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ON THE ROOTS OF LACUNARY POLYNOMIALS

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(*communicated by J. Borwein*)

Abstract. We prove estimates for the roots of lacunary polynomials. They are deduced from the study of the equation $x^n - x^{n-1} = a$, where $n \in \mathbf{N}$, $n \geqslant 2$, $a > 0$.

Let $P(X) = \sum_{i=0}^{m} a_i X^i$ be a nonzero complex polynomial. We are interested to find bounds for the absolute values of the roots of P in function of the coefficients a_i. Such estimates were obtained by P. Montel [4] and more recently by M. Mignotte [3]. Other bounds are given by evaluations valid for arbitrary polynomials (good references are to be found, for example, in the monographs of P. Henrici [2] and P. Borwein–T. Erdélyi [1]).

In this paper we obtain such bounds in the following way:

To the polynomial P we associate convenient $a > 0$ and $n \in \mathbf{N} \setminus \{0, 1\}$, with $a = a(a_0, a_1, \ldots, a_m) > 0$. Therefore we obtain bounds for the unique root $\xi > 1$ of the equation $x^n - x^{n-1} = a$. This allows us to describe bounds for the roots of the original polynomial P. In particular, this method gives good estimates for the case of lacunary polynomials, i.e. for polynomials with a certain number of consecutive zero coefficients.

1. Bounds for the Solutions of $x^n - x^{n-1} \leqslant a$

We shall study separately the cases $0 < a < 1$ and $a \geqslant 1$.

1.1. The equation $x^n - x^{n-1} - a = 0$ for $0 < a < 1$.

Our evaluation follows from the study of the condition $(z-c)(z+1)^m \geqslant 1$, where $c, z > 0$, $m \in \mathbb{N}$, $m \geqslant 1$.

LEMMA 1.1. *If* $z > 0$, $c > 0$, $m \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ *and* $(z-c)(z+1)^m = 1$, *then*

$$c < z < \min\{d_1, d_2\},$$

where

$$d_1 = d_1(c, m) = \frac{mc - 1 + \sqrt{(1 - mc)^2 + 4(1 + c)m}}{2m}, \qquad d_2 = d_2(c, m) = c + \sqrt{t_2}$$

and

$$t_2 = \frac{2}{(1 + c)\left(m + \sqrt{m^2 + \frac{2}{3}m(m-1)(m-2)(m-3)(1 + c)}\right)}.$$

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Proof. First observe that we must take $z > c$ and that the function $z \mapsto (z-c)(z+1)^m$ is increasing for $z > c$.

The inequality $(z-c)(z+1)^m \geqslant 1$ is fulfilled if the following condition is satisfied:

$$(z-c)(1+mz) \geqslant 1, \tag{1}$$

Now look to the previous condition. It is equivalent to $mz^2 + (1-mc)z - c - 1 \geqslant 0$ and it holds when

$$z \geqslant d_1 := \frac{mc - 1 + \sqrt{(1-mc)^2 + 4(1+c)m}}{2m}. \tag{2}$$

To obtain another evaluation for z, we put $t = z - c$. Then $(z+c)(z+1)^m \geqslant 1$ becomes

$$t(1+c+t)^m \geqslant 1. \tag{3}$$

Note that (3) is satisfied if one has:

$$t\left((1+c)mt + \frac{(1+c)^3 \cdot m(m-1)(m-2)(m-3)}{6} t^3\right) \geqslant 1. \tag{4}$$

The condition (4) is fulfilled if

$$\begin{aligned} t^2 \geqslant t_2 &:= \frac{-m(1+c) + \sqrt{m^2(1+c)^2 + \frac{2}{3}m(m-1)(m-2)(m-3)(1+c)^3}}{\frac{m(m-1)(m-2)(m-3)(1+c)^3}{3}} \\ &= \frac{2}{(1+c)\left(m + \sqrt{m^2 + \frac{2}{3}m(m-1)(m-2)(m-3)(1+c)}\right)}. \end{aligned}$$

The corresponding upper bound for z is

$$d_2 := c + \sqrt{t_2}. \tag{5}$$

Therefore $z < \min\{d_1, d_2\}$. $\square$

THEOREM 1.2. *If $0 < a < 1$, then the unique root $\xi > 1$ of the equation $x^n - x^{n-1} - a = 0$ satisfies*

$$1 < \xi < \min\{b_1, b_2\},$$

where

$$b_1 = a^{\frac{1}{n}}\big(1 + d_1(c, n-1)\big), \quad b_2 = a^{\frac{1}{n}}\big(1 + d_2(c, n-1)\big), \quad c = a^{-1/n} - 1.$$

Proof. Suppose $0 < a < 1$. Let $b = a^{-1/n}$ and $x = a^{\frac{1}{n}}y$. The equation becomes

$$y^n - by^{n-1} - 1 = 0. \tag{6}$$

Let $b = c + 1$. Note that $c > 0$ because $a^{-1/n} > 1$. Put $y = 1 + z$. It follows

$$(1+z)^n - (1+c)(1+z)^{n-1} = 1,$$

i.e.

$$(z-c)(z+1)^{n-1} = 1. \tag{7}$$

The conclusion follows from Lemma 1.1. □

EXAMPLES.

Let $L(c,m) = \min\{d_1, d_2\}$ and $S(a,n) = \min\{b_1, b_2\}$. Then $L(c,m)$, respectively $S(a,n)$, is attained in both cases, as follows from the following table.

c	m	$L(c,m)$	Attained for
0.12	4	0.468	d_1
1.02	7	1.132	d_1
0.2	21	0.267	d_2

a	n	$S(a,n)$	Attained for
0.57	5	1.311	b_1
0.003	8	1.052	b_1
0.02	22	1.057	b_2

1.2. The equation $x^n - x^{n-1} - a = 0$ for $a \geqslant 1$.

We first study the condition $(z+c)(z+1)^m \geqslant 1$, where z, $c > 0$, $m \in \mathbb{N}^*$.

LEMMA 1.3. *Let $m \in \mathbb{N}^*$, $c > 0$. If $z \geqslant 0$ and $(c+z)(1+z)^m = 1$, then $z \leqslant K(c,m)$, where*

$$K(c,m) = \min\{m_1, m_2, m_3\}$$

and

$$m_1 = \begin{cases} c^{-1/m} - 1, & \text{if} \quad 0 < c < 1, \\ 1 - c^{-1/m}, & \text{if} \quad c \geqslant 1, \end{cases}$$

$$m_2 = \frac{\log\big((1.514065m)/\log(m+1)\big)}{m},$$

$$m_3 = \frac{-c + \sqrt{c^2 + 2cm}}{4cm}.$$

Proof. Observe first that the function $z \mapsto (z+c)(z+1)^m$ is increasing for $z > 0$. Note that if $0 < c < 1$, then for $z = c^{-1/m} - 1 \geqslant 0$ one has $(z+c)(1+z)^m = (z+c)/c \geqslant 1$.

On the other hand the inequality $(z+c)(1+z)^m \geqslant 1$ holds if one of the following two conditions is satisfied:

$$z(1+z)^m = 1, \tag{8}$$

$$2\sqrt{zc}\,(1+z)^m \geqslant 1. \tag{9}$$

Suppose that (8) is satisfied. Put $y = z+1$, then (8) becomes

$$y^{m+1} - y^m - 1 = 0. \tag{10}$$

Let $\xi = \xi_{m+1}$ be the real root > 1 of the equation (10). We define λ_{m+1} by the condition

$$\frac{\log\bigl(\lambda_{m+1} m/\log(m+1)\bigr)}{m} = \xi_{m+1} - 1.$$

Therefore

$$\lambda_{m+1} = \frac{\log(m+1)}{m} \exp\{m(\xi_{m+1} - 1)\}.$$

We may estimate ξ_{m+1} with arbitrary precision and deduce an approximate value of λ_{m+1}, for example for $m < 100$. A computation (done in Reduce) gives $\lambda_{m+1} \leqslant 1.514065$ (the maximum is reached for $m = 11$).

Let $g(z) = (c+z)(1+z)^m$.

Suppose now $m \geqslant 100$. Because $z > 0$ we have

$$g(z) \geqslant z \exp\left(m\bigl(z - \tfrac{1}{2}z^2\bigr)\right).$$

It follows that, for $1.514065(m+1)/m \leqslant \lambda \leqslant \log(m+1)$, one has

$$g\left(\frac{\log(\lambda m/\log(m+1))}{m}\right) \geqslant \frac{\log(\lambda m/\log(m+1))}{m} \cdot \frac{\lambda m}{\log(m+1)} \cdot \exp\left(-\frac{\log^2 m}{2m}\right)$$

$$= \frac{\lambda}{\log(m+1)} \cdot \log\Bigl(\frac{\lambda m}{\log(m+1)}\Bigr) \cdot \exp\Bigl(-\frac{\log^2 m}{2m}\Bigr)$$

$$> 0.899\,\lambda \cdot \frac{\log(m+1) - \log\log(m+1)}{\log(m+1)} > 0.6734\,\lambda.$$

Therefore $g\left(\dfrac{\log(\lambda m/\log(m+1))}{m}\right) \geqslant 1$ if $m \geqslant 100$ and $\lambda \geqslant 1.514065$.

It follows that the real solution $\xi > 1$ of the equation (10) satisfies

$$\xi < 1 + \frac{\log\bigl(1.514065 m/\log(m+1)\bigr)}{m}.$$

Therefore the corresponding solution $z > 0$ of (8) satisfies

$$z < \frac{\log\big(1.514065m/\log(m+1)\big)}{m}.$$

The condition (9) is equivalent to

$$4cz(1+z)^{2m} \geqslant 1. \tag{11}$$

But (11) is satisfied if the following inequality is fulfilled:

$$4cz\left(1+\binom{2m}{1}z\right) \geqslant 1, \tag{12}$$

The relation (12) is satisfied if $z \geqslant \dfrac{-c+\sqrt{c^2+2cm}}{4cm}$.
Thus we obtained the stated estimations. □

EXAMPLES.

c	m	$K(c,m)$	Attained for
0.98	8	0.002	m_1
1.09	5	0.017	m_1
0.2	18	0.062	m_2
0.3	4	0.266	m_3

THEOREM 1.4. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and $a > 1$. Then the unique root $\xi > 1$ of the trinomial $x^n - x^{n-1} - a$ satisfies*

$$a^{1/n} < \left(\frac{1+\sqrt{4a+1}}{2}\right)^{2/n} \leqslant \xi \leqslant a^{\frac{1}{n}}\Big(1+K(1-a^{-\frac{1}{n}}, n-1)\Big) := T(a,n).$$

Proof. We observe that, by Lemma 1 from [1], the polynomial has a unique root $\xi > 1$.

Note that $a > 1$. Let $b = a^{-1/n}$ and $x = a^{1/n}y$. Consider $y = 1+z$. The equation $x^n - x^{n-1} - a = 0$ becomes

$$y^n - by^{n-1} - 1 = 0.$$

Put $b = 1-c$. Note that $0 < b < 1$, so $0 < c = 1-b < 1$. It follows

$$(1+z)^n - (1-c)(1+z)^{n-1} = 1,$$

i.e.

$$(z+c)(1+z)^{n-1} = 1. \tag{14}$$

Therefore we study the inequality

$$(c+z)(1+z)^m \geqslant 1,$$

where c and z are real numbers with $0 < c < 1$, $z > 0$, $m \in \mathbb{N}^*$.

We establish now a lower bound for the root ξ. Note that 1 is such a bound. On the other hand, because $\xi > 1$, and $n \geqslant 2$

$$a = \xi^n - \xi^{n-1} \leqslant \xi^n - \xi^{n/2},$$

therefore

$$\xi \geqslant \left(\frac{1+\sqrt{4a+1}}{2}\right)^{2/n},$$

which is another lower bound for the root ξ. □

EXAMPLES. Denote by $T(a,n)$ the upper bound of ξ given by Theorem 1.4. The last column refers to the corresponding formulas given in Lemma 1.3.

a	n	$T(a,n)$	Attained for
898×10^9	9	21.416	m_1
6.2	123	1.023	m_2
5.9	4	1.975	m_3

COROLLARY 1.5. *Let $a > 0$, $n \in \mathbb{N}$, $n \geqslant 2$. Then the unique root $\xi > 1$ of $x^n - x^{n-1} - a = 0$ fulfills the conditions*

$$\max\left\{1, a^{\frac{1}{n}}\right\} \leqslant \xi \leqslant M(a,n) := \begin{cases} S(a,n), & \text{if } 0 < a < 1, \\ T(a,n), & \text{if } a > 1. \end{cases}$$

2. A Bound for the Roots of Complex Polynomials

PROPOSITION 2.1. *Let $P = a_n X^n + a_{n-k} X^{n-k} + \cdots + a_0 \in \mathbb{C}[X]$, $k \geqslant 1$, $K = K(P) = \max\limits_{0 \leqslant i \leqslant n-k} |a_i/a_n|$. If $z \in \mathbb{C}$ is a root of P, then*

$$|z| \leqslant M(K,k),$$

where $M(a,n)$ is defined in Corollary 1.5 for $n \geqslant 2$ and $M(a,1) = a+1$.

Proof. Let

$$H_1 = \max_{0 \leqslant i \leqslant n-k} |a_i|.$$

If $z \in \mathbb{C}$, $|z| > 1$, then

$$\begin{aligned}|P(z)| \geqslant& |a_n| \cdot |z|^n - (|a_0| + |a_1| \cdot |z| + \cdots + |a_{n-1}| \cdot |z|^{n-k})\\ \geqslant& |a_n| \cdot |z|^n - H_1 \cdot (1 + |z| + \cdots + |z|^{n-k})\\ >& |a_n| \cdot |z|^n - \frac{H_1 |z|^{n-k+1}}{|z|-1}\\ =& |a_n| \frac{|z|^{n+1} - |z|^n - \frac{H_1 |z|^{n-k+1}}{|a_n|}}{|z|-1}\\ =& |a_n| \frac{|z|^{n+1} - |z|^n - K|z|^{n-k+1}}{|z|-1}.\end{aligned}$$

Consider the upper bound $M(K,k)$ of the root > 1 of the equation $x^k - x^{k-1} = K$ given by Corollary 1.5 if $k \geqslant 2$ and which is obvious if $k = 1$:

$$\text{If} \quad |z| \geqslant M(K,k), \quad \text{then} \quad P(z) \neq 0.$$

Therefore $M(K,k)$ is an upper bound for the moduli of the roots of P. □

REMARK. If $|z| > 1 + K$, then

$$|z|^{n+1} - |z|^n - K = |z|^n(|z| - 1) - K > (1+K)^n \cdot K - K > 0.$$

If P is a monic integer polynomial, then $K = H(P) = \max_{0 \leqslant i \leqslant n} |a_i|$, so $1 + K = 1 + H(P)$ which is the upper bound for the roots of P, obtained by Cauchy (1829).

REMARK. Following M. Mignotte [3], instead of K one may consider a convenient Hölder norm L_p^q.

3. Applications to Lacunary Polynomials

We now use the estimate of the unique root > 1 of a polynomial $x^n - x^{n-1} - a$ to derive bounds for the absolute values of the roots of lacunary polynomials. A polynomial $P(X) = \sum_{i=0}^n a_i X^i \in \mathbb{C}[X]$ is called *lacunary* if there exists $k \geqslant 2$ such that $a_{n-1} = \cdots = a_{n-k+1} = 0$ and $a_{n-k} \neq 0$.

3.1. Complex lacunary polynomials.

THEOREM 3.1. *Let* $P(X) = a_n X^n + a_{n-k} X^{n-k} + \cdots + a_1 X + a_0 \in \mathbb{C}[X]$, *with* $a_n, a_{n-k} \neq 0$ *and* $k \geqslant 2$. *Let* $p, q \in [1, \infty]$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and denote*

$$L_p = \sqrt[p]{\sum_{j=0}^{n-k} |a_j|^p}.$$

If $w \in \mathbb{C}$ *is a root of* P*, then*

$$|\omega| \leqslant \left(M\left(\left(\frac{L_p^q}{|a_n|}\right)\right), k\right)^{\frac{1}{q}},$$

where $M(a,n)$ *is given by Corollary 1.5.*

Proof. By M. Mignotte [3], if $\omega \in \mathbb{C}$ is a root of P, then

$$|\omega|^k - |\omega|^{k-1} - \frac{L_p^q}{|a_n|} < 0.$$

We now take $a = L_p^q/|a_n|$ and $n = k$ in Corollary 1.5. □

3.2. Lacunary polynomials with nonnegative coefficients.

THEOREM 3.2. *Let* $P(X) = a_n X^n + a_{n-k} X^{n-k} + \cdots + a_1 X + a_0 \in \mathbb{R}[X]$. *If*

$$0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-k}, \quad a_n > 0,$$

then all roots of P *lie in the domain*

$$\{z \in \mathbb{C}\,; |z| \leqslant 1\} \cup \{z \in \mathbb{C}\,; a_n|z|^k - a_{n-k} \leqslant |a_n z^{k-1} - a_{n-k}|\}.$$

Proof.
Let $\omega \in \mathbb{C}$ be a root of P. We put $\sigma = |\omega|$.
We observe that

$$\begin{aligned}(1-\omega)P(\omega) &= a_0 + a_1\omega + \cdots + a_{n-k}\omega^{n-k} + a_n\omega^n - a_0\omega - a_1\omega^2 - \cdots \\ &\quad - a_{n-k-1}\omega^{n-k} - a_{n-k}\omega^{n-k+1} - a_n\omega^{n+1} \\ &= a_0 + (a_1 - a_0)\omega + (a_2 - a_1)\omega^2 + \cdots \\ &\quad + (a_{n-k} - a_{n-k-1})\omega^{n-k} + (a_n\omega^{k-1} - a_{n-k})\omega^{n-k+1} - a_n\omega^{n+1}.\end{aligned}$$

Therefore, for $\sigma = |\omega| > 1$,

$$\begin{aligned}|(1-\omega)P(\omega)| &\geqslant a_n|\omega|^{n+1} - |a_0 + (a_1 - a_0)\omega + (a_2 - a_1)\omega^2 \\ &\quad + \cdots + (a_{n-k} - a_{n-k-1})\omega^{n-k} + (a_n\omega^{k-1} - a_{n-k})\omega^{n-k+1}| \\ &\geqslant a_n\sigma^{n+1} - (|a_0| + |a_1 - a_0|\sigma + |a_2 - a_1|\sigma^2 + \cdots \\ &\quad + |a_{n-k} - a_{n-k-1}|\sigma^{n-k} + |a_n\omega^{k-1} - a_{n-k}|\sigma^{n-k+1}) \\ &\geqslant a_n\sigma^{n+1} - (|a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_{n-k} - a_{n-k-1}| \\ &\quad + |a_n\omega^{k-1} - a_{n-k}|)\sigma^{n-k+1} \\ &= a_n\sigma^{n+1} - (a_{n-k} + |a_n\omega^{k-1} - a_{n-k}|)\sigma^{n-k+1} \\ &= (a_n\sigma^k - a_{n-k} - |a_n\omega^{k-1} - a_{n-k}|)\sigma^{n-k+1}.\end{aligned}$$

If

$$a_n|\omega|^k - a_{n-k} > |a_n\omega^{k-1} - a_{n-k}|,$$

then $|(1-\omega)P(\omega)| > 0$, so ω cannot be a root of P.

Therefore

$$|\omega| \leqslant 1 \quad \text{or} \quad a_n|\omega|^k - a_{n-k} \leqslant |a_n\omega^{k-1} - a_{n-k}|.$$

□

REMARK. Theorem 3.2 is a lacunary version of Eneström–Kakeya criterion, cf. [1], 1.2.

COROLLARY 3.3. *If $z \in \mathbb{C}$ is a root of the polynomial P, then*

$$|z| \leqslant M\left(2\frac{a_{n-k}}{a_n}, k\right).$$

Proof. First note that $|a_n z^{k-1} - a_{n-k}| \leqslant a_n|z|^{k-1} + a_{n-k}$. Hence $a_n|z|^k - a_n|z|^{k-1} - 2a_{n-k} \leqslant 0$, i.e.

$$|z| \leqslant 1 \quad \text{or} \quad |z|^k - |z|^{k-1} - 2\frac{a_{n-k}}{a_n} \leqslant 0.$$

□

REMARK. *Let $z \in \mathbb{C}$ be a root of the polynomial*

$$P(X) = a_nX^n + a_{n-k}X^{n-k} + \cdots + a_1X + a_0 \in \mathbb{R}[X].$$

a) If

$$0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-k} \leqslant \frac{a_n}{2},$$

then

$$|z| < S\left(\frac{2a_{n-k}}{a_n}, k\right).$$

b) If

$$0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-k} \quad \text{and} \quad a_{n-k} > \frac{a_n}{2},$$

then

$$|z| < T\left(\frac{2a_{n-k}}{a_n}, k\right).$$

PROPOSITION 3.4. *Let* $P(X) = a_nX^n + a_{n-k}X^{n-k} + \cdots + a_1X + a_0 \in \mathbb{R}[X]$, *where* $a_n,\ a_{n-k},\ \ldots,\ a_1,\ a_0 > 0$. *If* $z \in \mathbb{C}$ *is a root of* P, *then*

$$|z| \leqslant \alpha M\left(\frac{2a_{n-k}}{\alpha^k a_n}, k\right),$$

where $\alpha = \max_{1\leqslant i\leqslant n-k}\left(\frac{a_{i-1}}{a_i}\right)$.

Proof. For $\alpha > 0$ consider the associated polynomial

$$P_\alpha(Y) = P(\alpha Y) = a_n\alpha^nY^n + \sum_{i=0}^{n-k} a_i\alpha^iY^i.$$

If

$$0 < a_0 \leqslant a_1\alpha \leqslant \cdots \leqslant a_{i-1}\alpha^{i-1} \leqslant a_i\alpha^i \leqslant \cdots \leqslant a_{n-k}\alpha^{n-k} \tag{15}$$

then, by Corollary 3.3, the roots of P_α are bounded by $M\left(\frac{2a_{n-k}}{\alpha^k a_n}, n\right)$.

But (15) is fulfilled if and only if

$$\alpha \geqslant \frac{a_{i-1}}{a_i}, \quad \forall i \in \{1, 2, \ldots, n-k\}.$$

We take

$$\alpha = \max_{1\leqslant i\leqslant n-k}\left(\frac{a_{i-1}}{a_i}\right)$$

and the roots of P are bounded by $\alpha M\left(\frac{2a_{n-k}}{\alpha^k a_n}, k\right)$. $\square$

4. Another Estimate for the Roots of Lacunary Polynomials

PROPOSITION 4.1. *Let* $P(X) = X^n + a_{n-k}X^{n-k} + \cdots + a_1X + a_0 \in \mathbb{C}[X] \setminus \mathbb{C}$. *Then the moduli of the roots of* P *are bounded by*

$$B := \max\left\{1, \left(|a_0| + |a_1| + \cdots + |a_{n-k}|\right)^{\frac{1}{k}}\right\}.$$

Proof. Let $z \in \mathbb{C}$ be a root of P. If $|z| > 1$ and if $|z| > \left(|a_0| + |a_1| + \cdots + |a_{n-k}|\right)^{\frac{1}{k}}$, then

$$|z|^n - \left(|a_0| + |a_1| + \cdots + |a_{n-k}|\right)|z|^{n-k} > 0.$$

On the other hand

$$\left|\sum_{j=0}^{n-k} a_jz^j\right| \leqslant \sum_{j=0}^{n-k} |a_j| \cdot |z|^j \leqslant \left(\sum_{j=0}^{n-k} |a_j|\right) \cdot |z|^{n-k}.$$

Therefore

$$|P(z)| \geqslant |z^n| - \left|\sum_{j=0}^{n-k} a_j z^j\right| \geqslant |z|^n - \left(\sum_{j=0}^{n-k} |a_j|\right) \cdot |z|^{n-k} > 0,$$

so $P(z) \neq 0$.

Therefore the absolute values of the roots of P are $\leqslant \max\{1, (|a_0| + |a_1| + \cdots + |a_{n-k}|)^{\frac{1}{k}}\}$. $\square$

REMARK. If in Proposition 4.1 we consider P an integer polynomial which is not a monomial, then we may take $B := (|a_0| + |a_1| + \cdots + |a_{n-k}|)^{\frac{1}{k}}$.

5. Examples

We consider the bounds for the moduli of the roots of lacunary polynomials obtained throughout this paper. In addition we refer also to the estimates from [3] and [4].

The following notations will be used:

$$A = A(\infty) = M\big(H(P), k\big),$$

$$A(p) = M\big(L_p^q, k\big)^{\frac{1}{q}},$$

$$B = \big(|a_0| + |a_1| + \cdots + |a_{n-k}|\big)^{\frac{1}{k}}, \qquad \text{(Prop. 4.1)}$$

$$C(p) = \left(1 + \frac{\log k}{k-1}\right)^{\frac{1}{q}} \left(\Big(\sum_{j=0}^{n-k} |a_j|^p\Big)^{\frac{1}{p}}\right)^{\frac{1}{k}}, \qquad \text{(M. Mignotte [3])}$$

$$C = C(\infty) = \Big(1 + \frac{\log k}{k-1}\Big) H^{\frac{1}{k}},$$

$$D(p) = \left(1 + \Big(\sum_{j=0}^{n-k} |a_j|^p\Big)^{\frac{q}{p}}\right)^{\frac{1}{q}}, \qquad \text{(P. Montel [4])}$$

$$E = M\left(\frac{2a_{n-k}}{\alpha^k a_n}, n\right), \qquad \text{(Prop. 3.4)}$$

Note that the bound E will be used only for polynomials with positive coefficients.

Consider the polynomials:

$$P_1(X) = X^{10} + 29X^6 + 27X^5 + 30X^4 - 28X^3 + 29X^2 + 28X - 9,$$
$$P_2(X) = X^{10} + 13X^6 + 15X^5 + 14X^4 - 9X^3 + 14X^2 - 13X + 14,$$
$$P_3(X) = X^{11} + 6X^6 + 9X^5 + 8X^4 - 7X^3 + 4X^2 + 7X + 6,$$
$$P_4(X) = X^{12} - 32X^4 - 17X^3 + 18X^2 + 18X + 19,$$
$$P_5(X) = X^{17} - 8X^4 - 3X^3 + 5X^2 + 7X - 6,$$
$$P_6(X) = X^4 + 3X + 2,$$
$$P_7(X) = X^5 + 3X^2 + 2X + 12,$$
$$P_8(X) = X^7 + 17X^4 + 18X^3 + 2X^2 + 3X + 12,$$
$$P_9(X) = X^9 + 321X^4 + 24X^3 + 3X^2 + 17X + 5.$$
$$P_{10}(X) = X^{11} + 2X^3 + 4X^2 + 5X + 2,$$
$$P_{11}(X) = X^{12} + 29X^6 + 27X^5 + 26X^4 + 26X^3 + 25X^2 + 25X + 24,$$
$$P_{12}(X) = X^{14} + 12X^3 + 13X^2 + 10X + 9.$$

The following estimates include the cases $p = 1.1$, $p = 2.9$ and $p = \infty$.

P	A	$A(2.9)$	$A(1.1)$	B	C	$C(2.9)$	$C(1.1)$	$D(2.9)$	$D(1.1)$	E
P_1	2.81	4224.65	4.56	3.66	3.42	3.94	5.12	53.11	151.48	—
P_2	2.49	954.42	3.72	3.09	2.87	3.30	4.33	26.25	77.16	—
P_3	1.98	81.74	2.43	2.15	2.17	2.36	2.92	13.87	39.47	—
P_4	1.79	46.89	1.97	1.78	2.0003	2.04	2.27	38.96	90.14	—
P_5	1.28	4.704	1.35	1.29	1.42	1.45	1.55	11.04	25.17	—
P_6	2.07	28.26	1.86	1.709	2.23	2.304	2.59	3.63	4.704	2.28
P_7	2.93	1084.06	3.04	2.57	3.54	3.55	3.89	12.27	15.83	6.30
P_8	3.31	6963.79	4.75	3.73	4.06	4.43	5.55	23.61	46.01	5.39
P_9	3.48	16448.63	4.13	3.26	4.447	4.448	4.53	321.11	353.89	6.53
P_{10}	1.42	6.56	1.44	1.37	1.58	1.62	1.76	6.21	11.55	2.04
P_{11}	2.08	247.61	2.75	2.3805	2.3809	2.61	3.13	51.10	152.51	2.26
P_{12}	1.405	9.16	1.49	1.41	1.56	1.61	1.72	18.22	38.83	1.48

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