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THE s–CONVEX ORDERS AMONG REAL RANDOM VARIABLES, WITH APPLICATIONS

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Abstract. In this paper, new classes of stochastic order relations are introduced. These can be seen as extensions of the usual convex order and are closely related to the orderings discussed in Lefèvre and Utev (1996), as well as to the stochastic dominances in economics and stop-loss orders in actuarial sciences. These classes are studied in detail, including properties, characterizations, sufficient conditions, and extrema with respect to these orderings in different sets of distribution functions. Some applications illustrate the theory.

1. Introduction and motivation

Let X and Y be, respectively, the waiting times of a typical customer in two adjacent businesses which sell similar items for about the same price. If $X \preceq_{\mathrm{st}} Y$ (that is, if X is smaller than Y in the stochastic dominance; see the exact definition below) then a customer will usually prefer the first business because then, in particular, $EX \leqslant EY$. Thus, in order not to lose customers, the second business can then be expected to improve its queueing procedure; in other words, the economic competition can be expected to lead to $EX = EY$. Given that $EX = EY$, if now $X \preceq_{\mathrm{cx}} Y$ (that is, if X is smaller than Y in the convex order; see the exact definition below) then a typical customer, in order to avoid possible long waits, will still prefer the first business because then, in particular, $EX^2 \leqslant EY^2$ (that is, $\mathrm{Var}(X) \leqslant \mathrm{Var}(Y)$). Again, economic competition can be expected then to lead to $EX^2 = EY^2$. In such a situation, when the first two moments of X and Y are, respectively, the same, neither the stochastic dominance, nor the convex order, can be used to compare these waiting times. Indeed, if $X \preceq_{\mathrm{st}} Y$ (respectively, $X \preceq_{\mathrm{cx}} Y$) and $EX = EY$ (respectively, $EX^2 = EY^2$), then X and Y are necessarily identically distributed (see Proposition 3.8 below). The aim of this study is to introduce new concepts of stochastic dominance which allow the comparison of random variables with identical first moments. In other words, we

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will study in this paper the generalization of the usual (that is, *non-monotone*) convex and concave stochastic orders to higher degrees of convexity and concavity. Such comparisons can be used, in particular, to derive extrema in specific sets of distribution functions. For example, we can obtain the "best" and the "worst" waiting times among those having the same first moments. As we will see, these stochastic extrema will furnish useful numerical bounds on quantities that are hard to compute.

Many attempts have been made in the past twenty years to extend the classical *increasing* convex and concave orders into hierarchical classes of stochastic order relations. Essentially, two approaches to this problem have been used: the first one is by Rolski (1976) and Fishburn (1976, 1980) who based their study on iterative integrals of the distribution and survival functions, and then identified classes of real functions that define their orderings as integral stochastic order relations (see (1) below for the definition of this notion); the second one, arising for instance in the economic and actuarial literature, is based on the expected utility theory and defines the stochastic orderings by means of various cones of functions using the operator expectation (see (1)). The second approach can be found, for example, in Levy (1992), Kaas, Van Heerwaarden and Goovaerts (1994) or Lefèvre and Utev (1996).

Quite surprisingly, to the best of our knowledge, the usual convex and concave orders seem not to have been generalized so far. Our aim here is to propose extensions of these orders, that we will call the s-convex and s-concave orders. We will examine their relationship to the s-increasing convex order (called stop-loss dominance of order $s-1$ in the actuarial literature and s-convex orders in Lefèvre and Utev (1996)) as well as to the s-increasing concave orders (called sth degree stochastic dominance in economics and in the financial literature, and s-concave orders in Lefèvre and Utev (1996)). We will then derive some properties and some characterizations of these orderings. Finally, we will obtain some extrema with respect to these new stochastic order relations. These extrema yield interesting applications in various fields of applied probability.

In the sequel, F_X will denote the distribution function of a random variable X, $\overline{F}_X$ its survival function, that is, $\overline{F}_X \equiv 1 - F_X$, and f_X its probability density function with respect to some dominating measure μ.

For any subinterval $\mathscr{S}$ of the real line $\mathbb{R}$ let $\mathscr{C}^s(\mathscr{S})$ denote the class of all the functions $\phi : \mathscr{S} \to \mathbb{R}$ such that their sth derivative, $\phi^{(s)}$, exists and is continuous on $\mathscr{S}$; $\mathscr{S}$ may be open, half-open, or closed, finite or infinite. We will denote respectively by a and b the left and right endpoints of $\mathscr{S}$ when they are necessarily finite. For any real value y let $y_+ = \frac{|y|+y}{2}$, that is, $y_+ = y$ if $y \geqslant 0$ and 0 otherwise (with the understanding that $y_+^0 = 1$ if $y \geqslant 0$ and 0 otherwise), and let $y_- = \frac{|y|-y}{2}$, that is $y_- = -y$ if $y \leqslant 0$ and $y_- = 0$ otherwise.

2. Preliminaries and definitions

In this section we introduce some new classes of partial orders defined on the set (or on a suitable subset thereof) of all distribution functions of real-valued random variables (that is, binary relations satisfying the reflexivity, the transitivity and the anti-symmetry properties on this set). As it is more intuitive to speak about random variables than about distribution functions, we will say that the random variables X and Y are ordered if their respective distribution functions F_X and F_Y are ordered. However, let us note that by dealing with random variables instead of with distribution functions, we lose the anti-symmetry property (see, for example, Stoyan (1983)).

The stochastic order relations that are studied in this paper are called integral stochastic orders (for instance, by Whitt (1986), Marshall (1991) and Müller (1997)). Such stochastic order relations are defined by a reference to a class $\mathscr{U}^{\mathscr{S}}$ of real functions $\phi : \mathscr{S} \to \mathbb{R}$, satisfying some desirable properties (usually $\mathscr{U}^{\mathscr{S}}$ is a convex cone), by saying that the random variable X is $\mathscr{U}^{\mathscr{S}}$-smaller than the random variable Y if

$$E\phi(X) \leqslant E\phi(Y) \qquad \text{for all} \quad \phi \in \mathscr{U}^{\mathscr{S}} \tag{1}$$

for which the expectations exist. In practice, $\mathscr{S}$ is the common support of the distribution functions of the random variables X and Y, or the smallest subset of $\mathbb{R}$ containing their respective supports.

Let us first recall the notions of the usual stochastic dominance as well as of the (increasing) convex and (increasing) concave orderings (see, for instance, Shaked and Shanthikumar (1994), Sections 1.A, 2.A and 3.A). Let X and Y be two random variables whose distribution functions have supports in $\mathscr{S}$. Then X is said to be smaller than Y in the usual stochastic dominance, denoted by $X \preceq_{\text{st}} Y$, if (1) holds with the class $\mathscr{U}^{\mathscr{S}}$ of the non-decreasing functions over $\mathscr{S}$; while X is said to be smaller than Y in the convex (respectively, concave) order, denoted by $X \preceq_{\text{cx}} Y$ (respectively, $X \preceq_{\text{cv}} Y$), if (1) holds with the class $\mathscr{U}^{\mathscr{S}}$ of the convex (respectively, concave) functions on $\mathscr{S}$. Furthermore, X is said to be smaller than Y in the increasing convex (respectively, increasing concave) sense, denoted by $X \preceq_{\text{icx}} Y$ (respectively, $X \preceq_{\text{icv}} Y$), if (1) holds with the class $\mathscr{U}^{\mathscr{S}}$ of the non-decreasing convex (respectively, non-decreasing concave) functions on $\mathscr{S}$.

The reader may wonder why, in the notation $\mathscr{U}^{\mathscr{S}}$, we make explicit the dependence on $\mathscr{S}$. This dependence is fundamental for the orders studied in this paper. Denuit, Lefèvre and Utev (1997) and Denuit and Lefèvre (1997) exploited a particular structure of the support (when it is an arithmetic grid) for the purpose of defining more efficient orders than those obtained by simply taking $\mathscr{S} = \mathbb{R}$.

In this paper we study some extensions of the stochastic dominance as well as of the convex and concave orders. In order to generalize these classical stochastic order relations, we opt for an extension of the non-decreasing and of the convex (concave) functions which generate $\preceq_{\text{st}}$ and $\preceq_{\text{cx}}$ ($\preceq_{\text{cv}}$). For this purpose it is useful to note that a real-valued function ϕ is non-decreasing on its domain $\mathscr{S}$ if, and only if,

$$\begin{vmatrix} 1 & 1 \\ \phi(x_0) & \phi(x_1) \end{vmatrix} \geqslant 0 \qquad \text{whenever} \quad x_0 < x_1 \in \mathscr{S}, \tag{2}$$

while it is convex if, and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ \phi(x_0) & \phi(x_1) & \phi(x_2) \end{vmatrix} \geqslant 0 \qquad \text{whenever} \quad x_0 < x_1 < x_2 \in \mathscr{S}. \tag{3}$$

A natural extension of (2) and (3) consists of considering real functions ϕ for which

$$\delta_s(x_0, x_1, \dots, x_s; \phi) \equiv \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_s \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{s-1} & x_1^{s-1} & \dots & x_s^{s-1} \\ \phi(x_0) & \phi(x_1) & \dots & \phi(x_s) \end{vmatrix} \geqslant 0$$
$$\text{whenever}\, x_0 < x_1 < \cdots < x_s \in \mathscr{S}. \tag{4}$$

This leads to the following definition. A real-valued function ϕ is said to be s-convex on its domain $\mathscr{S}$ if, and only if, for all choices of $s+1$ distinct points $x_0 < x_1 < \cdots < x_s$ in $\mathscr{S}$ we have $\delta_s(x_0, x_1, \dots, x_s; \phi) \geqslant 0$.

Now, a function ϕ is said to be concave if (3) holds with the sign reversed. We thus may define s-concavity as follows. A real-valued function ϕ is said to be s-concave on its domain $\mathscr{S}$ if, and only if, for all choices of $s+1$ distinct points $x_0 < x_1 < \cdots < x_s$ in $\mathscr{S}$ we have $(-1)^{s+1}\delta_s(x_0, x_1, \dots, x_s; \phi) \geqslant 0$. We mention that our definition of s-concavity differs from the one usually found in the literature (see, for example, Popoviciu (1933) and Bullen (1971)): classically, ϕ is s-concave if, and only if, $-\phi$ is s-convex, whereas here we have that $\phi(\cdot)$ is s-concave if, and only if, $-\phi(-\cdot)$ is s-convex. The only functions ϕ such that both ϕ and $-\phi$ are s-convex (respectively, s-concave) in the present sense are the polynomials of degree at most $s-1$. The 1-convex (respectively, 1-concave) functions are thus the non-decreasing functions, while the 2-convex (respectively, 2-concave) functions are the usual convex (respectively, concave) functions (the 0-convex functions should be the non-negative functions over $\mathscr{S}$, but we will only consider $s \geqslant 1$ in the sequel). Below we denote by $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}}$ (respectively, $\mathscr{U}_{s\text{-cv}}^{\mathscr{S}}$) the class of all the s-convex (respectively, s-concave) functions $\phi : \mathscr{S} \to \mathbb{R}$.

The s-convex functions are classical in interpolation theory where they are called convex of order s by Popoviciu (1933), and convex with respect to the Tchebycheff system $\{1, x, x^2, \dots, x^{s-1}\}$ by Karlin and Novikoff (1963) and by Karlin and Studden (1966). See Roberts and Varberg (1973) for other terminology.

We now list some basic properties of $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}}$. The class $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}}$ is a convex cone (that is, if ϕ_1 and ϕ_2 both belong to $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}}$, then, for all non-negative α and β we have that $\alpha\phi_1 + \beta\phi_2$ also belongs to $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}}$) which is closed in the topology of pointwise convergence. If $\phi \in \mathscr{U}_{s\text{-cx}}^{\mathscr{S}}$ where $s \geqslant 2$, then $\phi^{(k)}$ exists, is continuous and it belongs to $\mathscr{U}_{(s-k)\text{-cx}}^{\mathscr{S}}$ for $1 \leqslant k \leqslant s-2$. In particular, $\phi^{(s-2)}$ exists, is convex and therefore it has left and right derivatives in the interior of $\mathscr{S}$, both of them being non-decreasing (see Bullen (1971), Theorem 7(a) and Corollary 15(a), as well as Roberts and Varberg (1973), Theorem B on page 5 and Theorem A on page 238). If $\phi^{(s)}$ exists, then $\phi \in \mathscr{U}^{\mathscr{S}}_{s\text{-cx}}$ if, and only if, $\phi^{(s)} \geqslant 0$. Moreover, $\mathscr{U}^{\mathscr{S}}_{s\text{-cx}}$ consists of all limits of sequences of functions in the class $\mathscr{C}^s(\mathscr{S})$ for which $\phi^{(s)} \geqslant 0$ (see Karlin and Studden (1966), Chapter XI, Example 1.4). The s-convex functions ϕ on $\mathscr{S}$ can also be characterized by their position with respect to the graphs of the associated Lagrange interpolation polynomials of degree $s-1$ (see Popoviciu (1933) or Bullen (1971, Theorem 5)).

Let us denote by ψ_s the function defined by

$$\psi_s(x) = x^s, \tag{5}$$

$s = 1, 2, \ldots$. Also, let $\psi_{s-1,t,+}$ denote the function defined by

$$\psi_{s-1,t,+}(x) = (x-t)_+^{s-1}. \tag{6}$$

Popoviciu (1942) showed that the functions $\pm\psi_k$, $k = 0, 1, \ldots, s-1$, and $\psi_{s-1,t,+}$, $t \in \mathscr{S}$, span $\mathscr{U}^{\mathscr{S}}_{s\text{-cx}}$. More precisely, for $n \geqslant s$, let the function φ_n be of the form

$$\varphi_n(x) = \sum_{j=0}^{s-1} \alpha_j x^j + \sum_{j=0}^{n-s} \beta_j (x-t_j)_+^{s-1}, \tag{7}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are real constants, $\beta_0, \beta_2, \ldots, \beta_{n-s}$ are non-negative constants, and $t_0 < t_1 < \cdots < t_{n-s} \in \mathscr{S}$. Then every $\phi \in \mathscr{U}^{\mathscr{S}}_{s\text{-cx}}$ is the uniform limit of a sequence $\{\varphi_n,\ n \geqslant s\}$, where the φ_n's are of the form (7).

Let us now shift our attention to monotone s-convex functions. A function ϕ is said to be increasing convex if, and only if, it is simultaneously non-decreasing and convex, that is, if, and only if, (2) and (3) are simultaneously satisfied. Analogously, a function ϕ is said to be increasing concave if it is non-decreasing and concave, that is, if (2) and (3) with the sign reversed are satisfied. In order to extend the notion of increasing convexity (respectively, concavity), it is natural to require that the $\delta_k(x_0, x_1, \ldots, x_k; \phi)$'s (defined in (4)) are non-negative for $k = 1, 2, \ldots, s$ (respectively, are of alternating sign for $k = 1, 2, \ldots, s$ (of non-positive sign for even k and of non-negative sign for odd k)). This leads to the following definition. A real-valued function ϕ is said to be s-increasing convex on its domain $\mathscr{S}$ if, and only if, for all choices of $k+1$ distinct points $x_0 < x_1 < \cdots < x_k$ in $\mathscr{S}$, we have $\delta_k(x_0, x_1, \ldots, x_k; \phi) \geqslant 0$, $k = 1, 2, \ldots, s$. Analogously, we say that ϕ is s-increasing concave if, and only if, $(-1)^{k+1}\delta_k(x_0, x_1, \ldots, x_k; \phi) \geqslant 0$, $k = 1, 2, \ldots, s$.

We denote by $\mathscr{U}^{\mathscr{S}}_{s\text{-icx}}$ (respectively, $\mathscr{U}^{\mathscr{S}}_{s\text{-icv}}$) the class of the s-increasing convex (respectively s-increasing concave) functions on $\mathscr{S}$. It is easily seen that

$$\mathscr{U}^{\mathscr{S}}_{s\text{-icx}} = \bigcap_{k=1}^{s} \mathscr{U}^{\mathscr{S}}_{k\text{-cx}} \qquad \text{and} \qquad \mathscr{U}^{\mathscr{S}}_{s\text{-icv}} = \bigcap_{k=1}^{s} \mathscr{U}^{\mathscr{S}}_{k\text{-cv}}. \tag{8}$$

The 1-increasing convex (respectively, 1-increasing concave) functions are the non-decreasing functions, while the 2-increasing convex (respectively, 2-increasing concave) functions are the usual increasing convex (respectively, increasing concave) functions.

The higher degree convex functions have already been utilized in the statistical literature. See, for instance, Oja (1981), Cambanis and Simons (1982), and Johnson, Kotz and Balakrishnan (1995, page 677).

Consider the case in which a (the left endpoint of $\mathscr{S}$) is finite. Let us denote by $\psi_{s,a} : \mathscr{S} \to \mathbb{R}^+$ the function defined by $\psi_{s,a}(x) = (x-a)^s$, $s = 1, 2, \ldots$. Popoviciu (1942) showed that in this case (when a is finite)

$$\text{the functions } \psi_{k,a},\ k = 0, \ldots, s-1, \text{ and } \psi_{s-1,t,+},\ t \in \mathscr{S}, \text{ span } \mathscr{U}^{\mathscr{S}}_{s\text{-icx}}. \quad (9)$$

In this paper we study the classes of stochastic order relations which are given in Definition 2.1 below. In the sequel, s denotes an integer greater or equal to 1 and all the random variables are assumed to possess a finite $(s-1)$ st moment.

DEFINITION 2.1. Let X and Y be two random variables that take on values in $\mathscr{S}$. Then X is said to be smaller than Y in the s-convex (respectively, s-concave, s-increasing convex, s-increasing concave) order, denoted by $X \preceq^{\mathscr{S}}_{s\text{-cx}} Y$ (respectively, $X \preceq^{\mathscr{S}}_{s\text{-cv}} Y$, $X \preceq^{\mathscr{S}}_{s\text{-icx}} Y$, $X \preceq^{\mathscr{S}}_{s\text{-icv}} Y$), if (1) holds with $\mathscr{U}^{\mathscr{S}} = \mathscr{U}^{\mathscr{S}}_{s\text{-cx}}$ (respectively, $\mathscr{U}^{\mathscr{S}} = \mathscr{U}^{\mathscr{S}}_{s\text{-cv}}$, $\mathscr{U}^{\mathscr{S}} = \mathscr{U}^{\mathscr{S}}_{s\text{-icx}}$, $\mathscr{U}^{\mathscr{S}} = \mathscr{U}^{\mathscr{S}}_{s\text{-icv}}$).

Note that $\preceq^{\mathscr{S}}_{1\text{-cx}} \Longleftrightarrow \preceq^{\mathscr{S}}_{1\text{-icx}} \Longleftrightarrow \preceq_{\text{st}}$, $\preceq^{\mathscr{S}}_{2\text{-cx}} \Longleftrightarrow \preceq_{\text{cx}}$, $\preceq^{\mathscr{S}}_{2\text{-cv}} \Longleftrightarrow \preceq_{\text{cv}}$, $\preceq^{\mathscr{S}}_{2\text{-icx}} \Longleftrightarrow \preceq_{\text{icx}}$, and $\preceq^{\mathscr{S}}_{2\text{-icv}} \Longleftrightarrow \preceq_{\text{icv}}$.

From now on, we will focus on the s-convex and the s-increasing convex orders, since, for any real-valued random variables X and Y, we have that

$$X \preceq^{\mathscr{S}}_{s\text{-icx}} Y \Longleftrightarrow -Y \preceq^{-\mathscr{S}}_{s\text{-icv}} -X, \quad (10)$$

where $-\mathscr{S} = \{x \in \mathbb{R} | -x \in \mathscr{S}\}$, as $\phi(\cdot) \in \mathscr{U}^{\mathscr{S}}_{s\text{-icx}} \Longleftrightarrow -\phi(-\cdot) \in \mathscr{U}^{-\mathscr{S}}_{s\text{-icv}}$, and

$$X \preceq^{\mathscr{S}}_{s\text{-cx}} Y \Longleftrightarrow \begin{cases} X \preceq^{\mathscr{S}}_{s\text{-cv}} Y & \text{when } s \text{ is odd,} \\ Y \preceq^{\mathscr{S}}_{s\text{-cv}} X & \text{when } s \text{ is even,} \end{cases} \quad (11)$$

since $\mathscr{U}^{\mathscr{S}}_{s\text{-cx}} = \mathscr{U}^{\mathscr{S}}_{s\text{-cv}}$ when s is odd, and $\phi \in \mathscr{U}^{\mathscr{S}}_{s\text{-cx}} \Longleftrightarrow -\phi \in \mathscr{U}^{\mathscr{S}}_{s\text{-cv}}$ when s is even. So, all the results obtained for the convex case can be easily adapted to the concave one. Note that, when $s = 1$ and 2, then (10) and (11) reduce to the well-known results concerning stochastic dominance, convex and concave orders, as well as increasing convex and increasing concave orders (see, for example, Theorems 1.A.3(a) and 3.A.11 of Shaked and Shanthikumar (1994)).

3. Characterizations and properties

In this section we first give several characterizations which generalize known characterizations of the usual convex order. We will need the following terminology and lemma. Let $\phi : \mathscr{S} \to \mathbb{R}$ and let $x_0 < x_1 < \cdots < x_s \in \mathscr{S}$, and define recursively, starting from

$$[x_i; \phi] = \phi(x_i), \qquad i = 0, 1, \ldots, s,$$

the *divided differences* of order s by

$$[x_0, x_1, \ldots, x_s; \phi] = \frac{[x_1, x_2, \ldots, x_s; \phi] - [x_0, x_1, \ldots, x_{s-1}; \phi]}{x_s - x_0}; \tag{1}$$

see, for example, Pečarić, Proschan and Tong (1992, page 14).

The following lemma shows that, in order to identify functions in $\mathcal{U}^{\mathcal{S}}_{s\text{-cx}}$ or in $\mathcal{U}^{\mathcal{S}}_{s\text{-icx}}$, one can check the signs of some divided differences rather than the signs of the determinants in (4). Its proof can be found, for example, in Denuit, Lefèvre and Utev (1997). Recall the notation ψ_s from (5).

LEMMA 3.1. *Let $\phi : \mathcal{S} \to \mathbb{R}$, let s be a positive integer, and let $x_0 < x_1 < \cdots < x_s \in \mathcal{S}$. Then*

$$[x_0, x_1, \ldots, x_s; \phi] = \frac{\delta_s(x_0, x_1, \ldots, x_s; \phi)}{\delta_s(x_0, x_1, \ldots, x_s; \psi_s)},$$

where $\delta_s(x_0, x_1, \ldots, x_s; \psi_s)$ is the well-known Vandermonde's determinant, so that

$$[x_0, x_1, \ldots, x_s; \phi] = \frac{\delta_s(x_0, x_1, \ldots, x_s; \phi)}{\Pi^s_{i,j=0; i>j}(x_i - x_j)}.$$

This leads to the characterization below which extends to the s-convex orders the relations (1.A.1) and (2.A.5) in Shaked and Shanthikumar (1994).

THEOREM 3.2. (shifted truncated moments characterization) *Let X and Y be two random variables that take on values in $\mathcal{S}$. Then*

$$X \preceq^{\mathcal{S}}_{s\text{-}cx} Y \Longleftrightarrow \begin{cases} EX^k = EY^k, \quad k = 1, 2, \ldots, s-1, \quad \text{and} \\ E(X-t)^{s-1}_+ \leqslant E(Y-t)^{s-1}_+ \quad \text{for all } t \in \mathcal{S}, \end{cases} \tag{2}$$

and

$$X \preceq^{\mathcal{S}}_{s\text{-}cx} Y \Longleftrightarrow \begin{cases} EX^k = EY^k, \quad k = 1, 2, \ldots, s-1, \quad \text{and} \\ (-1)^s \left[E(t-Y)^{s-1}_+ - E(t-X)^{s-1}_+\right] \geqslant 0 \quad \text{for all } t \in \mathcal{S}. \end{cases} \tag{3}$$

Proof. We first prove the "$\Longrightarrow$"-part of (2). It is easily seen that the functions $\pm\psi_k$, $k = 1, \ldots, s-1$, all belong to $\mathcal{U}^{\mathcal{S}}_{s\text{-cx}}$ (just by noting that $\delta_s(x_0, x_1, \ldots, x_s; \psi_k) = 0$ for $k = 1, 2, \ldots, s-1$). What remains to be proven is that the function $\psi_{s-1,t,+}$ (see (6)) also belongs to $\mathcal{U}^{\mathcal{S}}_{s\text{-cx}}$ for any $t \in \mathcal{S}$. In fact, we will prove that $\psi_{s-1,t,+} \in \mathcal{U}^{\mathcal{S}}_{s\text{-icx}}$. The result is obvious for $s = 1$. We now proceed by induction. Suppose that the property holds for $\psi_{k-1,t,+}$ and let us establish it for $\psi_{k,t,+}$. Note that $\psi_{k,t,+}(x) = (x-t)\psi_{k-1,t,+}(x)$ for any x. Let $x_0 < x_1 < \cdots < x_{k+1} \in \mathcal{S}$. If $x_{k+1} < t$, then $\delta_{k+1}(x_0, x_1, \ldots, x_{k+1}; \psi_{k,t,+}) = 0$. Similarly, if $x_0 > t$ we have

$$\delta_{k+1}(x_0, x_1, \ldots, x_{k+1}; \psi_{k,t,+}) = \sum_{j=0}^{k} \binom{k}{j} (-t)^{k-j} \delta_{k+1}(x_0, x_1, \ldots, x_{k+1}; \psi_j) = 0.$$

If $x_0 \leqslant t \leqslant x_{k+1}$ we use the following formula (that can be found in Popoviciu (1940) and can be seen as an analog of the Leibniz formula for the divided differences). Let ϕ_1 and ϕ_2 be two real-valued functions defined on $\mathscr{S}$, and $y_0 < y_1 < \cdots < y_k \in \mathscr{S}$, then

$$[y_0, y_1, \ldots, y_k; \phi_1\phi_2] = \sum_{i=0}^{k}[y_0, y_1, \ldots, y_i; \phi_1][y_i, y_{i+1} \ldots, y_k; \phi_2].$$

We then get

$$\begin{aligned}
[x_0, x_1, \ldots, x_{k+1}; \psi_{k,t,+}] &= (x_0 - t)[x_0, x_1, \ldots, x_{k+1}; \psi_{k-1,t,+}] \\
&\quad + [x_1, x_2, \ldots, x_{k+1}; \psi_{k-1,t,+}] \\
&= (x_0 - t)\frac{[x_1, x_2, \ldots, x_{k+1}; \psi_{k-1,t,+}] - [x_0, x_1, \ldots, x_k; \psi_{k-1,t,+}]}{x_{k+1} - x_0} \\
&\quad + [x_1, x_2, \ldots, x_{k+1}; \psi_{k-1,t,+}] \\
&= \frac{(x_{k+1} - t)[x_1, x_2, \ldots, x_{k+1}; \psi_{k-1,t,+}]}{x_{k+1} - x_0} \\
&\quad + \frac{(t - x_0)[x_0, x_1, \ldots, x_k; \psi_{k-1,t,+}]}{x_{k+1} - x_0}.
\end{aligned}$$

The fact that $\psi_{k,t,+} \in \mathscr{U}^{\mathscr{S}}_{(k+1)\text{-icx}}$ now follows from Lemma 3.1.

In order to prove the "$\Longleftarrow$"-part of (2), let ϕ be an s-convex function. We know that ϕ is the uniform limit of the sequence $\{\varphi_n,\ n \geqslant s\}$ defined in (7). Thus

$$\begin{aligned}
|E\varphi_n(X) - E\phi(X)| &= \left|\int_{x\in\mathscr{S}} (\varphi_n(x) - \phi(x))\, dF_X(x)\right| \\
&\leqslant \int_{x\in\mathscr{S}} |\varphi_n(x) - \phi(x)|\, dF_X(x) \\
&\leqslant \sup_{x\in\mathscr{S}} |\varphi_n(x) - \phi(x)| \to 0 \text{ as } n \to \infty
\end{aligned}$$

since φ_n converges uniformly to ϕ. Similarly, $|E\varphi_n(Y) - E\phi(Y)| \to 0$ as $n \to \infty$. Now, every function φ_n in (7) is a linear combination, with non-negative coefficients, of the functions in the set $\{\pm\psi_0, \pm\psi_1, \ldots, \pm\psi_{s-1}, \psi_{s-1,t,+},\ t \in \mathscr{S}\}$. It follows that $E\phi(X) \leqslant E\phi(Y)$ for every $\phi \in \mathscr{U}^{\mathscr{S}}_{s\text{-cx}}$.

In order to prove (3) start from $(x - t)^{s-1} = ((x - t)_+ - (x - t)_-)^{s-1}$ to get

$$E(X - t)_+^{s-1} = \sum_{j=0}^{s-1}\binom{s-1}{j}EX^j(-t)^{s-j-1} + (-1)^s E(t - X)_+^{s-1} \tag{4}$$

(see (2.5) in Rachev and Rüschendorf (1990)). The characterization (3) now follows from (2) using the identity (4) and a similar identity involving Y.

□

In the actuarial literature (where the random variables under interest are mostly non-negative) the s-increasing convex order $\preceq^{\mathscr{S}}_{s\text{-icx}}$ (see Definition 2.1) is often defined or characterized by requiring (2) to hold with the inequalities $EX^k \leqslant EY^k$ replacing the equalities $EX^k = EY^k$, $k = 1, 2, \ldots, s-1$ (see, for example, Kaas, Van Heerwaarden and Goovaerts (1994)). This is indeed a reasonable definition for non-negative random variables. However, for more general random variables the situation is somewhat more complex. A discussion on the s-increasing convex order for general random variables is given later in this section (see Remark 3.6 below).

For any distribution function F let us denote $F^{[0]}(t) = F(t)$, and, for $k \geqslant 1$, denote $F^{[k]}(t) = \int_{x=-\infty}^{t} F^{[k-1]}(x)\,dx$. Similarly, denote $\overline{F}^{[0]}(t) = \overline{F}(t)$, and, for $k \geqslant 1$, denote $\overline{F}^{[k]}(t) = \int_{x=t}^{\infty} \overline{F}^{[k-1]}(x)\,dx$. The next result characterizes the order $\preceq^{\mathscr{S}}_{s\text{-cx}}$ by means of these iterated integrals; it extends Theorem 2.A.1 in Shaked and Shanthikumar (1994) to the s-convex case.

THEOREM 3.3. (iterated integrals characterization) *Let X and Y be two random variables that take on values in $\mathscr{S}$. Then*

$$X \preceq^{\mathscr{S}}_{s\text{-}cx} Y \Longleftrightarrow \begin{cases} EX^k = EY^k, \quad k = 1, 2, \ldots, s-1, \quad \textit{and} \\ (-1)^s \left[F_Y^{[s-1]}(t) - F_X^{[s-1]}(t)\right] \geqslant 0 \quad \textit{for all } t \in \mathbb{R}, \end{cases}$$

and

$$X \preceq^{\mathscr{S}}_{s\text{-}cx} Y \Longleftrightarrow \begin{cases} EX^k = EY^k, \quad k = 1, 2, \ldots, s-1, \quad \textit{and} \\ \overline{F}_Y^{[s-1]}(t) - \overline{F}_X^{[s-1]}(t) \geqslant 0 \quad \textit{for all } t \in \mathbb{R}. \end{cases}$$

Proof. These results follow from Theorem 3.2 using the identities

$$F_Y^{[s-1]}(t) - F_X^{[s-1]}(t) = \frac{E(t-Y)_+^{s-1} - E(t-X)_+^{s-1}}{(s-1)!} \tag{5}$$

and

$$\overline{F}_Y^{[s-1]}(t) - \overline{F}_X^{[s-1]}(t) = \frac{E(Y-t)_+^{s-1} - E(X-t)_+^{s-1}}{(s-1)!}, \tag{6}$$

which are easily proven by induction and Fubini's Theorem.

□

The next characterization extends results proposed by Müller (1996) for the usual stochastic order.

THEOREM 3.4. (monotonicity characterization) *Let X and Y be two random variables that take on values in $\mathscr{S}$. Then*

$$X \preceq^{\mathscr{S}}_{s\text{-}cx} Y \Longleftrightarrow \begin{cases} EX^k = EY^k, \quad k = 1, 2, \ldots, s-1, \quad \textit{and} \\ E(Y-t)_+^s - E(X-t)_+^s \quad \textit{is non-increasing in } t \in \mathbb{R}, \end{cases}$$

and

$$X \preceq^{\mathscr{S}}_{s\text{-}cx} Y \Longleftrightarrow \begin{cases} EX^k = EY^k, \quad k = 1, 2, \ldots, s-1, \quad \textit{and} \\ (-1)^s \left[E(t-Y)_+^s - E(t-X)_+^s\right] \quad \textit{is non-decreasing in } t \in \mathbb{R}. \end{cases}$$

Proof. These results follow from Theorem 3.3 using the identities

$$s! \int_{x=t}^{\infty} \left[\overline{F}_Y^{[s-1]}(x) - \overline{F}_X^{[s-1]}(x) \right] dx = E(Y-t)_+^s - E(X-t)_+^s$$

and

$$s! \int_{x=-\infty}^{t} \left[F_Y^{[s-1]}(x) - F_X^{[s-1]}(x) \right] dx = E(t-Y)_+^s - E(t-X)_+^s. \quad \square$$

In the next result it is shown that the s-convex orders can be characterized by means of $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}} \cap \mathscr{C}^s(\mathscr{S})$.

THEOREM 3.5. (continuous functions characterization) *Let X and Y be two random variables that take on values in $\mathscr{S}$. Then $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$ if, and only if,* (1) *holds with $\mathscr{U}^{\mathscr{S}} = \mathscr{U}_{s\text{-}cx}^{\mathscr{S}} \cap \mathscr{C}^s(\mathscr{S})$.*

Proof. The necessity part is obvious from Definition 2.1 using the fact that $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}} \cap \mathscr{C}^s(\mathscr{S}) \subseteq \mathscr{U}_{s\text{-cx}}^{\mathscr{S}}$. To prove the converse we note that the functions $\pm\psi_k$ (see (5)), $k = 1, 2, \ldots, s-1$, belong to $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}} \cap \mathscr{C}^s(\mathscr{S})$ and that $\psi_{s,t,+}$ (see (6)), $t \in \mathscr{S}$, can be obtained as the limit of a sequence $\{\varrho_n,\ n \geqslant 1\}$ of functions in $\mathscr{U}_{s\text{-cx}}^{\mathscr{S}} \cap \mathscr{C}^s(\mathscr{S})$. For example, we can define ϱ_n, for $x \in \mathbb{R}$, by

$$\varrho_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} (x-\xi)^{s-1} e^{-\frac{n^2}{2}(\xi-t)^2} d\xi, \qquad n \geqslant 1,$$

and note that

$$\varrho_n^{(s)}(x) = \frac{1}{\sqrt{2\pi}} (s-1)! e^{-\frac{n^2}{2}(x-t)^2} > 0$$

for any real x. The result then follows from (2).

$\square$

REMARK 3.6. Having the above characterizations of the s-convex orders, let us examine some possible characterizations of the s-increasing convex orders, and their relationship to similar orders in the literature. When a is finite, then from (9) it is seen that the analog of characterization (2) in the s-increasing convex case becomes

$$X \preceq_{s\text{-icx}}^{\mathscr{S}} Y \Longleftrightarrow \begin{cases} E(X-a)^k \leqslant E(Y-a)^k, \quad k = 1, 2, \ldots, s-1, \qquad \text{and} \\ E(X-t)_+^{s-1} \leqslant E(Y-t)_+^{s-1} \quad \text{for all } t \in \mathscr{S}. \end{cases} \tag{7}$$

However, this result has no meaning when $a = -\infty$. Thus, in our framework, if we want a characterization such as (7) to hold for $\preceq_{s\text{-icx}}^{\mathscr{S}}$ then we need a to be finite. This observation enables us to understand the links and the differences that exist between the s-increasing convex orders and the stochastic order relations introduced by Rolski (1976) (see also Remark 2.3 in Shaked and Wong (1995)). ◀

In the rest of this section we give some properties of the s-convex orders which are analogs of results in Sections 1.A and 2.A of Shaked and Shanthikumar (1994). First we examine the relations that exist between the moments of two s-convex ordered random variables. The following result follows at once from Definition 2.1.

PROPOSITION 3.7. *Let X and Y be random variables that take on values in $\mathscr{S}$. If $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$ then*

$$EX^k \leqslant EY^k \quad \text{for } k \geqslant s \text{ such that } k-s \text{ is even.}$$

If, moreover, X and Y are non-negative then

$$EX^k \leqslant EY^k \quad \text{for } k \geqslant s.$$

A useful property of the s-convex order is stated next; it generalizes Theorem 1.A.7 in Shaked and Shanthikumar (1994).

PROPOSITION 3.8. *Let X and Y be two random variables that take on values in $\mathscr{S}$. If $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$ and if $EX^s = EY^s$ then X and Y are identically distributed.*

Proof. From Theorem 3.3 it follows that

$$(-1)^s \left[F_Y^{[s-1]}(t) - F_X^{[s-1]}(t)\right] \geqslant 0 \quad \text{and} \quad \overline{F}_Y^{[s-1]}(t) - \overline{F}_X^{[s-1]}(t) \geqslant 0 \quad \text{for all } t \in \mathbb{R}.$$

Using (5) and (6) we get, after some computation, that

$$\int_{-\infty}^{0} (-1)^s \left[F_Y^{[s-1]}(t) - F_X^{[s-1]}(t)\right] dt + \int_0^\infty \left[\overline{F}_Y^{[s-1]}(t) - \overline{F}_X^{[s-1]}(t)\right] dt = \frac{EY^s - EX^s}{s!} = 0,$$

where the last equality follows from the assumption of the proposition. Thus it is seen that $F_Y^{[s-1]}(t) - F_X^{[s-1]}(t) = 0$ for $t \leqslant 0$, and that $\overline{F}_Y^{[s-1]}(t) - \overline{F}_X^{[s-1]}(t) = 0$ for $t \geqslant 0$. Differentiating these equalities $s-1$ times we obtain, respectively, $F_Y(t) - F_X(t) = 0$ for $t \leqslant 0$, and $\overline{F}_Y(t) - \overline{F}_X(t) = 0$ for $t \geqslant 0$.

□

Let us now examine the supports of the distribution functions of two s-convex ordered random variables. To this end, let us define, for any real-valued random variable X, the quantities ℓ_X and u_X, the left and the right endpoints of the support of its distribution function, by $\ell_X = \inf\{x \in \mathbb{R} | F_X(x) > 0\}$ and $u_X = \sup\{x \in \mathbb{R} | F_X(x) < 1\}$.

PROPOSITION 3.9. *Let X and Y be two random variables that take on values in $\mathscr{S}$. If $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$ then $u_X \leqslant u_Y$. Also, if s is even then $\ell_X \geqslant \ell_Y$, and if s is odd then $\ell_X \leqslant \ell_Y$.*

Proof. Suppose that $u_X > u_Y$. Let $t \in \mathbb{R}$ be such that $u_Y < t < u_X$. We then get $E(Y-t)_+^{s-1} = 0 < E(X-t)_+^{s-1}$, which contradicts (2). If s is odd and if we assume that $\ell_Y < \ell_X$ then, by choosing a real t such that $\ell_Y < t < \ell_X$, we get $E(t-X)_+^{s-1} = 0 < E(t-Y)_+^{s-1}$, which contradicts (3). If s is even and if we assume that $\ell_X < \ell_Y$ then, by choosing a real t such that $\ell_X < t < \ell_Y$, we get $E(t-X)_+^{s-1} > 0 = E(t-Y)_+^{s-1}$, which contradicts (3).

□

PROPOSITION 3.10. *Let X and Y be two random variables that take on values in $\mathscr{S}$. Then*

$$X \preceq_{s\text{-}cx}^{\mathscr{S}} Y \iff \begin{cases} -X \preceq_{s\text{-}cx}^{-\mathscr{S}} -Y & \text{when } s \text{ is even}, \\ -Y \preceq_{s\text{-}cx}^{-\mathscr{S}} -X & \text{when } s \text{ is odd}. \end{cases}$$

Proof. If s is even, we have that $\phi(\cdot) \in \mathscr{U}_{s\text{-cx}}^{\mathscr{S}} \iff \phi(-\cdot) \in \mathscr{U}_{s\text{-cx}}^{-\mathscr{S}}$, while if s is odd then $\phi(\cdot) \in \mathscr{U}_{s\text{-cx}}^{\mathscr{S}} \iff -\phi(-\cdot) \in \mathscr{U}_{s\text{-cx}}^{-\mathscr{S}}$. Hence the results follow from Definition 2.1.

□

Finally we list some closure properties of the s-convex orders. For any random variable Z and event A, we denote below by $[Z|A]$ any random variable whose distribution function is the conditional distribution of Z given A.

PROPOSITION 3.11. *Let X, Y and be two random variables that take on values in $\mathscr{S}$, and let Θ be a random variable that takes on values in $\mathscr{T} \subseteq \mathbb{R}$.*

(i) If $[X|\Theta=\theta] \preceq_{s\text{-}cx}^{\mathscr{S}} [Y|\Theta=\theta]$ for all $\theta \in \mathscr{T}$ then $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$; that is, the s-convex orders are closed under mixtures.

(ii) Let $\phi : \mathscr{S} \times \mathscr{T} \to \tilde{\mathscr{S}}$ be a measurable function. If X and Y are independent of Θ, and if $\phi(X,\theta) \preceq_{s\text{-}cx}^{\tilde{\mathscr{S}}} \phi(Y,\theta)$ for all $\theta \in \mathscr{T}$, then $\phi(X,\Theta) \preceq_{s\text{-}cx}^{\tilde{\mathscr{S}}} \phi(Y,\Theta)$.

(iii) If $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$ then $cX \preceq_{s\text{-}cx}^{c\mathscr{S}} cY$ whenever $c > 0$, where $c\mathscr{S} = \{x \in \mathbb{R} | x/c \in \mathscr{S}\}$.

(iv) If $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$ then $cX \preceq_{s\text{-}cx}^{c\mathscr{S}} cY$ whenever $c < 0$ and s is even, and $cY \preceq_{s\text{-}cx}^{c\mathscr{S}} cX$ whenever $c < 0$ and s is odd.

(v) If $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$ then $X + d \preceq_{s\text{-}cx}^{\mathscr{S}+d} Y + d$ for all $d \in \mathbb{R}$, where $\mathscr{S} + d = \{x \in \mathbb{R} | x - d \in \mathscr{S}\}$; that is, the s-convex orders are shift-invariant.

(vi) If $X_1, X_2, \ldots, X_n$ (respectively, $Y_1, Y_2, \ldots, Y_n$) are independent random variables that take on values in $\mathscr{S}$, such that $X_i \preceq_{s\text{-}cx}^{\mathscr{S}} Y_i$, $i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n} X_i \preceq_{s\text{-}cx}^{\mathscr{R}} \sum_{i=1}^{n} Y_i,$$

where $\mathscr{R}$ denotes the union of the supports of the distribution functions of the two sums; that is, the s-convex orders are closed under convolutions.

(vii) If $X_1, X_2, \ldots$ (respectively, $Y_1, Y_2, \ldots$) are independent random variables that take on values in $\mathscr{S}$, such that $X_i \preceq_{s\text{-}cx}^{\mathscr{S}} Y_i$, $i = 1, 2, \ldots$, then, for any positive

integer-valued random variable N *which is independent of the* X_i*'s and of the* Y_j*'s, one has*

$$\sum_{i=1}^{N} X_i \preceq_{s\text{-}cx}^{\tilde{\mathscr{R}}} \sum_{i=1}^{N} Y_i,$$

where $\tilde{\mathscr{R}}$ *denotes the union of the supports of the distribution functions of the two compound sums; that is, the* s*-convex orders are closed under compounding.*

$(viii)$ *Let* $\{X_j,\ j \geqslant 1\}$ *and* $\{Y_j,\ j \geqslant 1\}$ *be two sequences of random variables that take on values in* $\mathscr{S}$*, such that* $X_j \to X$ *and* $Y_j \to Y$ *in distribution as* $j \to \infty$*. If* $E(X)_+^{s-1}$ *and* $E(Y)_+^{s-1}$ *are finite, if* $E(X_j)_+^{s-1} \to E(X)_+^{s-1}$ *and* $E(Y_j)_+^{s-1} \to E(Y)_+^{s-1}$ *as* $j \to \infty$*, and if* $X_i \preceq_{s\text{-}cx}^{\mathscr{S}} Y_i$ *for all integers* i*, then* $X \preceq_{s\text{-}cx}^{\mathscr{S}} Y$*; that is, the* s*-convex orders are preserved under limits.*

Proof. In order to prove (i) we note that

$$EX^k = \int_{\theta\in\mathscr{T}} E[X^k \big| \Theta = \theta]\, dF_\Theta(\theta) = \int_{\theta\in\mathscr{T}} E[Y^k \big| \Theta = \theta]\, dF_\Theta(\theta) = EY^k,$$
$$k = 1, 2, \ldots, s-1,$$

where F_Θ denotes the distribution function of Θ. Moreover, for any real t we have

$$\begin{aligned} E(X-t)_+^{s-1} &= \int_{\theta\in\mathscr{T}} E[(X-t)_+^{s-1} \big| \Theta = \theta]\, dF_\Theta(\theta) \\ &\leqslant \int_{\theta\in\mathscr{T}} E[(Y-t)_+^{s-1} \big| \Theta = \theta]\, dF_\Theta(\theta) \\ &= E(Y-t)_+^{s-1}, \end{aligned}$$

and (i) follows from Theorem 3.2. Statement (ii) is a special case of (i). Statements (iii), (iv) and (v) are obvious from Theorem 3.2 and Proposition 3.10. To prove (vi) we first apply (ii) with $\phi(x,\theta) = x + \theta$ to the random variables X_1, Y_1 and X_2. We then get $X_1 + X_2 \preceq_{s\text{-cx}}^{\mathscr{S}+\mathscr{S}} Y_1 + X_2$, where $\mathscr{S}+\mathscr{S} = \{x \in \mathbb{R} | x = y + z \text{ where } y, z \in \mathscr{S}\}$. Repetition of this argument yields (vi). Using (vi) we get

$$\left[\sum_{i=1}^{N} X_i \big| N = n\right] \preceq_{s\text{-cx}}^{\mathscr{R}} \left[\sum_{i=1}^{N} Y_i \big| N = n\right]$$

for any integer n, and hence (vii) follows from (i). Finally, Rolski (1976) showed that if $X_k \to X$ in distribution and $E(X_k)_+^{s-1} \to E(X)_+^{s-1} < \infty$, then, for every $t \in \mathbb{R}$, one has $E(X_k - t)_+^{s-1} \to E(X-t)_+^{s-1}$, and this proves $(viii)$.

□

4. Conditions which imply the s-convex orders

In this section we give some conditions, by means of the number of sign changes of some functions, which imply the s-convex orders. These conditions are often very easy to verify, especially when s is small. They will be used in Section 5. for the purpose of obtaining extrema with respect to the s-convex orders.

The following notations and terminology are used below. For a real-valued function ϕ defined on $\mathbb{R}$, define the number of sign-changes of ϕ on $\mathbb{R}$ by

$$S^-(\phi) = \sup S^-[\phi(x_1), \phi(x_2), \ldots, \phi(x_n)],$$

where the supremum is extended over all sets $x_1 < x_2 < \cdots < x_n \in \mathbb{R}$, n is arbitrary but finite and $S^-[y_1, y_2, \ldots, y_n]$ is the number of sign changes of the indicated sequence $\{y_1, y_2, \ldots, y_n\}$, zero terms being discarded (see, for example, Karlin (1968)). The functions ϕ_1 and ϕ_2 are said to cross each other k times if $S^-(\phi_1 - \phi_2) = k$, $k = 0, 1, 2, \ldots$. We denote by $\mathscr{B}_s(\mathscr{S}; \mu_1, \mu_2, \ldots, \mu_{s-1})$ the class of all the random variables X whose distribution functions have support in $\mathscr{S}$ and with prescribed first $s-1$ moments $EX^k = \mu_k$, $k = 1, \ldots, s-1$.

We will need the following two lemmas. For the first lemma recall the notation $\overline{F}^{[k]}$ which was defined before Theorem 3.3.

LEMMA 4.1. *Let X and Y be two random variables in $\mathscr{B}_s(\mathscr{S}; \mu_1, \mu_2, \ldots, \mu_{s-1})$. Then*

$$\lim_{t \to -\infty} \left[\overline{F}_Y^{[k]}(t) - \overline{F}_X^{[k]}(t)\right] = 0 \qquad \textit{for} \quad k = 1, \ldots, s-1.$$

Proof. Combining (6), (4), and the equality of the first $s-1$ moments, we get

$$\overline{F}_Y^{[k]}(t) - \overline{F}_X^{[k]}(t) = (-1)^{k+1} \frac{E(t-Y)_+^k - E(t-X)_+^k}{k!} \qquad \text{for} \quad k = 1, 2, \ldots, s-1,$$

and the result follows.

□

The next lemma can be proven using the variation diminishing property of totally positive kernels (see, for instance, Karlin (1968)). However, for the sake of completeness we provide here a direct simple proof of it.

LEMMA 4.2. *Let g be a differentiable real function defined on $\mathbb{R}$, and let g' be its derivative. Then*

$$S^-(g) \leqslant S^-(g') + 1. \tag{1}$$

If, moreover, $\lim_{x \to -\infty} g(x) = 0$ *and* $\lim_{x \to \infty} g(x) = 0$ *then*

$$S^-(g) \leqslant \max(0, S^-(g') - 1). \tag{2}$$

Proof. Suppose $S^-(g') = i$ and suppose that g' exhibits opposite signs on the consecutive intervals $I_0, I_1, \ldots, I_i$. Then g is monotone on each of the intervals, and therefore it can have at most one sign change on each of the $i+1$ intervals, and (1) follows. When $\lim_{x \to -\infty} g(x) = 0$ then g cannot have a sign change on I_0. Similarly, when $\lim_{x \to \infty} g(x) = 0$ then g cannot have a sign change on I_i. Thus (2) follows.

□

We now show that if $S^-(F_X - F_Y)$ is small enough then X and Y are comparable in the s-convex sense.

THEOREM 4.3. *Let X and Y be two random variables in $\mathscr{B}_s(\mathscr{S};\mu_1,\mu_2,\dots,\mu_{s-1})$. If $S^-(F_X-F_Y)\leqslant s-1$ and if the last sign of F_X-F_Y is a $+$, then $X\preceq^{\mathscr{S}}_{s\text{-}cx} Y$.*

Proof. By Theorem 3.3 it suffices to show that $\overline{F}_Y^{[s-1]}(t)-\overline{F}_X^{[s-1]}(t)\geqslant 0$ for $t\in\mathbb{R}$, or, equivalently, that $S^-\left(\overline{F}_Y^{[s-1]}-\overline{F}_X^{[s-1]}\right)=0$ and that $\overline{F}_Y^{[s-1]}-\overline{F}_X^{[s-1]}\geqslant 0$. Now, by Lemma 4.1 and (2) we have that $S^-\left(\overline{F}_Y^{[s-1]}-\overline{F}_X^{[s-1]}\right)\leqslant\max\left(0,S^-\left(\overline{F}_Y^{[s-2]}-\overline{F}_X^{[s-2]}\right)-1\right)\leqslant\cdots\leqslant\max\left(0,S^-\left(\overline{F}_Y^{[0]}-\overline{F}_X^{[0]}\right)-(s-1)\right)=0$. Since the last sign of $\overline{F}_X-\overline{F}_Y$ is a $-$, it follows that the last sign of $\overline{F}_X^{[s-1]}-\overline{F}_Y^{[s-1]}$ is a $-$, and therefore $\overline{F}_X^{[s-1]}\leqslant\overline{F}_Y^{[s-1]}$ everywhere.

□

In the next theorem we obtain a sufficient condition for $X\preceq^{\mathscr{S}}_{s\text{-cx}} Y$ by means of the number of crossings of the density functions f_X and f_Y. The proof of the next result is similar to the proof of Theorem 4.3 and is therefore omitted.

THEOREM 4.4. *Let X and Y be two random variables in $\mathscr{B}_s(\mathscr{S};\mu_1,\mu_2,\dots,\mu_{s-1})$. If $S^-(f_X-f_Y)\leqslant s$, and if the last sign of f_Y-f_X is a $+$, then $X\preceq^{\mathscr{S}}_{s\text{-}cx} Y$.*

REMARK 4.5. If the two random variables X and Y belong to $\mathscr{B}_s(\mathscr{S};\mu_1,\mu_2,\dots,\mu_{s-1})$, and if $X\neq_{\text{st}} Y$, then $S^-(f_X-f_Y)\geqslant s$ (this can be shown, for example, by modifying the proof of Lemma 4.2 of Denuit and Lefèvre (1997)). Thus, in Theorem 4.4 we may replace the assumption $S^-(f_X-f_Y)\leqslant s$ by the assumption $S^-(f_X-f_Y)=s$. Similarly, in Theorem 4.3 we may replace the assumption $S^-(F_X-F_Y)\leqslant s$ by the assumption $S^-(F_X-F_Y)=s$. ◀

REMARK 4.6. It is worthwhile to mention that with the aid of Theorem 4.4 it is possible to extend to the real case Theorem 2.3 (*iii*) of Kaas and Hesselager (1995). For a random variable X in $\mathscr{B}_s(\mathscr{S};\mu_1,\mu_2,\dots,\mu_{s-1})$ with a differentiable density function f_X let us define ρ_X by $\rho_X(t)=\frac{d}{dt}\log f_X(t)$, $t\in\mathbb{R}$. Similarly, for another random variable Y in $\mathscr{B}_s(\mathscr{S};\mu_1,\mu_2,\dots,\mu_{s-1})$ with a differentiable density function f_Y we define ρ_Y. Observe that

$$S^-(f_X-f_Y)=S^-\left(\log\frac{f_X}{f_Y}\right)\leqslant S^-\left(\frac{d}{dt}\log\frac{f_X}{f_Y}\right)+1=S^-(\rho_X-\rho_Y)+1,$$

where the inequality follows from (1). Thus from Theorem 4.4 it follows that if $S^-(\rho_X-\rho_Y)\leqslant s-1$, and if the last sign of $\rho_Y-\rho_X$ is a $+$, then $X\preceq^{\mathscr{S}}_{s\text{-cx}} Y$. ◀

5. Bounds and extrema with respect to the s-convex orders

Recall from Section 4 that $\mathscr{B}_s([a,b];\mu_1,\mu_2,\dots,\mu_{s-1})$ denotes the class of all random variables X whose distribution functions have supports in $[a,b]$ and with prescribed first $s-1$ moments $EX^k=\mu_k$, $k=1,\dots,s-1$. In this section we first obtain the minimum and the maximum in the s-convex sense within the class $\mathscr{B}_s([a,b];\mu_1,\mu_2,\ldots,\mu_{s-1})$. That is, we identify the random variables $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ that belong to $\mathscr{B}_s([a,b];\mu_1,\mu_2,\ldots,\mu_{s-1})$ such that $X_{\min}^{(s)} \preceq_{s\text{-cx}}^{[a,b]} X \preceq_{s\text{-cx}}^{[a,b]} X_{\max}^{(s)}$ for all X in $\mathscr{B}_s([a,b];\mu_1,\mu_2,\ldots,\mu_{s-1})$. The theory of Tchebycheff systems, described in Karlin and Studden (1966), may be used to solve this problem. Here, however, we will derive the extrema from the sufficient conditions that we derived in Section 4.

For $s = 1$ the bounds are trivial. This is because the 1-convex ordering is the standard stochastic dominance. Therefore, it is obvious that with respect to $\preceq_{1\text{-cx}}^{[a,b]}$ we have $X_{\min}^{(1)} = a$ and $X_{\max}^{(1)} = b$ almost surely.

The next theorem, which is the same as Theorem 2.A.9 in Shaked and Shanthikumar (1994), gives the extrema for the case $s = 2$. We include it here for clarity, especially because its simple proof motivates the proof of the following Theorem 5.2.

THEOREM 5.1. *Let $X \in \mathscr{B}_2([a,b];\mu_1)$. Consider the random variables $X_{min}^{(2)}$ and $X_{max}^{(2)}$ in $\mathscr{B}_2([a,b];\mu_1)$ defined by $X_{min}^{(2)} = \mu_1$ almost surely, and*

$$X_{max}^{(2)} = \begin{cases} a & \text{with probability } r_1 = \frac{b-\mu_1}{b-a}, \\ b & \text{with probability } r_2 = \frac{\mu_1-a}{b-a}. \end{cases}$$

Then $X_{min}^{(2)} \preceq_{2\text{-cx}}^{[a,b]} X \preceq_{2\text{-cx}}^{[a,b]} X_{max}^{(2)}$.

Proof. It is easily seen that the distribution functions of $X_{\min}^{(2)}$ and of any X in $\mathscr{B}_2([a,b];\mu_1)$ intersect at most once, and therefore, by Theorem 4.3, $X_{\min}^{(2)}$ is the minimum in the $\preceq_{2\text{-cx}}^{[a,b]}$ sense.

Now consider $X_{\max}^{(2)}$. The numbers r_1 and r_2 are probabilities since $r_1 + r_2 = 1$ and r_1, $r_2 \geqslant 0$. Moreover, $EX_{\max}^{(2)} = \mu_1$, and it is easily seen that the distribution functions of $X_{\max}^{(2)}$ and of any X in $\mathscr{B}_2([a,b];\mu_1)$ intersect at most once (see Figure 5.1). Thus, Theorem 4.3 yields the result.

□

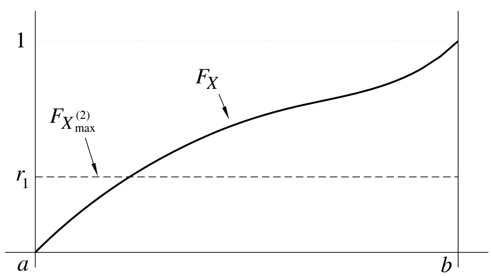

Figure 5.1. A typical distribution and the distribution of $X_{\max}^{(2)}$.

It is of interest to note that we still have that $X_{\min}^{(2)} \preceq_{2\text{-cx}}^{\mathbb{R}} X$ for all $X \in \mathscr{B}_2(\mathbb{R};\mu_1)$. However, the maximum in the 2-convex sense becomes meaningless if either a or b are infinite.

The next result describes the stochastic bounds for the case $s = 3$.

THEOREM 5.2. *Let $X \in \mathscr{B}_3([a,b];\mu_1,\mu_2)$. Consider the random variables $X_{min}^{(3)}$ and $X_{max}^{(3)}$ in $\mathscr{B}_3([a,b];\mu_1,\mu_2)$ defined by*

$$X_{min}^{(3)} = \begin{cases} a & \text{with probability } p_1 = \frac{\mu_2-\mu_1^2}{(a-\mu_1)^2+\mu_2-\mu_1^2}, \\ \mu_1 + \frac{\mu_2-\mu_1^2}{\mu_1-a} & \text{with probability } p_2 = \frac{(a-\mu_1)^2}{(a-\mu_1)^2+\mu_2-\mu_1^2}, \end{cases} \tag{1}$$

and

$$X_{max}^{(3)} = \begin{cases} \mu_1 - \frac{\mu_2-\mu_1^2}{b-\mu_1} & \text{with probability } q_1 = \frac{(b-\mu_1)^2}{(b-\mu_1)^2+\mu_2-\mu_1^2}, \\ b & \text{with probability } q_2 = \frac{\mu_2-\mu_1^2}{(b-\mu_1)^2+\mu_2-\mu_1^2}. \end{cases} \tag{2}$$

Then $X_{min}^{(3)} \preceq_{3\text{-}cx}^{[a,b]} X \preceq_{3\text{-}cx}^{[a,b]} X_{max}^{(3)}$.

Proof. The numbers p_1 and p_2 in (1) are such that $p_1 + p_2 = 1$ and p_1, $p_2 \geqslant 0$. Moreover, we see that $EX_{\min}^{(3)} = \mu_1$ and that $E[X_{\min}^{(3)}]^2 = \mu_2$. Let us prove now that the support of the distribution function of $X_{\min}^{(3)}$ is contained in $[a,b]$. After some simplifications it is seen that we need to show that $\mu_2 - a\mu_1 \leqslant b\mu_1 - ab$. Since the Vandermonde's determinant is always non-negative and $X \in [a,b]$ almost surely, we have that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & X & b \\ a^2 & X^2 & b^2 \end{vmatrix} \geqslant 0 \qquad \text{almost surely.} \tag{3}$$

By expanding the determinant in (3) we get that

$$(b^2 - a^2)X - (b-a)X^2 - ab(b-a) \geqslant 0 \qquad \text{almost surely,} \tag{4}$$

hence the desired result follows by taking the expectation in (4). As it can be easily checked that the distribution functions of $X_{\min}^{(3)}$ and of any X in $\mathscr{B}_3([a,b];\mu_1,\mu_2)$ cannot cross more than twice (see Figure 5.2), it follows that $X_{\min}^{(3)}$ is indeed the minimum by Theorem 4.3. Now, the numbers q_1 and q_2 in (2) are such that $q_1 + q_2 = 1$ and q_1, $q_2 \geqslant 0$. In addition, $EX_{\max}^{(3)} = \mu_1$ and $E[X_{\max}^{(3)}]^2 = \mu_2$. It remains to verify that the support of the distribution function of $X_{\max}^{(3)}$ is contained in $[a,b]$. This is equivalent to $ab - a\mu_1 \leqslant b\mu_1 - \mu_2$, which was shown above to be true. As it is easily seen, the distribution functions of $X_{\max}^{(3)}$ and of any X in $\mathscr{B}_3([a,b];\mu_1,\mu_2)$ cannot cross more than twice (see Figure 5.3). By Theorem 4.3 it now follows that $X_{\max}^{(3)}$ is the maximum. □

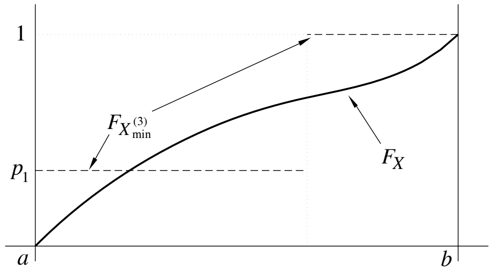

Figure 5.2. A typical distribution and the distribution of $X^{(3)}_{\min}$.

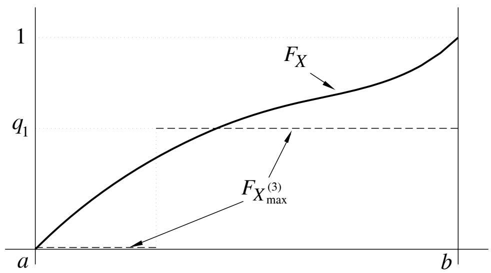

Figure 5.3 A typical distribution and the distribution of $X^{(3)}_{\max}$.

Note that within $\mathscr{B}_3([a,\infty);\mu_1,\mu_2)$ we still have $X^{(3)}_{\min} \preceq^{[a,\infty)}_{3\text{-cx}} X$ for any X, however $X^{(3)}_{\max}$ is now meaningless. When, in addition, $a=0$, we get a result derived by Lefèvre and Utev (1996) (see their Property 2.10).

In Theorem 5.1 (respectively, Theorem 5.2) it is assumed that only the first moment (respectively, the first two moments) of X is (respectively, are) known. When more is known about X it is sometimes possible to improve the bounds on X. For example, Denuit and Lefèvre (1997) studied this problem under the assumption that X takes on values in $\{0,1,\ldots,n\}$, $n>0$, and has prescribed $s-1$ first moments. Naturally the extrema obtained in that case are better than those obtained in Theorems 5.1 and 5.2. Below we will see how the bounds in Theorems 5.1 and 5.2 can be improved when X is known to have a unimodal density function. As a special case of the results below we will obtain improved bounds on X when X is known to have a decreasing density function.

Denote by $\mathscr{B}^*_s([a,b];\mu_1,\mu_2,\ldots,\mu_{s-1};m)$ the class of all random variables X with unimodal density functions, whose distribution functions have support in $[a,b]$, with prescribed first $s-1$ moments $EX^k=\mu_k$, $k=1,\ldots,s-1$, and with a mode m. In the sequel we will denote by $\text{Unif}[\alpha,\beta]$ the uniform distribution function on the interval $[\alpha,\beta]$. Furthermore, we will denote by $p\,\text{Unif}[\alpha_1,\beta_1]+(1-p)\,\text{Unif}[\alpha_2,\beta_2]$ the distribution function which is the mixture of the uniform distributions on $[\alpha_1,\beta_1]$ and on $[\alpha_2,\beta_2]$ with weights p and $1-p$, respectively. The notation $\text{Unif}[\alpha,\alpha]$ will stand for the distribution which is degenerate at the point α.

Let us first examine the case $s = 1$, that is, the standard stochastic dominance $\preceq_{\mathrm{st}}$. Let $X \in \mathscr{B}_1^*([a,b];m)$. Consider the random variables $X_{\min}^{(1)*}$ and $X_{\max}^{(1)*}$ in $\mathscr{B}_1^*([a,b];m)$ with the respective distribution functions

$$F_{X_{\min}^{(1)*}} = \mathrm{Unif}[a,m] \quad \text{and} \quad F_{X_{\max}^{(1)*}} = \mathrm{Unif}[m,b]. \tag{5}$$

Then, $X_{\min}^{(1)*} \preceq_{1\text{-cx}}^{[a,b]} X \preceq_{1\text{-cx}}^{[a,b]} X_{\max}^{(1)*}$. The latter assertion is easily deduced from Theorem 4.4, since f_X and $f_{X_{\min}^{(1)*}}$, as well as f_X and $f_{X_{\max}^{(1)*}}$, cross each other at most once for any $X \in \mathscr{B}_1^*([a,b];m)$. Moreover, it is easily seen that $X_{\min}^{(1)} \preceq_{1\text{-cx}}^{[a,b]} X_{\min}^{(1)*}$ and $X_{\max}^{(1)*} \preceq_{1\text{-cx}}^{[a,b]} X_{\max}^{(1)}$. A typical graph of the density function of the bound $X_{\max}^{(1)*}$, described in (5), is depicted in Figure 5.4. The graph of the density function of $X_{\min}^{(1)*}$ is similar, but over the interval $[a,m]$.

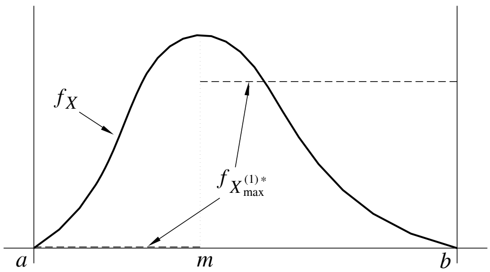

Figure 5.4 A typical unimodal density and the density of $X_{\max}^{(1)*}$.

To treat the case $s \geqslant 2$, let us recall Khinchine's Theorem which states that a random variable Y has a unimodal density function with a mode 0 if, and only if, there exist independent random variables U and Z, such that U is uniformly distributed on $[0,1]$ and the product UZ has distribution function F_Y (see, for example, Theorem 1.3 in Dharmadhikari and Joag-Dev (1988)).

THEOREM 5.3. *Let $X \in \mathscr{B}_2^*([a,b];\mu_1;m)$. Consider the random variables $X_{min}^{(2)*}$ and $X_{max}^{(2)*}$ in $\mathscr{B}_2^*([a,b];\mu_1;m)$ with the respective distribution functions*

$$F_{X_{min}^{(2)*}} = \mathrm{Unif}[\min(2\mu_1 - m, m), \max(2\mu_1 - m, m)]$$

and

$$F_{X_{max}^{(2)*}} = \frac{b + m - 2\mu_1}{b-a}\,\mathrm{Unif}[a,m] + \frac{2\mu_1 - a - m}{b-a}\,\mathrm{Unif}[m,b].$$

Then $X_{min}^{(2)*} \preceq_{2\text{-}cx}^{[a,b]} X \preceq_{2\text{-}cx}^{[a,b]} X_{max}^{(2)*}$.

Proof. Define $Y = X - m$. Then $Y \in \mathscr{B}_2^*([a^*,b^*];\mu_1 - m;0)$, where $a^* = a - m$ and $b^* = b - m$. By Khinchine's Theorem $Y =_{\mathrm{st}} UZ$, where U and Z are independent random variables, U is uniformly distributed on $[0,1]$, and "$=_{\mathrm{st}}$" denotes equality in distribution. Since $EY = EUEZ = \frac{1}{2}EZ$, it is seen that $EZ = \tilde{\mu}_1$, where

$\tilde{\mu}_1 = 2\mu_1 - 2m$. Therefore $Z \in \mathscr{B}_2([a^*,b^*];\tilde{\mu}_1)$. From Theorem 5.1 we see that $Z^{(2)}_{\min} \preceq^{[a^*,b^*]}_{\text{2-cx}} Z \preceq^{[a^*,b^*]}_{\text{2-cx}} Z^{(2)}_{\max}$, where $Z^{(2)}_{\min} = \tilde{\mu}_1 = 2\mu_1 - 2m$ almost surely, and

$$Z^{(2)}_{\max} = \begin{cases} a^* & \text{with probability } \frac{b^*-\tilde{\mu}_1}{b^*-a^*}, \\ b^* & \text{with probability } \frac{\tilde{\mu}_1-a^*}{b^*-a^*}, \end{cases} = \begin{cases} a-m & \text{with probability } \frac{b+m-2\mu_1}{b-a}, \\ b-m & \text{with probability } \frac{2\mu_1-a-m}{b-a}. \end{cases}$$

Now let U be a uniform $[0,1]$ random variable which is independent of $Z^{(2)}_{\min}$ and of $Z^{(2)}_{\max}$. Define $Y^{(2)*}_{\min} = Z^{(2)}_{\min}U$ and $Y^{(2)*}_{\max} = Z^{(2)}_{\max}U$. By Khinchine's Theorem it follows that $Y^{(2)*}_{\min}$, $Y^{(2)*}_{\max} \in \mathscr{B}^*_2([a^*,b^*];\mu_1 - m;0)$; and by Proposition 3.11 (ii) and (iii) it is seen that $Y^{(2)*}_{\min} \preceq^{[a^*,b^*]}_{\text{2-cx}} Y \preceq^{[a^*,b^*]}_{\text{2-cx}} Y^{(2)*}_{\max}$. If we set $X^{(2)*}_{\min} = Y^{(2)*}_{\min} + m$ and $X^{(2)*}_{\max} = Y^{(2)*}_{\max} + m$, then, from Proposition 3.11 (v) we get that $X^{(2)*}_{\min} \preceq^{[a,b]}_{\text{2-cx}} X \preceq^{[a,b]}_{\text{2-cx}} X^{(2)*}_{\max}$. A straightforward calculation shows that $X^{(2)*}_{\min}$ and $X^{(2)*}_{\max}$ indeed have the distribution functions stated in Theorem 5.3.

□

When we compare $X^{(2)}_{\min}$ and $X^{(2)}_{\max}$ of Theorem 5.1 with $X^{(2)*}_{\min}$ and $X^{(2)*}_{\max}$ of Theorem 5.3 we see that $X^{(2)}_{\min} \preceq^{[a,b]}_{\text{2-cx}} X^{(2)*}_{\min}$ and that $X^{(2)*}_{\max} \preceq^{[a,b]}_{\text{2-cx}} X^{(2)}_{\max}$. Thus, unimodality (when it is known to hold) yields improved bounds in the $\preceq^{[a,b]}_{\text{2-cx}}$ sense. The lower bound in Theorem 5.3 generalizes Theorem 2.A.18(a) of Shaked and Shanthikumar (1994). Typical graphs of the density functions of the bounds $X^{(2)*}_{\min}$ and $X^{(2)*}_{\max}$ of Theorem 5.3 are depicted in Figures 5.5 and 5.6.

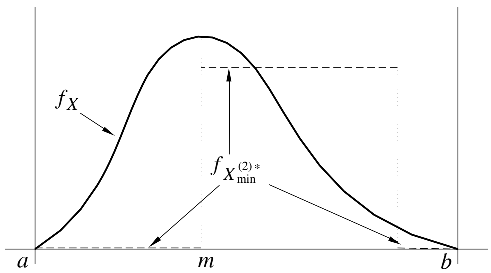

Figure 5.5 A typical unimodal density and the density of $X^{(2)*}_{\min}$.

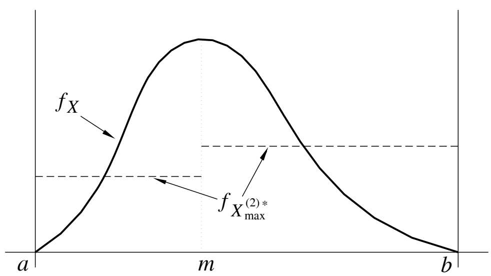

Figure 5.6. A typical unimodal density and the density of $X^{(2)*}_{\max}$.

It is worthwhile to point out that in the above proof, each point mass of $Z_{\min}^{(2)}$ and of $Z_{\max}^{(2)}$ gives rise to a uniform distribution for the mixtures that make up the distribution functions of $X_{\min}^{(2)*}$ and of $X_{\max}^{(2)*}$. This observation gives an insight into the previous proof, and into the proof of the following Theorem 5.4.

THEOREM 5.4. *Let $X \in \mathscr{B}_3^*([a,b];\mu_1,\mu_2;m)$. Consider the random variables $X_{min}^{(3)*}$ and $X_{max}^{(3)*}$ in $\mathscr{B}_3^*([a,b];\mu_1,\mu_2;m)$ with the respective distribution functions*

$$F_{X_{min}^{(3)*}} = \frac{3\mu_2 + 2\mu_1 m - m^2 - 4\mu_1^2}{(a+m-2\mu_1)^2 + 3\mu_2 + 2\mu_1 m - m^2 - 4\mu_1^2}\,\mathrm{Unif}[a,m] + \frac{(a+m-2\mu_1)^2}{(a+m-2\mu_1)^2 + 3\mu_2 + 2\mu_1 m - m^2 - 4\mu_1^2}\,\mathrm{Unif}[\underline{c},\overline{c}],$$

where $\underline{c} = \min(m, \frac{3\mu_2+am-2a\mu_1-2m\mu_1}{2\mu_1-a-m})$ *and* $\overline{c} = \max(m, \frac{3\mu_2+am-2a\mu_1-2m\mu_1}{2\mu_1-a-m})$, *and*

$$F_{X_{max}^{(3)*}} = \frac{3\mu_2 + 2\mu_1 m - m^2 - 4\mu_1^2}{(b+m-2\mu_1)^2 + 3\mu_2 + 2\mu_1 m - m^2 - 4\mu_1^2}\,\mathrm{Unif}[m,b] + \frac{(b+m-2\mu_1)^2}{(b+m-2\mu_1)^2 + 3\mu_2 + 2\mu_1 m - m^2 - 4\mu_1^2}\,\mathrm{Unif}[\underline{d},\overline{d}],$$

where $\underline{d} = \min(m, \frac{2b\mu_1+2\mu_1 m-bm-3\mu_2}{b+m-2\mu_1})$ *and* $\overline{d} = \max(m, \frac{2b\mu_1+2\mu_1 m-bm-3\mu_2}{b+m-2\mu_1})$. *Then* $X_{min}^{(3)*} \preceq_{3\text{-}cx}^{[a,b]} X \preceq_{3\text{-}cx}^{[a,b]} X_{max}^{(3)*}$.

Proof. The proof of Theorem 5.4 is similar to the proof of Theorem 5.3, though the computations are less simple. Define $Y = X - m$. Then $Y \in \mathscr{B}_3^*([a^*,b^*];\mu_1 - m, \mu_2 - 2m\mu_1 + m^2; 0)$, where $a^* = a - m$ and $b^* = b - m$. By Khinchine's Theorem $Y =_{\mathrm{st}} UZ$, where U and Z are independent random variables and U is uniformly distributed on $[0,1]$. A straightforward computation shows that the first two moments of Z are $\tilde{\mu}_1 = 2\mu_1 - 2m$ and $\tilde{\mu}_2 = 3\mu_2 - 6m\mu_1 + 3m^2$. Therefore $Z \in \mathscr{B}_3([a^*,b^*];\tilde{\mu}_1,\tilde{\mu}_2)$. From Theorem 5.2 we see that $Z_{\min}^{(3)} \preceq_{3\text{-cx}}^{[a^*,b^*]} Z \preceq_{3\text{-cx}}^{[a^*,b^*]} Z_{\max}^{(3)}$, where

$$Z_{\min}^{(3)} = \begin{cases} a - m & \text{with probability } \frac{3\mu_2+2m\mu_1-m^2-4\mu_1^2}{(a+m-2\mu_1)^2+3\mu_2+2m\mu_1-m^2-4\mu_1^2}, \\ 2\mu_1 - 2m + \frac{3\mu_2+2m\mu_1-m^2-4\mu_1^2}{2\mu_1-a-m} & \text{with probability } \frac{(a+m-2\mu_1)^2}{(a+m-2\mu_1)^2+3\mu_2+2m\mu_1-m^2-4\mu_1^2}, \end{cases}$$

and

$$Z_{\max}^{(3)} = \begin{cases} 2\mu_1 - 2m - \frac{3\mu_2+2m\mu_1-m^2-4\mu_1^2}{b+m-2\mu_1} & \text{with probability } \frac{(b+m-2\mu_1)^2}{(b+m-2\mu_1)^2+3\mu_2+2m\mu_1-m^2-4\mu_1^2}, \\ b - m & \text{with probability } \frac{3\mu_2+2m\mu_1-m^2-4\mu_1^2}{(b+m-2\mu_1)^2+3\mu_2+2m\mu_1-m^2-4\mu_1^2}. \end{cases}$$

Now let U be a uniform $[0,1]$ random variable which is independent of $Z^{(3)}_{\min}$ and of $Z^{(3)}_{\max}$. Define $Y^{(3)*}_{\min} = Z^{(3)}_{\min}U$ and $Y^{(3)*}_{\max} = Z^{(3)}_{\max}U$. By Khinchine's Theorem it follows that $Y^{(3)*}_{\min}$, $Y^{(3)*}_{\max} \in \mathscr{B}^*_3([a^*,b^*];\mu_1-m,\mu_2-2m\mu_1+m^2;0)$; and by Proposition 3.11 (ii) and (iii) it is seen that $Y^{(3)*}_{\min} \preceq^{[a^*,b^*]}_{\text{3-cx}} Y \preceq^{[a^*,b^*]}_{\text{3-cx}} Y^{(3)*}_{\max}$. If we set $X^{(3)*}_{\min} = Y^{(3)*}_{\min} + m$ and $X^{(3)*}_{\max} = Y^{(3)*}_{\max} + m$, then, from Proposition 3.11 (v) we see that $X^{(3)*}_{\min} \preceq^{[a,b]}_{\text{3-cx}} X \preceq^{[a,b]}_{\text{3-cx}} X^{(3)*}_{\max}$. A straightforward calculation shows that $X^{(3)*}_{\min}$ and $X^{(3)*}_{\max}$ indeed have the distribution functions stated in Theorem 5.4.

□

The bounds $X^{(3)*}_{\min}$ and $X^{(3)*}_{\max}$ of Theorem 5.4 (when unimodality is known to hold) improve, in the $\preceq^{[a,b]}_{\text{3-cx}}$ sense, the bounds $X^{(3)}_{\min}$ and $X^{(3)}_{\max}$ of Theorem 5.2, in the same fashion as the bounds of Theorem 5.3 are an improvement of the bounds of Theorem 5.1.

An interesting special case of Theorem 5.4 is the situation in which a random variable X has a monotone density on $[a,b]$. For example, if $m=a$ in Theorem 5.4 then X has a non-increasing density on $[a,b]$. We thus obtain the following corollary.

COROLLARY 5.5. *Let X be a random variable with first two moments μ_1 and μ_2 and with a non-increasing density function on $[a,b]$. Consider the random variables $X^{(3)**}_{min}$ and $X^{(3)**}_{max}$ (which have the first two moments μ_1 and μ_2 and which have non-increasing density functions on $[a,b]$) with the respective distribution functions*

$$F_{X^{(3)**}_{min}} = \frac{3\mu_2+2\mu_1 a-a^2-4\mu_1^2}{3a^2-6a\mu_1+3\mu_2}\,\text{Unif}[a,a] + \frac{4(a-\mu_1)^2}{3a^2-6a\mu_1+3\mu_2}\,\text{Unif}[a,\frac{3\mu_2+a^2-4a\mu_1}{2(\mu_1-a)}]$$

and

$$F_{X^{(3)**}_{max}} = \frac{3\mu_2+2\mu_1 a-a^2-4\mu_1^2}{b^2+2ab-2a\mu_1-4b\mu_1+3\mu_2}\,\text{Unif}[a,b] + \frac{(a+b-2\mu_1)^2}{b^2+2ab-2a\mu_1-4b\mu_1+3\mu_2}\,\text{Unif}[a,\frac{2b\mu_1+2a\mu_1-ab-3\mu_2}{a+b-2\mu_1}].$$

Then $X^{(3)**}_{min} \preceq^{[a,b]}_{\text{3-cx}} X \preceq^{[a,b]}_{\text{3-cx}} X^{(3)**}_{max}$.

Note that in Corollary 5.5 the distribution of $X^{(3)**}_{\min}$ is a mixed distribution with an atom at a and with a continuous component on $[a,\frac{3\mu_2+a^2-4a\mu_1}{2(\mu_1-a)}]$. A typical graph of the density function of the bound $X^{(3)**}_{\max}$ of Corollary 5.5 is depicted in Figure 5.7.

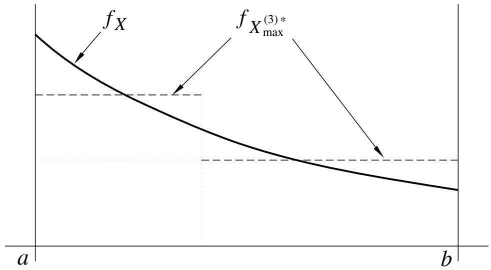

Figure 5.7. A typical non-increasing density and the density of $X^{(3)**}_{\max}$.

6. Applications

6.1. Queueing theory

For any non-degenerate non-negative random variable X, with a survival function $\overline{F}_X$, and with a finite mean EX, let us define the associated random variable $\widetilde{X}$ whose survival function $\overline{F}_{\widetilde{X}}$ is given by

$$\overline{F}_{\widetilde{X}}(x) = \frac{1}{EX}\int_{t=x}^{\infty} \overline{F}_X(t)\,dt, \qquad x \geqslant 0.$$

The random variable $\widetilde{X}$ is often called in the literature the stationary forward recurrence time, or the equilibrium age or residual lifetime. The following theorem shows that transforming X and Y to $\widetilde{X}$ and $\widetilde{Y}$ reduces the degree of the s-convex order.

THEOREM 6.1. *Let X and Y be two non-negative random variables. If $X \preceq^{\mathbb{R}^+}_{s\text{-}cx} Y$ for some $s \geqslant 2$, then $\widetilde{X} \preceq^{\mathbb{R}^+}_{(s-1)\text{-}cx} \widetilde{Y}$.*

Proof. First we obtain the relationship between the iterated integrals of X and of $\widetilde{X}$. Since

$$\overline{F}_{\widetilde{X}}(x) = \frac{1}{EX}\int_{t=x}^{\infty} \overline{F}_X(t)\,dt = \frac{1}{EX}\,\overline{F}^{[1]}_X(x)$$

we obtain

$$\overline{F}^{[s-2]}_{\widetilde{X}}(x) = \frac{1}{EX}\,\overline{F}^{[s-1]}_X(x) \qquad \text{and, similarly,} \qquad \overline{F}^{[s-2]}_{\widetilde{Y}}(x) = \frac{1}{EX}\,\overline{F}^{[s-1]}_Y(x) \tag{1}$$

since $EX = EY$. The relationship between the moments of X and of $\widetilde{X}$ is given by

$$E\widetilde{X}^k = \frac{EX^{k+1}}{(k+1)EX} \qquad \text{and, similarly,} \qquad E\widetilde{Y}^k = \frac{EY^{k+1}}{(k+1)EY}, \qquad k = 1, 2, \dots, s-2,$$

as can be easily verified. Since $EX^k = EY^k$, $k = 2, 3, \dots, s-1$, we get that $E\widetilde{X}^k = E\widetilde{Y}^k$, $k = 1, 2, \dots, s-2$. From Theorem 3.3 and (1) we see that $\overline{F}^{[s-2]}_{\widetilde{Y}}(t) - \overline{F}^{[s-2]}_{\widetilde{X}}(t) \geqslant 0$ for $t \geqslant 0$. Thus the stated result follows from Theorem 3.3.

□

Consider now a stable M/GI/1 queue with independent and identically distributed service times $X_1, X_2, \dots,$ and arrival rate λ. Let W denote the stationary waiting time that is associated with the queue. Then W is expressible as

$$W =_{\text{st}} \sum_{n=1}^{N} \widetilde{X}_n,$$

where N is geometrically distributed on $\{0, 1, \dots\}$ with parameter λEX_1, and is independent of the $\widetilde{X}_n$'s (see, for instance, Stoyan (1983, page 82)).

Consider also a second stable M/GI/1 queue with independent and identically distributed service times $Y_1, Y_2, \dots,$ and arrival rate λ, and let V denote the stationary waiting time that is associated with this second queue. Then V is expressible as

$$V =_{\text{st}} \sum_{n=1}^{M} \widetilde{Y}_n,$$

where M is geometrically distributed on $\{0, 1, \dots\}$ with parameter λEY_1, and is independent of the $\widetilde{Y}_n$'s.

If $X_1 \preceq^{\mathbb{R}^+}_{s\text{-cx}} Y_1$ for some $s \geqslant 2$ then, from Theorem 6.1 we see that $\widetilde{X}_1 \preceq^{\mathbb{R}^+}_{(s-1)\text{-cx}} \widetilde{Y}_1$. Also we have then that $N =_{\text{st}} M$ since $EX_1 = EY_1$. Thus, from Proposition 3.11 (vii) we get that

$$W \preceq^{\mathbb{R}^+}_{(s-1)\text{-cx}} V. \tag{2}$$

This result extends Theorem 5.2.3 in Stoyan (1983) to the s-convex stochastic orders (see also Daley and Rolski (1984) and Makowski (1994)).

Inequality (2) can be used in several ways. For example, suppose that we are interested in the queue with the service times $X_1, X_2, \dots,$ where the distribution function of X_1 is either unknown or analytically complicated. However, suppose that the first two moments of X_1 are known or can be approximated, and suppose also that X_1 is known to have a unimodal density function with a known mode (in particular, it may be known that the density of X_1 is decreasing on the support of its distribution function). Then, using the stochastic bounds of Theorem 5.4 we have that $X^{(3)*}_{\min} \preceq^{\mathbb{R}^+}_{3\text{-cx}} X_1 \preceq^{\mathbb{R}^+}_{3\text{-cx}} X^{(3)*}_{\max}$. Therefore, by (2) we see that the stationary waiting time W that is associated with the queue of interest is bounded as $W^*_{\min} \preceq^{\mathbb{R}^+}_{2\text{-cx}} W \preceq^{\mathbb{R}^+}_{2\text{-cx}} W^*_{\max}$, where $W^*_{\min}$ and $W^*_{\max}$ are the stationary waiting times of the queues with the relatively simple service times that are distributed, respectively, as $X^{(3)*}_{\min}$ and $X^{(3)*}_{\max}$.

Inequality (2) is also useful when it can be shown directly that $X_1 \preceq^{\mathbb{R}^+}_{3\text{-cx}} Y_1$. For example, Kaas and Hesselager (1995) have proved that if X, Y and Z are random variables distributed respectively according to the gamma, the inverse Gaussian and the lognormal laws with the same means and variances, then $X \preceq^{\mathbb{R}^+}_{3\text{-cx}} Y$ and $X \preceq^{\mathbb{R}^+}_{3\text{-cx}} Z$. Then, by (2), the stationary waiting times of the queues with service times that are distributed, respectively, as X, Y and Z, are ordered, correspondingly, according to the 2-convex order.

6.2. Insurance

Consider the following situation. A manufacturer buys, for the price of d dollars, a sophisticated piece of equipment with a random lifelength X. He would like to avoid the serious financial loss that may be caused by the destruction of the piece of equipment and therefore asks an insurer to cover this risk. The insurance contract provides for a payment of d if the piece of equipment breaks down before time b, where b is the useful lifetime of the piece of equipment; otherwise at age b the piece of equipment is scrapped and a new one is provided by the insurer. If the insurance company computes the premium on the basis of the "mean-value" principle (that is, the premium charged is equal to the expectation of the actual financial loss suffered by the insurer), then the premium should be

$$\pi = dEv^X, \tag{3}$$

where v is the discount rate corresponding to a yearly interest rate of r (that is, $v = 1/(1+r)$ is the present value of an amount of 1 dollar paid in 1 year).

If the distribution function of X is mostly unknown, then π cannot be computed accurately. Suppose, however, that the mean and the variance of X are known, as well as the support $[0, b]$ of its distribution function. Furthermore, suppose that X is known to have a unimodal density function with a known mode (in particular, it may be known that the density of X is decreasing on the support of its distribution function). In such a situation, the bounds of Theorem 5.4 are very useful. They enable the actuary to compute accurate bounds on the premium π based only on the partial information that the insurer possesses.

We list below some bounds on π using the s-convex extrema on X when $s = 2$ and $s = 3$. For simplicity we let $d = 1$ in all the computations below. The function ϕ, defined by $\phi(x) = v^x$, is 2-convex. Thus, for $s = 2$, when the mode is unknown, we get from Theorem 5.1

$$\pi_{\min} = v^{\mu_1} \qquad \text{and} \qquad \pi_{\max} = \frac{b-\mu_1}{b} + v^b\,\frac{\mu_1}{b};$$

and when the mode m is known we get from Theorem 5.3

$$\pi^*_{\min} = \left|\frac{v^{2\mu_1 - m} - v^m}{2(\mu_1 - m)\log(v)}\right| \qquad \text{and}$$

$$\pi^*_{\max} = \frac{b - 2\mu_1 + m}{b}\,\frac{v^m - 1}{m\log(v)} + \frac{2\mu_1 - m}{b}\,\frac{v^b - v^m}{(b-m)\log(v)},$$

provided $b \neq m$ and $\mu_1 \neq m$ (when $\mu_1 = m$ then $\pi^*_{\min} = v^m$). Some values of these bounds are given in Table 6.1.

μ_1	m	Lower bound	Upper bound
4	unknown	0.822702	0.845565
	3	0.823029	0.831017
	4	0.822702	0.830370
	5	0.823029	0.829739
5	unknown	0.783526	0.806957
	4	0.783837	0.791321
	5	0.783526	0.791321
	6	0.783837	0.791321
6	unknown	0.746215	0.768348
	5	0.746511	0.752903
	6	0.746215	0.753519
	7	0.746511	0.754120

Table 6.1. Bounds on the premium (3) using $s=2$ when $b=10$, $d=1$ and $r=5\%$.

The function ϕ, defined by $\phi(x)=-v^x$, is 3-convex. Thus, for $s=3$, when the mode is unknown, we get from Theorem 5.2

$$\pi_{\min}=v^{\mu_1-\frac{\sigma^2}{b-\mu_1}}\frac{(b-\mu_1)^2}{(b-\mu_1)^2+\sigma^2}+v^b\frac{\sigma^2}{(b-\mu_1)^2+\sigma^2}\qquad\text{and}\qquad\pi_{\max}=\frac{\sigma^2}{\mu_2}+v^{\frac{\mu_2}{\mu_1}}\frac{\mu_1^2}{\mu_2};$$

and when the mode m is known we get from Theorem 5.4

$$\pi^*_{\min}=\gamma\left|\frac{v^m-v^{\frac{2\mu_1 b-mb+2m\mu_1-3\mu_2}{b-2\mu_1+m}}}{\frac{2mb-4\mu_1 m+m^2-2\mu_1 b+3\mu_2}{b-2\mu_1+m}\ln(v)}\right|+(1-\gamma)\frac{v^b-v^m}{(b-m)\ln(v)},$$

where

$$\gamma=\frac{(b+m-2\mu_1)^2}{(b+m-2\mu_1)^2+3\mu_2+2m\mu_1-m^2-4\mu_1^2},$$

and

$$\pi^*_{\max}=\frac{3\sigma^2-(m-\mu_1)^2}{3\mu_2-2\mu_1 m}\frac{v^m-1}{m\ln(v)}+\left(1-\frac{3\sigma^2-(m-\mu_1)^2}{3\mu_2-2\mu_1 m}\right)\left|\frac{v^{\frac{3\mu_2-2m\mu_1}{2\mu_1-m}}-v^m}{\frac{3\mu_2-4m\mu_1+m^2}{2\mu_1-m}\ln(v)}\right|.$$

Some values of these bounds are given in Table 6.2. The accuracy of these bounds, especially when the mode is known, is quite remarkable.

μ_1	σ	m	Lower bound	Upper bound
4	1	unknown	0.823595	0.823744
		3	0.823632	0.823704
		4	0.823620	0.823722
		5	0.823646	0.823718
	1.5	unknown	0.824718	0.825035
		3	0.824780	0.824951
		4	0.824782	0.824971
		5	0.824816	0.824980
5	1	unknown	0.784390	0.784536
		4	0.784419	0.784488
		5	0.784411	0.784511
		6	0.784432	0.784503
	1.5	unknown	0.785478	0.785790
		4	0.785530	0.785695
		5	0.785536	0.785723
		6	0.785562	0.785729
6	1	unknown	0.747052	0.747194
		5	0.747074	0.747140
		6	0.747070	0.747167
		7	0.747086	0.747154
	1.5	unknown	0.748107	0.748411
		5	0.748153	0.748307
		6	0.748160	0.748340
		7	0.748178	0.748343

Table 6.2. Bounds on the premium (3) using $s = 3$ when $b = 10$, $d = 1$ and $r = 5\%$.

6.3. Statistics

Let $\{f(\cdot;\theta),\ \theta \in \Omega\}$ be a family of density functions such that $f(x;\theta)$ is STP (strictly totally positive; see, for example, Karlin (1968)), where $\Omega \subseteq \mathbb{R}$. Let P be a mixing probability measure on Ω with more than s support points (in particular, the support of P may be a continuum), and let Q_s be another mixing probability measure on Ω with exactly s support points. Let Y be a random variable with the density function f_Y defined by

$$f_Y(x) = \int_{\theta\in\Omega} f(x;\theta)\, dP(\theta),$$

and denote the distribution function of Y by F_Y. Similarly, let X be a random variable with the density function f_X defined by

$$f_X(x) = \int_{\theta\in\Omega} f(x;\theta)\, dQ_s(\theta),$$

and denote the distribution function of X by F_X. When P is unknown, then it is reasonable to estimate (that is, to approximate) f_Y by f_X, where Q_s is such that

$$EX^k = EY^k, \qquad k = 1, 2, \ldots, 2s-1$$

(see, Lindsay and Roeder (1997) and references therein for more details). Lindsay and Roeder (1997) have shown that if X and Y are as above, and if $X \neq_{\text{st}} Y$, then $S^-(f_X - f_Y) = 2s$ (this is a natural conclusion in light of Remark 4.5) and the last sign of $f_Y - f_X$ is a $+$. They also showed that if the above X and Y are such that $X \neq_{\text{st}} Y$ then $S^-(F_X - F_Y) = 2s-1$ and the last sign of $F_X - F_Y$ is a $+$. From Theorems 4.3 and 4.4 it follows that the above X and Y satisfy

$$X \preceq^{\mathscr{S}}_{2s\text{-cx}} Y,$$

where $\mathscr{S}$ is the union of the supports of the distribution functions of X and Y. Thus Y, and its "estimate" X, satisfy (1) with $\mathscr{U}^{\mathscr{S}} = \mathscr{U}^{\mathscr{S}}_{2s\text{-cx}}$, and this fact yields a host of potentially useful inequalities. Further developments of these observations will be reported elsewhere.

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