasurable (cf. [BS]). We shall also need the average of f^*, and so define

$$f^{**}(t) = t^{-1}\int_0^t f^*(s)ds, \qquad t > 0.$$

Now let $m \in \mathbb{N}$ and define (logarithmic) functions $\ell_1, \ldots, \ell_m$ on $(0,\infty)$ by

$$\ell_1(t) = \ell(t) = 1 + |\log t|, \quad \ell_m(t) = 1 + \log \ell_{m-1}(t) \quad (m > 1). \tag{2.1}$$

It is easy to see that for all $t \in (0,\infty) \setminus \{1\}$,

$$\left.\begin{aligned} &\ell_1'(t) = t^{-1}\operatorname{sgn}(t-1), \\ &\ell_m'(t) = \Big(\prod_{j=1}^{m-1} \ell_j(t)\Big)^{-1} t^{-1}\operatorname{sgn}(t-1) \qquad (m>1). \end{aligned}\right\} \tag{2.2}$$

Let $p, q \in (0,\infty]$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. The generalized *Lorentz-Zygmund space* $L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathscr{R})$ consists of all functions $f \in \mathscr{M}(\mathscr{R},\mu)$ such that the quantity

$$\|f\|_{p,q;\alpha_1,\ldots,\alpha_m} := \Big\| t^{1/p-1/q}\Big(\prod_{j=1}^m \ell_j^{\alpha_j}(t)\Big) f^*(t)\Big\|_{q,(0,\infty)} \tag{2.3}$$

is finite, where $\|\cdot\|_{q,(a,b)}$ is the usual L^q-(quasi-) norm on an interval $(a,b) \subseteq \mathbb{R}$. We shall sometimes write

$$L^p(\log L)^{\alpha_1} \ldots (\underbrace{\log\log\ldots\log}_{m \text{ times}} L)^{\alpha_m}(\mathscr{R})$$

instead of $L_{p,p;\alpha_1,\ldots,\alpha_m}(\mathscr{R})$. When each $\alpha_j = 0$, the space $L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathscr{R})$ coincides with the classical Lorentz space $L^{p,q}(\mathscr{R})$, which is just $L^p(\mathscr{R})$ when $p = q$; if $m = 1$, $L_{p,q;\alpha_1}(\mathscr{R})$ is the Lorentz-Zygmund space $L^{p,q}(\log L)^{\alpha_1}(\mathscr{R})$ introduced in [BR] and which, when $p = q$, is the Zygmund space $L^p(\log L)^{\alpha_1}(\mathscr{R})$. If $\mu(\mathscr{R}) < \infty$, then

$$L_{p,p;\alpha_1,\ldots,\alpha_m}(\mathscr{R}) = \Big\{f \in \mathscr{M}(\mathscr{R},\mu);\ \int_{\mathscr{R}} \Big[|f|\prod_{j=1}^m \ell_j^{\alpha_j}(e+|f|)\Big]^p d\mu < \infty\Big\}.$$

The spaces $L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathscr{R})$ were studied in [EGO II-IV], [EOP], and [OP], where more information can be found.

Throughout the paper the symbol I_σ, $\sigma \in (0,n)$, is used to denote the *kernel of the Riesz potential*, i.e. $I_\sigma(x) = |x|^{\sigma-n}$, $x \in \mathbb{R}^n$. The *Bessel kernel* g_σ, $\sigma > 0$, is defined to be that function on $\mathbb{R}^n$ whose Fourier transform is

$$\widehat{g}_\sigma(x) = (2\pi)^{-n/2}(1+|x|^2)^{-\sigma/2},$$

where the Fourier transform of a function f is given by

$$\widehat{f}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot y} f(y)dy.$$

It is known that g_σ is a positive, integrable function which is analytic except at the origin (cf. [AS] or [Z]).

Let $\sigma > 0$, $p \in (1,\infty)$, $q \in [1,\infty]$, and $\alpha_1,\ldots,\alpha_m \in \mathbb{R}$. The *logarithmic Bessel potential space* $H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$ is defined by

$$H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) := \{u = g_\sigma * f;\, f \in L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)\},$$

and is equipped with the (quasi-) norm

$$\|u\|_{\sigma;p,q;\alpha_1,\ldots,\alpha_m} := \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}.$$

Note that for $\alpha_j = 0$, $j = 1,\ldots,m$, $H^\sigma L_{p,p;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$ is simply the (fractional) Sobolev space of order σ.

When $k \in \mathbb{N}$, $p,q \in (1,\infty)$, and $\alpha_1,\ldots,\alpha_m \in \mathbb{R}$ then, by [EGO IV, Theorem 4.2], the space $H^k L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$ equals to

$$W^k L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) := \{u;\ D^\alpha u \in L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \text{ if } |\alpha| \leqslant k\},$$

equipped with the (quasi-) norm

$$\sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{p,q;\alpha_1,\ldots,\alpha_m},$$

and the corresponding (quasi-) norms are equivalent.

By a Young function Φ we mean a continuous, non-negative, strictly increasing and convex function on $[0,\infty)$ satisfying

$$\lim_{t\to 0_+} \Phi(t)/t = \lim_{t\to+\infty} t/\Phi(t) = 0.$$

Given a Young function Φ and any measurable subset Ω of $\mathbb{R}^n$, $L_\Phi(\Omega)$ will denote the corresponding Orlicz space, equipped with the Orlicz norm $\|\cdot\|_\Phi = \|\cdot\|_{\Phi,\Omega}$; for details of such spaces we refer to [A].

Let Φ_1 and Φ_2 be Young functions. Recall that Φ_2 *dominates* Φ_1 *globally* if there exists a positive constant k such that

$$\Phi_1(t) \leqslant \Phi_2(kt) \tag{2.4}$$

holds for all $t \geqslant 0$. Similarly, Φ_2 dominates Φ_1 *near infinity* (*near the origin*) if there exist positive constants k and T such that (2.4) holds for all $t \in (T,\infty)$ (for all $t \in (0,T)$). Two Young functions are said to be *equivalent globally* (*near infinity* or *near the origin*) if each dominates the other globally (near infinity or near the origin). It is easy to see that if Φ_2 dominates (is equivalent to) Φ_1 near infinity and near the origin, then Φ_2 dominates (is equivalent to) Φ_1 globally.

We have from [A, Theorem 8.12]: If Φ_1 and Φ_2 are equivalent globally (or near infinity and $|\Omega| < \infty$), then

$$L_{\Phi_1}(\Omega) = L_{\Phi_2}(\Omega) \tag{2.5}$$

and the corresponding norms are equivalent. In particular, (2.5) holds if there are T_0 and T_∞, $0 < T_0 < T_\infty < \infty$, such that

$$\Phi_1(t) = \Phi_2(t) \quad \text{for all} \quad t \in (0, T_0) \cup (T_\infty, \infty).$$

Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding $id : X \to Y$ is continuous. The norm of the embedding is

$$\|id\| = \|id\|_{X \to Y} = \sup_{\|f\|_X \leqslant 1} \|f\|_Y.$$

For two non-negative expressions (i.e. functions or functionals) F_1 and F_2 we shall write $F_1 \lesssim F_2$ whenever $F_1 \leqslant CF_2$ for some constant $C \in (0, \infty)$ independent of the variables in the expressions F_1 and F_2. If $F_1 \lesssim F_2$ and $F_2 \lesssim F_1$, we write $F_1 \approx F_2$.

We shall adopt the convention that $a/\infty = 0$ and $a/0 = \infty$ for all $a \in (0, \infty)$. If $p \in [1, \infty]$, the conjugate number p' is given by $1/p + 1/p' = 1$.

If $m \in \mathbb{N}$, we define

$$\exp_m = \underbrace{\exp \circ \exp \circ \ldots \circ \exp}_{m \text{ times}}.$$

For the formal reason we put

$$\ell_0(t) = \max(t, t^{-1}), \qquad t \in (0, \infty), \tag{2.6}$$

and, if $m = 1$,

$$\prod_{j=1}^{m-1} \ell_j(t) = 1 \qquad , \qquad t \in (0, \infty).$$

For $\rho \in (0, \infty)$ and $x \in \mathbb{R}^n$ let $B_n(x, \rho)$ denote the open ball in $\mathbb{R}^n$ of radius ρ and center x. The symbol $\mathscr{B}(B_n(0,1))$ stands for the set of all bounded measurable functions on $\mathbb{R}^n$ with supports in $\overline{B_n(0,1)}$ (the closure of $B_n(0,1)$). By κ_n we denote the surface area of the unit ball in $\mathbb{R}^n$.

3. Main theorems and examples

Our first result concerns global limiting embeddings of logarithmic Bessel potential spaces into Orlicz spaces and reads as

3.1. THEOREM. *Let* $\sigma \in (0,n)$, $p \in [1,\infty]$, *and* $m \in \mathbb{N}$. *Let* $\alpha_m < 1/p'$, $\alpha = \alpha_m - 1/p'$ *and, if* $m > 1$, *let* $\alpha_j = 1/p'$ *for* $j = 1,\dots,m-1$. *Suppose that* $q \in [p,\infty)$ *and that one of the following conditions is satisfied:*

$$q > n/\sigma; \tag{3.1}$$

$$q = n/\sigma, \quad p > 1, \quad m > 1; \tag{3.2}$$

$$q = n/\sigma, \quad p > 1, \quad m = 1, \quad \alpha_m \geqslant 0; \tag{3.3}$$

$$q = n/\sigma, \quad p = 1, \quad \alpha_m \geqslant 0; \tag{3.4}$$

Then

$$H^\sigma L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow L_\Phi(\mathbb{R}^n), \tag{3.5}$$

where the Young function Φ *is given by*

$$\Phi(t) = \begin{cases} t^q & \text{for all small enough} \quad t \geqslant 0 \\ \exp_m t^{-\frac{1}{\alpha}} & \text{for all large enough} \quad t > 0. \end{cases} \tag{3.6}$$

3.2. COROLLARY. *Let all the assumptions of Theorem 3.1 be satisfied. Then*

$$H^\sigma L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).$$

3.3. REMARK. Let all the assumptions of Theorem 3.1 be satisfied. Using the method of [EGO III], one can prove:

(1) (i) The embedding (3.5) is not compact.
(2) (ii) The space $H^\sigma_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$ is not continuously embedded in any Orlicz space $L_\Psi(\mathbb{R}^n)$, where Ψ dominates Φ near infinity.

The next theorem provides estimates of norms of the embeddings in Corollary 3.2. These estimates are sharp in their dependence on q provided that q is large enough.

3.4. THEOREM. *Let* $\sigma \in (0,n)$, $p \in [1,\infty]$, *and* $m \in \mathbb{N}$. *Let* $\alpha_m < 1/p'$, $\alpha = \alpha_m - 1/p'$ *and, if* $m > 1$, *let* $\alpha_j = 1/p'$ *for* $j = 1,\dots,m-1$. *Put*

$$X(\mathbb{R}^n) = H^\sigma L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n).$$

Then for all sufficiently large q,

$$X(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \tag{3.7}$$

and

$$\|id\|_{X(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \approx \ell_{m-1}^{-\alpha}(q). \tag{3.8}$$

3.5. EXAMPLE. Suppose that $p \in (1,\infty)$, $\sigma = n/p$, and

$$\begin{cases} \text{either} & q \in (p,\infty), \quad \beta \in (-\infty, 1/p') \\ \text{or} & q = p, \qquad \beta \in [0, 1/p'). \end{cases} \tag{3.9}$$

Then, by Theorem 3.1,

$$H^{n/p}L^p(\log L)^\beta(\mathbb{R}^n) \hookrightarrow L_\Phi(\mathbb{R}^n), \tag{3.10}$$

where

$$\Phi(t) = \begin{cases} t^q & \text{for small} \quad t \geqslant 0, \\ \exp t^{\frac{n}{n(1-\beta)-\sigma}} & \text{for large} \quad t > 0. \end{cases}$$

If, in addition, $p = n > 1$, we have from (3.9) and (3.10) that

$$W^1 L^n (\log L)^\beta (\mathbb{R}^n) = H^1 L^n (\log L)^\beta (\mathbb{R}^n) \hookrightarrow L_\Phi(\mathbb{R}^n), \tag{3.11}$$

where

$$\Phi(t) = \begin{cases} t^q & \text{for small} \quad t \geqslant 0, \\ \exp t^{\frac{n}{n(1-\beta)-1}} & \text{for large} \quad t > 0, \end{cases}$$

provided that

$$\begin{cases} \text{either} \quad q \in (n, \infty) \quad \text{and} \quad \beta \in (-\infty, 1/n') \\ \text{or} \quad q = n, \quad \text{and} \quad \beta \in [0, 1/n'). \end{cases} \tag{3.12}$$

In particular, if $\beta = 0$, we have

$$W^{1,n}(\mathbb{R}^n) = W^1 L^n (\mathbb{R}^n) \hookrightarrow L_\Phi(\mathbb{R}^n), \tag{3.13}$$

where

$$\Phi(t) = \begin{cases} t^q & \text{for small} \quad t \geqslant 0, \\ \exp t^{n'} & \text{for large} \quad t > 0, \end{cases}$$

provided that $q \in [n, \infty)$, which is the result corresponding to that of [A, Section 8.26] or [EE].

Let $\|id_1\|$, $\|id_2\|$, and $\|id_3\|$, respectively, stands for the norm of the embedding

$$H^{n/p} L^p (\log L)^\beta (\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n),$$
$$W^1 L^n (\log L)^\beta (\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n),$$

and

$$W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).$$

Then, by Theorem 3.4, for all large q,

$$\|id_1\| \approx q^{1/p'-\beta} \qquad \text{if} \quad \beta < 1/p',$$
$$\|id_2\| \approx q^{1/n'-\beta} \qquad \text{if} \quad \beta < 1/n',$$

and

$$\|id_3\| \approx q^{1/n'}.$$

3.6. EXAMPLE. Let $p \in (1,\infty)$, $\sigma = n/p$, $q \in [p,\infty)$ and $\beta \in (-\infty, 1/p')$. Then, by Theorem 3.1,

$$H^{n/p}L^p(\log L)^{1/p'}(\log\log L)^{\beta}(\mathbb{R}^n) \hookrightarrow L_{\Phi}(\mathbb{R}^n), \tag{3.14}$$

where

$$\Phi(t) = \begin{cases} t^q & \text{for small} \quad t \geqslant 0, \\ \exp\exp t^{\frac{n}{n(1-\beta)-\sigma}} & \text{for large} \quad t > 0. \end{cases}$$

If, moreover, $p = n > 1$, we have from (3.14) that

$$W^1L^n(\log L)^{1/n'}(\log\log L)^{\beta}(\mathbb{R}^n) \hookrightarrow L_{\Phi}(\mathbb{R}^n), \tag{3.15}$$

where

$$\Phi(t) = \begin{cases} t^q & \text{for small} \quad t \geqslant 0, \\ \exp\exp t^{\frac{n}{n(1-\beta)-1}} & \text{for large} \quad t > 0, \end{cases}$$

provided that $q \in [n,\infty)$ and $\beta \in (-\infty, 1/n')$. In particular, if $\beta = 0$, we obtain

$$W^1L^n(\log L)^{1/n'} \hookrightarrow L_{\Phi}(\mathbb{R}^n), \tag{3.16}$$

where

$$\Phi(t) = \begin{cases} t^q & \text{for small} \quad t \geqslant 0, \\ \exp\exp t^{n'} & \text{for large} \quad t > 0, \end{cases}$$

provided that $q \in [n,\infty)$.

Let $\|id_1\|$, $\|id_2\|$, and $\|id_3\|$, respectively, stands for the norm of the embedding

$$H^{n/p}L^p(\log L)^{1/p'}(\log\log L)^{\beta}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n),$$
$$W^1L^n(\log L)^{1/n'}(\log\log L)^{\beta}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n),$$

and

$$W^1L^n(\log L)^{1/n'}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).$$

Then, by Theorem 3.4, for all large q,

$$\|id_1\| \approx (\log q)^{1/p'-\beta} \quad \text{if} \quad \beta < 1/p',$$
$$\|id_2\| \approx (\log q)^{1/n'-\beta} \quad \text{if} \quad \beta < 1/n',$$

and

$$\|id_3\| \approx (\log q)^{1/n'}.$$

4. Proofs of Theorem 3.1 and Corollary 3.2

To prove Theorem 3.1, we need the following lemmas.

4.1. LEMMA. *Let Φ be a Young function and let X be a (quasi-) normed linear space satisfying $X \subset \mathscr{M}(\mathbb{R}^n, \mu_n)$. Assume that there exist t_0 and t_∞, $0 < t_0 < t_\infty < \infty$, such that for all $u \in X$ with $\|u\|_X \leqslant 1$,*

$$\int_0^{t_0} \Phi(u^*(t))dt \lesssim 1 \quad \textit{and} \quad \int_{t_\infty}^{\infty} \Phi(u^*(t))dt \lesssim 1. \tag{4.1}$$

Then

$$X \hookrightarrow L_\Phi(\mathbb{R}^n). \tag{4.2}$$

Proof. Let $u \in X$, $\|u\|_X \leqslant 1$. We have (cf. [BS])

$$\int_{\mathbb{R}^n} \Phi(|u(x)|)dx = \int_0^{t_0} \Phi(u^*(t))dt + \int_{t_0}^{t_\infty} \Phi(u^*(t))dt + \int_{t_\infty}^{\infty} \Phi(u^*(t))dt.$$

Using monotonicity of $\Phi \circ u^*$, we obtain

$$\int_{t_0}^{t_\infty} \Phi(u^*(t))dt \leqslant (t_\infty - t_0)\ \Phi(u^*(t_0)) \leqslant \frac{t_\infty - t_0}{t_0} \int_0^{t_0} \Phi(u^*(t))dt.$$

Consequently,

$$\int_{\mathbb{R}^n} \Phi(|u(x)|)dx \lesssim \int_0^{t_0} \Phi(u^*(t))dt + \int_{t_\infty}^{\infty} \Phi(u^*(t))dt,$$

which, together with (4.1), yields

$$\int_{\mathbb{R}^n} \Phi(|u(x)|)dx \lesssim 1.$$

Since, by Young's inequality

$$\|u\|_\Phi \leqslant \int_{\mathbb{R}^n} \Phi(|u(x)|)dx + 1,$$

we obtain that $\|u\|_\Phi \lesssim 1$ and (4.2) follows. □

The next lemma provides us with an estimate for the non-increasing rearrangement of the Bessel kernel g_σ.

4.2. LEMMA. *Let $\sigma \in (0, n)$. Then there is $B \in (0, \infty)$ such that*

$$g_\sigma^*(t) \lesssim t^{\sigma/n-1} \exp(-Bt^{1/n}) \quad , \quad t > 0, \tag{4.3}$$

and

$$g_\sigma^{**}(t) \lesssim \begin{cases} t^{\sigma/n-1}, & t \in (0,1], \\ t^{-1}, & t \in (1, \infty). \end{cases} \tag{4.4}$$

Proof. The estimate (4.3) is proved in [EGO II, Lemma 3.5]. Moreover, we have from (4.3) that

$$g_\sigma^{**}(t) = t^{-1}\int_0^t g_\sigma^*(\tau)d\tau \lesssim t^{\sigma/n-1} \quad \text{if} \quad t \in (0,1],$$

and

$$g_\sigma^{**}(t) = t^{-1}\Big[\int_0^1 g_\sigma^*(\tau)d\tau + \int_1^t g_\sigma^*(\tau)d\tau\Big] \approx t^{-1} \quad \text{if} \quad t \in (1,\infty). \qquad \square$$

Using the well-known criterion for the validity of Hardy's inequality (cf. [OK]), one can prove the following lemma.

4.3. LEMMA. *Let* $\sigma \in (0,n)$, $p \in [1,\infty]$, $m \in \mathbb{N}$, *and* $\alpha_1,\dots,\alpha_m \in \mathbb{R}$. *Suppose that* $q \in [p,\infty]$ *and that one of the following conditions is satisfied:*

$$q > \frac{n}{\sigma};$$
$$q = \frac{n}{\sigma}, \quad \alpha_1 > 0;$$
$$q = \frac{n}{\sigma}, \quad \alpha_1 = 0, \quad \alpha_2 > 0;$$
$$\vdots$$
$$q = \frac{n}{\sigma}, \quad \alpha_1 = \alpha_2 = \ldots = \alpha_{m-2} = 0, \quad \alpha_{m-1} > 0;$$
$$q = \frac{n}{\sigma}, \quad \alpha_1 = \alpha_2 = \ldots = \alpha_{m-2} = \alpha_{m-1} = 0, \quad \alpha_m \geqslant 0.$$

Then there is a constant $C \in (0,\infty)$ *such that for all* $h \in \mathscr{M}^+(1,\infty)$,

$$\Big\| t^{-1}\int_1^t h(\tau)d\tau \Big\|_{q,(1,\infty)} \leqslant C\, \Big\| t^{\frac{\sigma}{n}-\frac{1}{p}}\Big(\prod_{j=1}^m \ell_j^{\alpha_j}(t)\Big) h(t)\Big\|_{p,(1,\infty)} \tag{4.5}$$

and

$$\Big\| \int_t^\infty h(\tau)d\tau \Big\|_{q,(1,\infty)} \leqslant C\, \Big\| t^{\frac{\sigma}{n}+\frac{1}{p'}}\Big(\prod_{j=1}^m \ell_j^{\alpha_j}(t)\Big) h(t)\Big\|_{p,(1,\infty)}. \tag{4.6}$$

The next lemma provides an estimate of f^* for f from generalized Lorentz-Zygmund space.

4.4. LEMMA. *Let* $m \in \mathbb{N}$, $\alpha_1,\dots,\alpha_m \in \mathbb{R}$ *and let* $r \in (0,\infty)$ *and* $p \in (0,\infty]$, *or* $r = \infty = p$. *Then there exists a constant* $c \in (0,\infty)$ *such that for every* $f \in L_{r,p;\alpha_1,\dots,\alpha_m}$ *and all* $t \in (0,\infty)$,

$$f^*(t) \leqslant c\, t^{-1/r}\Big(\prod_{j=1}^m \ell_j^{-\alpha_j}(t)\Big)\|f\|_{r,p;\alpha_1,\dots,\alpha_m}.$$

The proof is similar to that of [EGO I, Lemma 3.3], where the case $r \in (1,\infty)$, $p \in [1,\infty]$ was treated. $\square$

Proof of Theorem 3.1. Put $L = L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$ and $X = H^\sigma L$. Let $u \in X$, $\|u\|_X \leqslant 1$. Then $u = g_\sigma * f$, where $f \in L$ and $\|f\|_L = \|u\|_X \leqslant 1$.

The proof will be given in three steps:

Step 1. We prove that for all such u,

$$\int_0^1 \exp_m(u^*(t)^{-\frac{1}{\alpha}})\,dt \lesssim 1 \tag{4.7}$$

and

$$\int_1^\infty u^*(t)^q dt \lesssim 1. \tag{4.8}$$

The estimate (4.7) with $m = 2$ was proved in [EGO II, Lemmas 4.2 and 4.1]; the proof for a general $m \in \mathbb{N}$ is analogous.

Since $1 \geqslant \|f\|_L$, the estimate (4.8) will be proved if we show that

$$\|u^*(t)\|_{q,(1,\infty)} \lesssim \|f\|_L. \tag{4.9}$$

We have by O'Neil's lemma (see [O, Lemma 1.5] or [Z, Lemma 1.8.8]) that

$$u^*(t) \leqslant u^{**}(t) \leqslant t g_\sigma^{**}(t) f^{**}(t) + \int_t^\infty g_\sigma^*(\tau) f^*(\tau)\,d\tau.$$

Consequently,

$$\begin{aligned}\|u^*(t)\|_{q,(1,\infty)} &\leqslant \|t g_\sigma^{**}(t) f^{**}(t)\|_{q,(1,\infty)} + \Big\|\int_t^\infty g_\sigma^*(\tau) f^*(\tau)\,d\tau\Big\|_{q,(1,\infty)} \\ &=: N_1 + N_2.\end{aligned} \tag{4.10}$$

Using (4.4), we obtain

$$\begin{aligned}N_1 &\lesssim \|f^{**}(t)\|_{q,(1,\infty)} = \Big\| t^{-1}\Big(\int_0^1 f^*(\tau)d\tau + \int_1^t f^*(\tau)d\tau\Big)\Big\|_{q,(1,\infty)} \\ &\leqslant \Big(\int_0^1 f^*(\tau)d\tau\Big)\|t^{-1}\|_{q,(1,\infty)} + \Big\|t^{-1}\int_1^t f^*(\tau)d\tau\Big\|_{q,(1,\infty)} \\ &=: N_{11} + N_{12}.\end{aligned} \tag{4.11}$$

Since $q \in (1,\infty)$, we have $\|t^{-1}\|_{q,(1,\infty)} \approx 1$, and thus, by Hölder's inequality,

$$\begin{aligned}N_{11} &\lesssim \int_0^1 f^*(\tau)d\tau \\ &= \int_0^1 \Big[\tau^{\frac{\sigma}{n}-\frac{1}{p}}\Big(\prod_{j=1}^m \ell_j^{\alpha_j}(\tau)\Big) f^*(\tau)\Big]\Big[\tau^{\frac{1}{p}-\frac{\sigma}{n}}\prod_{j=1}^m \ell_j^{-\alpha_j}(\tau)\Big]d\tau \\ &\leqslant \|f\|_L \Big\|\tau^{1-\frac{\sigma}{n}-\frac{1}{p'}}\prod_{j=1}^m \ell_j^{-\alpha_j}(\tau)\Big\|_{p',(0,1)} \approx \|f\|_L.\end{aligned} \tag{4.12}$$

Applying Lemma 4.3 (the estimate (4.5)), we obtain

$$N_{12} \lesssim \Big\| t^{\frac{\sigma}{n}-\frac{1}{p}} \Big(\prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \Big) f^*(t) \Big\|_{p,(1,\infty)} \leqslant \|f\|_L.$$

Together with (4.12) this yields

$$N_1 \lesssim \|f\|_L. \tag{4.13}$$

Using Lemma 4.3 (the inequality (4.6)), the estimate (4.3), and the fact that

$$t^{\frac{\sigma}{n}} \exp(-Bt^{\frac{1}{n}}) \lesssim 1 \quad \text{for all} \quad t \in (1,\infty),$$

we arrive at

$$\begin{aligned} N_2 &\lesssim \Big\| t^{\frac{\sigma}{n}+\frac{1}{p'}} \Big(\prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \Big) g_\sigma^*(t) f^*(t) \Big\|_{p,(1,\infty)} \\ &\lesssim \Big\| t^{\frac{\sigma}{n}+\frac{1}{p'}} \Big(\prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \Big) t^{\frac{\sigma}{n}-1} \exp(-Bt^{\frac{1}{n}}) f^*(t) \Big\|_{p,(1,\infty)} \\ &\lesssim \Big\| t^{\frac{\sigma}{n}-\frac{1}{p}} \Big(\prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \Big) f^*(t) \Big\|_{p,(1,\infty)} \leqslant \|f\|_L \end{aligned} \tag{4.14}$$

and (4.9) follows from (4.10), (4.13), and (4.14).

Step 2. We prove that there is $A \in (0,\infty)$ such that

$$u^*(1) \leqslant A \quad \text{for every} \quad u \in X \quad \text{with} \quad \|u\|_X \leqslant 1. \tag{4.15}$$

Take $u \in X$ with $\|u\|_X \leqslant 1$. Then $u = g_\sigma * f$, with $\|f\|_L \leqslant 1$ and O'Neil's lemma, together with (4.4), implies

$$u^*(1) \lesssim f^{**}(1) + \int_1^\infty g_\sigma^*(\tau) f^*(\tau)\, d\tau. \tag{4.16}$$

We have from (4.12) that

$$f^{**}(1) \leqslant A_1 \|f\|_L, \tag{4.17}$$

where $A_1 = \|\tau^{1-\frac{\sigma}{n}-\frac{1}{p'}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(\tau)\|_{p',(0,1)}$. By Lemma 4.4 there is a constant $c \in (0,\infty)$ (independent of f) such that for all $t > 0$,

$$f^*(t) \leqslant c t^{-\frac{\sigma}{n}} \Big(\prod_{j=1}^{m} \ell_j^{-\alpha_j}(t) \Big) \|f\|_L. \tag{4.18}$$

Since $\|f\|_L \leqslant 1$, we have

$$f^*(t) \leqslant c t^{-\frac{\sigma}{n}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(t) \qquad \text{for all} \quad t > 0.$$

Together with (4.3) this yields

$$\int_1^\infty g_\sigma^*(\tau)f^*(\tau)d\tau \leqslant c\int_1^\infty \tau^{-1}\Big(\prod_{j=1}^m \ell_j^{-\alpha_j}(\tau)\Big)\exp(-B\tau^{\frac{1}{n}})d\tau =: A_2 < \infty. \quad (4.19)$$

Thus, we have from (4.16), (4.17), and (4.19) that (4.15) holds with $A = A_1 + A_2$.

Step 3. By Lemma 4.1, to prove Theorem 3.1, it is enough to verify (4.1) with some t_0 and t_∞ satisfying $0 < t_0 < t_\infty < \infty$.

We have from (3.5) that there are T_0 and T_∞, $0 < T_0 < T_\infty < \infty$ such that

$$\Phi(t) = \begin{cases} t^q & , \quad t \in [0, T_0] \\ \exp_m t^{-\frac{1}{\alpha}} & , \quad t \in [T_\infty, \infty). \end{cases} \quad (4.20)$$

Take some $t_0 \in (0,1)$ and $t_\infty \in (1,\infty)$. Let $u \in X$, $\|u\|_X \leqslant 1$.

If

$$u^*(t) > T_\infty \qquad \text{for all} \quad t \in (0, t_0), \quad (4.21)$$

then (4.20) and (4.7) imply

$$\int_0^{t_0} \Phi(u^*(t))dt = \int_0^{t_0} \exp_m(u^*(t)^{-\frac{1}{\alpha}})\, dt \leqslant \int_0^1 \exp_m(u^*(t)^{-\frac{1}{\alpha}})\, dt \lesssim 1.$$

(Since $|\{t > 0;\ u^*(t) > T_\infty\}|_1 = |\{x \in \mathbb{R}^n; |u(x)| > T_\infty\}|_n = \mu_u(T_\infty)$, where $\mu_u = \mu_u(\lambda)$ stands for the distribution function of u, we have that (4.21) is equivalent to $t_0 \leqslant \mu_u(T_\infty)$.)

If (4.21) does not hold, then $\mu_u(T_\infty) < t_0$ and (4.20), (4.7), and the fact that $u^*(\mu_u(T_\infty)) \leqslant T_\infty$ imply

$$\begin{aligned} \int_0^{t_0} \Phi(u^*(t))dt &= \int_0^{\mu_u(T_\infty)} \dots\, dt + \int_{\mu_u(T_\infty)}^{t_0} \dots\, dt \\ &= \int_0^{\mu_u(T_\infty)} \exp_m(u^*(t)^{-\frac{1}{\alpha}})\, dt + \int_{\mu_u(T_\infty)}^{t_0} \Phi(u^*(t))dt \\ &\leqslant \int_0^1 \exp_m(u^*(t)^{-\frac{1}{\alpha}})\, dt + t_0\Phi(u^*(\mu_u(T_\infty))) \\ &\lesssim 1 + t_0\Phi(T_\infty) \approx 1. \end{aligned}$$

It remains to verify the second estimate in (4.1). To this end we make use of (4.15).

If the constant A from (4.15) satisfies $A \leqslant T_0$, we obtain from (4.15), (4.20), and (4.8) that

$$\int_{t_\infty}^\infty \Phi(u^*(t))dt = \int_{t_\infty}^\infty u^*(t)^q dt \leqslant \int_1^\infty u^*(t)^q dt \lesssim 1$$

for all $u \in X$ with $\|u\|_X \leqslant 1$.

Assume that $A > T_0$. Taking $y \in (T_0, A]$, we have with $C_1 = \Phi(A)/T_0^q$ that

$$\Phi(y) \leqslant \Phi(A) = C_1 T_0^q \leqslant C_1 y^q.$$

Together with (4.20) this implies that

$$\Phi(y) \leqslant Cy^q \qquad \text{for all} \quad y \in [0, A], \tag{4.22}$$

where $C = \max\{1, C_1\}$. Finally, using (4.15), the monotonicity of u^*, (4.22), and (4.8), we obtain

$$\int_{t_\infty}^{\infty} \Phi(u^*(t))dt \leqslant \int_1^{\infty} \Phi(u^*(t))dt \leqslant C \int_1^{\infty} u^*(t)^q dt \lesssim 1$$

for all $u \in X$, $\|u\|_X \leqslant 1$, and the proof is complete. □

Proof of Corollary 3.2. Since the Young function Φ from (3.6) dominates globally the Young function Φ_1 given by $\Phi_1(t) = t^q$, $t \in [0, \infty)$, the result follows from (3.5) by [A, Theorem 8.12]. □

5. Proof of Theorem 3.4.

Note that the embedding (3.7) of Theorem 3.4 follows from Corollary 3.2. To prove the estimate (3.8), we need several lemmas.

5.1. LEMMA. *Let $m \in \mathbb{N}$ and $\nu > 0$. Then there is a constant $C_1 \in (0, \infty)$ such that for all $s \in (0, 1)$,*

$$\sup_{q \in [1,\infty)} \ell_{m-1}^{-\nu}(q) s^{1/q} \leqslant C_1 \ell_m^{-\nu}(s). \tag{5.1}$$

Proof. Let $m \in \mathbb{N}$ and $\nu > 0$. Let $s_0 \in (0, 1)$ *be fixed.*

First we show that

$$\sup_{q \in [1,\infty)} \ell_{m-1}^{-\nu}(q) s^{1/q} \lesssim \ell_m^{-\nu}(s) \quad \text{for all} \quad s \in [s_0, 1). \tag{5.2}$$

If $q \in [1, \infty)$, then $\ell_{m-1}(q) \geqslant 1$, which implies that

$$\ell_{m-1}^{-\nu}(q) \leqslant 1 \qquad \text{for all} \quad q \in [1, \infty). \tag{5.3}$$

The function ℓ_m is decreasing on $(0, 1)$ and hence

$$\ell_m^{-\nu}(s_0) \leqslant \ell_m^{-\nu}(s) \quad \text{for all} \quad s \in [s_0, 1). \tag{5.4}$$

Since $s^{1/q} \leqslant 1$ for all $q \in [1, \infty)$ and all $s \in (0, 1)$, we have from (5.3) and (5.4),

$$\sup_{q \in [1,\infty)} \ell_{m-1}^{-\nu}(q) s^{1/q} \leqslant 1 \approx \ell_m^{-\nu}(s_0) \leqslant \ell_m^{-\nu}(s) \quad \text{for all} \quad s \in [s_0, 1),$$

which is (5.2).

It remains to show that for some $s_0 \in (0, 1)$,

$$\sup_{q \in [1,\infty)} \ell_{m-1}^{-\nu}(q) s^{1/q} \lesssim \ell_m^{-\nu}(s) \quad \text{for all} \quad s \in (0, s_0). \tag{5.5}$$

Take $s_0 \in (0,1)$ such that

$$s_0 < e^{-\nu}. \tag{5.6}$$

We claim that for any $j \in \mathbb{N}$,

$$\ell_{j-1}\Big(\frac{\nu}{-\log s}\Big) \approx \ell_j(s) \qquad \text{for all} \quad s \in (0, s_0). \tag{5.7}$$

Note that it suffices to verify (5.7) for $j = 1, 2$; the other cases follow by the induction applying the definition (2.1) of ℓ_k $(k = 1, 2, \ldots)$. Since $0 < \nu/(-\log s) < \nu/(-\log s_0) < 1$ for $s \in (0, s_0)$, we have (cf. (2.6))

$$\ell_0\Big(\frac{\nu}{-\log s}\Big) = \frac{-\log s}{\nu} \approx 1 - \log s = \ell_1(s) \quad \text{for all} \quad s \in (0, s_0), \tag{5.8}$$

which is (5.7) with $j = 1$. The case $j = 2$ follows from the fact that

$$\lim_{s \to 0_+} \ell_1\Big(\frac{\nu}{-\log s}\Big)/\ell_2(s) = 1.$$

For a fixed $s \in (0, s_0)$ we define the function

$$F_s(\tau) = \ell_{m-1}^{-\nu}\Big(\frac{\tau}{-\log s}\Big)e^{-\tau}, \qquad \tau \in (0, -\log s], \tag{5.9}$$

and we want to prove that

$$F_s(\tau) \leqslant C\ell_m^{-\nu}(s), \quad \tau \in (0, -\log s] \tag{5.10}$$

with a constant C independent of s and τ.

As $\nu < -\log s$ (cf. (5.6)), we may split the interval $(0, -\log s]$ into the intervals $(0, \nu)$ and $[\nu, -\log s]$. Consider first $\tau \in (0, \nu)$. Then

$$\tau/(-\log s) < \nu/(-\log s) < 1. \tag{5.11}$$

Since the function $\ell_{m-1}^{-\nu}$ is increasing on (0,1), we have from (5.11) and (5.7) that for all $\tau \in (0, \nu)$,

$$F_s(\tau) = \ell_{m-1}^{-\nu}\Big(\frac{\tau}{-\log s}\Big)e^{-\tau} \leqslant \ell_{m-1}^{-\nu}\Big(\frac{\nu}{-\log s}\Big) \approx \ell_m^{-\nu}(s). \tag{5.12}$$

Using (2.2), we obtain for all $\tau \in (0, -\log s)$ that

$$\begin{aligned}\frac{dF_s}{d\tau}(\tau) &= e^{-\tau}\ell_{m-1}^{-\nu}\Big(\frac{\tau}{-\log s}\Big)\Big[-1 + \nu\Big(\prod_{j=1}^{m-1}\ell_j\Big(\frac{\tau}{-\log s}\Big)\Big)^{-1}\tau^{-1}\Big] \\ &\leqslant e^{-\tau}\ell_{m-1}^{-\nu}\Big(\frac{\tau}{-\log s}\Big)\Big[-1 + \frac{\nu}{\tau}\Big]\end{aligned}$$

and consequently, $F_s(\tau)$ is decreasing on $[\nu, -\log s]$. Thus, for all $\tau \in [\nu, -\log s]$,

$$F_s(\tau) \leqslant F_s(\nu) = \ell_{m-1}^{-\nu}\Big(\frac{\nu}{-\log s}\Big)e^{-\nu} \leqslant \ell_{m-1}^{-\nu}\Big(\frac{\nu}{-\log s}\Big) \approx \ell_m^{-\nu}(s). \tag{5.13}$$

The estimate (5.10) follows from (5.12) and (5.13).

Taking $\tau = \tau_{s,q} = (-\log s)/q$ with $q \in [1,\infty)$, we have $\tau_{s,q} \in (0, -\log s]$ and, by (5.10) and (5.9),

$$\ell_m^{-\nu}(s) \gtrsim F_s(\tau_{s,q}) = \ell_{m-1}^{-\nu}\Big(\frac{1}{q}\Big)e^{-\tau_{s,q}} = \ell_{m-1}^{-\nu}\Big(\frac{1}{q}\Big)s^{1/q} = \ell_{m-1}^{-\nu}(q)s^{1/q}$$

and (5.5) follows. □

The proof of the next lemma is analogous to that of [EGO II, Lemma 4.1], where the case $m = 2$ was treated.

5.2. LEMMA. *Let $\sigma \in (0,n)$, $p \in [1,\infty]$, and $m \in \mathbb{N}$. Let $\alpha_m < 1/p'$, $\alpha = \alpha_m - 1/p'$ and, if $m > 1$, let $\alpha_j = 1/p'$ for $j = 1,\dots,m-1$. Then there exists a positive constant C_2 such that for all $u \in H^\sigma L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$,*

$$\sup_{s\in(0,1)} \ell_m^{\alpha}(s)u^*(s) \leqslant C_2\|u\|_{\sigma;n/\sigma,p;\alpha_1,\dots,\alpha_m}. \tag{5.14}$$

An *upper estimate* of the norm of embedding (3.7) is given in the following assertion.

5.3. LEMMA. *Let all the assumptions of Theorem 3.1 be satisfied. Then there exists a constant $c \in (0,\infty)$ such that for all $u \in H^\sigma L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$,*

$$\|u\|_{q,\mathbb{R}^n} \leqslant c\, \ell_{m-1}^{-\alpha}(q)\|u\|_{\sigma;n/\sigma,p;\alpha_1,\dots,\alpha_m} \tag{5.15}$$

(where $\|\cdot\|_{q,\mathbb{R}^n}$ stands for the norm in the Lebesgue space $L^q(\mathbb{R}^n)$).

Proof. Put $X = H^\sigma L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$ and take $u \in X$. Then

$$\|u\|_{q,\mathbb{R}^n} = \Big(\int_0^\infty u^*(s)^q ds\Big)^{1/q} \leqslant I_1(u) + I_2(u), \tag{5.16}$$

where

$$I_1(u) = \|u^*\|_{q,(0,1)} \qquad \text{and} \qquad I_2(u) = \|u^*\|_{q,(1,\infty)}. \tag{5.17}$$

Using Lemmas 5.1 and 5.2, we have for all $u \in X$ and all $q \in [1,\infty)$,

$$\begin{aligned}\sup_{s\in(0,1)} s^{1/q}u^*(s) &\leqslant \ell_{m-1}^{-\alpha}(q) \sup_{s\in(0,1)} \big[\sup_{\rho\in[1,\infty)} \ell_{m-1}^{\alpha}(\rho)s^{1/\rho}\big]\, u^*(s)\\ &\leqslant C_1\ell_{m-1}^{-\alpha}(q) \sup_{s\in(0,1)} \ell_m^{\alpha}(s)\, u^*(s) \leqslant C_1C_2\ell_{m-1}^{-\alpha}(q)\|u\|_X.\end{aligned}$$

Consequently, for all $u \in X$ and all $q \in [1,\infty)$,

$$\begin{aligned}I_1(u) &= \Big(\int_0^1 s^{1/2}[s^{1/(2q)}u^*(s)]^q \frac{ds}{s}\Big)^{1/q} \\ &\leqslant C_1C_2\ell_{m-1}^{-\alpha}(2q)\|u\|_X\Big(\int_0^1 s^{-1/2}ds\Big)^{1/q} \lesssim \ell_{m-1}^{-\alpha}(q)\|u\|_X.\end{aligned} \tag{5.18}$$

Since $u \in X$, we have $u = g_\sigma * f$ with $f \in L := L_{n/\sigma,p;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$ and $\|u\|_X = \|f\|_L$. Using (4.10), (4.13), and (4.14), we obtain

$$I_2(u) \lesssim \|f\|_L = \|u\|_X$$

Together with (5.18), (5.17), and (5.16) this implies that for all $u \in X$ and all $q \in (\max\{p, n/\sigma\}, \infty)$,

$$\|u\|_{q,\mathbb{R}^n} \lesssim (\ell_{m-1}^{-\alpha}(q) + 1)\|u\|_X \leqslant 2\ell_{m-1}^{-\alpha}(q)\|u\|_X$$

and the result follows. □

To find a *lower estimate* of the norm of embedding (3.7) we need the following lemma.

5.4. LEMMA. *Let g be a positive function which is continuous on $(0,1]$ and non-increasing in some interval $(0,r_0] \subset (0,1]$. Let $\sigma \in (0,n)$, $p \in [1,\infty]$, $m \in \mathbb{N}$, and $\alpha_1,\dots,\alpha_m \in \mathbb{R}$. Then there exists a number $r_1 \in (0,r_0)$ such that the functions h_r, $r \in (0,r_1)$, defined in $\mathbb{R}^n$ by*

$$h_r(y) = \begin{cases} g(|y|) & , \quad r < |y| < 1 \\ 0 & , \quad \textit{otherwise} \end{cases} \tag{5.19}$$

satisfy

$$(I_\sigma * h_r)(x) = \kappa_n \int_{|x|}^{|x|/r} \Big(\frac{|x|}{\tau}\Big)^\sigma g\Big(\frac{|x|}{\tau}\Big) v_\sigma(\tau)d\tau + \kappa_n \int_r^1 t^{\sigma-1} g(t)dt, \qquad |x| < r, \tag{5.20}$$

where $v_\sigma \in L^1(0,\infty)$,

$$\|h_r\|_{n/\sigma,\infty;\alpha_1,\dots,\alpha_m} \lesssim \sup_{t\in[r,1)} t^\sigma g(t) \prod_{j=1}^m \ell_j^{\alpha_j}(t), \tag{5.21}$$

and, if $p \in [1,\infty)$,

$$\|h_r\|_{n/\sigma,p;\alpha_1,\dots,\alpha_m} \lesssim V_1(r) + V_2(r) \tag{5.22}$$

with

$$V_1(r) = \Big(\int_{2r}^1 \Big[t^\sigma g(t) \prod_{j=1}^m \ell_j^{\alpha_j}(t)\Big]^p \frac{dt}{t}\Big)^{1/p}$$

and

$$V_2(r) = r^\sigma g(r) \prod_{j=1}^m \ell_j^{\alpha_j}(r).$$

Proof is an easy modification of that of [EGO III, Lemma 4.1], where we replace the function $L(s)$ by

$$L_m(s) = s^{p\frac{\sigma}{n}-1} \prod_{j=1}^m \ell_j^{p\alpha_j}(s) \quad , \quad s \in (0,\infty) \quad (p \in [1,\infty)) \,,$$

and the function $\mathcal{L}(t)$ by

$$\mathcal{L}_m(t) = t^{\frac{\sigma}{n}} \prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \quad , \quad t \in (0,\infty). \qquad \square$$

5.5. COROLLARY. *Suppose that* $\sigma \in (0,n)$, $p \in [1,\infty]$, *and* $m \in \mathbb{N}$. *Let* $\alpha_m < 1/p'$ *and, if* $m > 1$, *let* $\alpha_j = 1/p'$ *for* $j = 1,\ldots,m-1$. *For* $\gamma < \alpha_m + 1/p$ *put*

$$g(t) = t^{-\sigma}\Big(\prod_{j=1}^{m-1} \ell_j(t)\Big)^{-1} \ell_m^{-\gamma}(t) \quad , \qquad t \in (0,1]. \tag{5.23}$$

Then there exists $r_1 \in (0,1]$ *such that for all* $r \in (0,r_1)$ *the function* h_r *defined by* (5.19) *satisfy:*

$$(I_\sigma * h_r)(x) \gtrsim \ell_m^{1-\gamma}(r) \quad , \qquad |x| < r, \tag{5.24}$$

$$\|h_r\|_{n/\sigma,p;\alpha_1,\ldots,\alpha_m} \lesssim \ell_m^{\alpha_m-\gamma+1/p}(r). \tag{5.25}$$

Proof is analogous to that of [EGO III, Examples 4.2] and thus it is omitted. $\square$

5.6. COROLLARY. *Let* $\sigma \in (0,n)$, $p \in [1,\infty]$, *and* $m \in \mathbb{N}$. *Let* $\alpha_m < 1/p'$, $\alpha = \alpha_m - 1/p'$ *and, if* $m > 1$, *let* $\alpha_j = 1/p'$ *for* $j = 1,\ldots,m-1$. *Then there exist a number* $r_1 \in (0,1)$ *and non-negative functions* $f_r \in \mathscr{B}(B_n(0,1))$, $r \in (0,r_1)$, *such that*

$$(I_\sigma * f_r)(x) \gtrsim \ell_m^{-\alpha}(r) \qquad \text{for all} \quad x \in B_n(0,r) \tag{5.26}$$

and

$$\|f_r\|_{n/\sigma,p;\alpha_1,\ldots,\alpha_m} \leqslant 1. \tag{5.27}$$

Proof. Take h_r from Corollary 5.5. Then, by (5.25) and (5.24), there are constants $c, C \in (0,\infty)$ independent of $r \in (0,r_1)$ such that

$$(I_\sigma * h_r)(x) \geqslant c\ell_m^{1-\gamma}(r) \quad , \quad |x| < r,$$

and

$$\|h_r\|_{n/\sigma,p;\alpha_1,\ldots,\alpha_m} \leqslant C\ell_m^{\alpha_m-\gamma+1/p}(r).$$

Hence, putting $f_r = C^{-1}\ell_m^{\gamma-1/p-\alpha_m}(r)h_r$, we obtain (5.26) and (5.27). $\square$

Now, we are able to establish a *lower estimate* of the norm of embedding (3.7).

5.7. LEMMA. *Let* $\sigma \in (0,n)$, $p \in [1,\infty]$, *and* $m \in \mathbb{N}$. *Let* $\alpha_m < 1/p'$, $\alpha = \alpha_m - 1/p'$ *and, if* $m > 1$, *let* $\alpha_j = 1/p'$ *for* $j = 1,\ldots,m-1$. *Then there is a constant* $C \in (0,\infty)$ *such that for any sufficiently large* $q \in (1,\infty)$ *there exists a function* $u_q \in H^\sigma L_{n/\sigma,p;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$ *satisfying*

$$\|u_q\|_{\sigma;n/\sigma,p;\alpha_1,\ldots,\alpha_m} \leqslant 1 \quad \text{and} \quad \|u_q\|_{q,\mathbb{R}^n} \geqslant C\ell_{m-1}^{-\alpha}(q).$$

Proof. By Corollary 5.6, there exist $r_1 \in (0,1)$ and functions $f_r \in \mathscr{B}(B_n(0,1))$, $r \in (0,r_1)$, such that (5.27) and (5.26) hold. Putting $F_r = g_\sigma * f_r$, $r \in (0,r_1)$, and using the fact that

$$(g_\sigma * f_r)(x) \approx (I_\sigma * f_r)(x) \quad , \qquad |x| < 1,$$

we obtain

$$\|F_r\|_{\sigma;n/\sigma,p;\alpha_1,\dots,\alpha_m} = \|f_r\|_{n/\sigma,p;\alpha_1,\dots,\alpha_m} \leqslant 1$$

and

$$F_r(x) \gtrsim \ell_m^{-\alpha}(r) \qquad \text{for all} \quad x \in B_n(0,r). \tag{5.28}$$

Choose $q_1 \in (1,\infty)$ such that $\exp(1-q_1) = r_1$. If $q \in (q_1,\infty)$, set

$$r = \exp(1-q). \tag{5.29}$$

Obviously, $r \in (0,r_1)$ and the function $u_q := F_r$ satisfies

$$\|u_q\|_{\sigma;n/\sigma,p;\alpha_1,\dots,\alpha_m} \leqslant 1$$

and (cf. (5.28))

$$\|u_q\|_{q,\mathbb{R}^n} \geqslant \Big(\int_{\{x;|x|<r\}} |u_q(x)|^q dx\Big)^{1/q} = \Big(\int_{\{x;|x|<r\}} |F_r(x)|^q dx\Big)^{1/q}$$
$$\gtrsim \ell_m^{-\alpha}(r)|B_n(0,1)|^{1/q} r^{n/q} \gtrsim \ell_m^{-\alpha}(r)$$

since $|B_n(0,1)|^{1/q} \to 1$ as $q \to \infty$ and, by (5.29), $r^{n/q} = e^{n(1-q)/q} \to e^{-n}$ as $q \to \infty$. □

Proof of Theorem 3.4. Theorem 3.4 follows from Lemmas 5.3 and 5.7. □

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