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MEASURES OF ALGEBRAIC SUMS OF SETS

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Abstract. A variety of measure–theoretic inequalities are derived for algebraic sum sets involving sets with fractal structure. The derivations are based on combinatorial inequalities which in turn are derived from canonical univariate algebraic inequalities for polynomials in noninteger powers. A systematic procedure is presented and some known results generalized.

1. Introduction

Measure–theoretic properties of algebraic sums of sets have been studied by a number of authors (see, for example, [2], [5–9], [12] and [15–17]). One typical problem is the following. If two sets E, F are very thin in some sense (for example, they have Lebesgue measure zero) how "thick" will their sum set $E+F$ be? By the algebraic sumset $E+F$ we signify the set

$$E+F \;=\; \{x+y : x\in E,\ y\in F\}.$$

Suppose m denotes Lebesgue measure and μ_c Cantor–Lebesgue measure, that is, the uniform distribution on the Cantor subset C of [0,1] formed by repeated removal of middle thirds. The following result has been established by Brown and Moran [5].

THEOREM A. *If* E, F *are Borel subsets of [0,1], then*

$$m(E+F) \geqslant 2\mu_c(E)^{\alpha}\mu_c(F)^{\alpha}, \tag{1.1}$$

where $\alpha = \log 3/\log 4$.

An immediate corollary is that μ_c is a basic measure (see [25] and [6]).

A related result has been derived by Oberlin [16]. Suppose that $E\subset[0,1]$ and $F\subset C$ are Borel sets. Then

$$m(E+F) \;\geqslant\; 2m(E)^{1-\log 2/\log 3}\mu_C(F).$$

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In [16] this result appears without the factor 2 on the right, since the sum $E+F$ is taken on the circle group $T \equiv [0,1]$. If sums are so evaluated, the factor 2 disappears from (1.1) also.

Brown and Moran's proof in [5] follows from the inequality

$$x^\alpha y^\alpha + \max\{x^\alpha(1-y)^\alpha, y^\alpha(1-x)^\alpha\} + (1-x)^\alpha(1-y)^\alpha \geqslant 1 \quad (0 \leqslant x, y \leqslant 1) \tag{1.2}$$

established by Woodall [26].

Woodall and subsequently Hajela and Seymour [11] derived a variety of interesting results in combinatorial geometry from the latter inequality, which is therefore of some interest in its own right. The history of these results and related ideas pertaining to a multivariate extension of (1.2) (see [14], [4]) is quite colourful. A brief account is given by Brown [2].

In [2] Brown also stated that (1.2) follows from a simpler canonical univariate inequality.

PROPOSITION B. *Suppose that* $s,\ t \geqslant 1$ *and* $s^{-1} + t^{-1} = \log 3/\log 2$. *Then*

$$1 + x + x^2 \geqslant (1+x^s)^{1/s}(1+x^t)^{1/t} \tag{1.3}$$

for all for $0 \leqslant x \leqslant 1$ *if and only if* $3(s+t) \leqslant 8$.

Proposition B has also proved seminal. Kemp [13] has given a simpler proof and noted that the domain $0 \leqslant x \leqslant 1$ can be replaced by $x \geqslant 0$, since (1.3) is invariant under the tranformation $x \to 1/x$. She proved also that for $x \geqslant 0$ (1.3) holds in the overlapping regions

$$s^{-1} + t^{-1} \leqslant \log 3/\log 2, \qquad s \geqslant s_1, \quad t \geqslant s_1,$$

and

$$s^{-1} + t^{-1} \leqslant 1.5, \qquad s \geqslant 1, \quad t \geqslant 1,$$

where $s_1 = 1.0246\ldots$ is the (unique) solution of $s^{-1}+t^{-1} = \log 3/\log 2$, $s+t = 8/3$, $s < t$.

The condition $s,t \geqslant 1$ in Proposition B derives from Lemma 4 of [7]. In [18] an improvement of this lemma was proved without the restriction on s,t (see also [19], [22]). From [18] it follows that this restriction is equivalent to $s,t > 0$. The results of [18] and [19] have been taken further by Alzer [1].

In the following section we establish a generalization of (1.1) which allows two different exponents α, β on the right–hand side. We show how this follows from a two–exponent version of (1.2), which will in turn be derived from Proposition B. In fact, these results can be strengthened by working from an appropriate generalization of Proposition B. So as to emphasize the simplicity of the approach, this strengthening is postponed to Section 3.

In the remainder of the paper we show how our technique leads directly to a number of other measure–theoretic inequalities of the type

$$m(E+F) \;\geqslant\; k\mu(E)^\alpha \nu(F)^\beta \tag{1.4}$$

for appropriately chosen measures μ, ν and constants α, β. In Section 4 we extend some canonical univariate inequalities established in [2]. Associated multivariate inequalities are derived in Section 5. The consequent measure–theoretic inequalities for sumsets are presented in Section 6.

To minimize the appearance of the intrusive constant k in specific instances of (1.4), we shall follow Oberlin and throughout take the addition of sets as being modulo [0,1].

2. Generalizing Theorem A

We begin with a generalization of Proposition B.

PROPOSITION 2.1. *Suppose* $0 < \alpha, \beta \leqslant 1$ *with* $\alpha + \beta = \log 3/\log 2$ *and* $3(\alpha^{-1} + \beta^{-1}) \leqslant 8$. *Then*

$$x^\alpha y^\beta + \max\{x^\alpha(1-y)^\beta, (1-x)^\alpha y^\beta\} + (1-x)^\alpha(1-y)^\beta \geqslant 1$$

for all $x, y \in [0,1]$.

Proof. As $0 < \alpha, \beta \leqslant 1$, the desired result holds when $x = 1$ or $y = 1$, so without loss of generality we may assume $x, y < 1$. Again by symmetry, we may assume without loss of generality that

$$x^\alpha(1-y)^\beta \geqslant (1-x)^\alpha y^\beta \tag{2.1}$$

or

$$\left(\frac{x}{1-x}\right)^\alpha \geqslant \left(\frac{y}{1-y}\right)^\beta.$$

We wish to show that

$$x^\alpha y^\beta + x^\alpha(1-y)^\beta + (1-x)^\alpha(1-y)^\beta \geqslant 1. \tag{2.2}$$

Obviously (2.2) holds when $y = 0$, since $\alpha < 1$. Also the second derivative of the left–hand side with respect to y is nonpositive since $\beta \leqslant 1$. Hence the left–hand side of (2.2) is concave in y and we need only check that (2.2) is satisfied when (2.1) holds with equality, that is, when

$$y = \frac{\left(\frac{x}{1-x}\right)^{\alpha/\beta}}{1 + \left(\frac{x}{1-x}\right)^{\alpha/\beta}}.$$

We need to verify that

$$x^\alpha\left(\frac{x}{1-x}\right)^\alpha + x^\alpha + (1-x)^\alpha \geqslant \left(1 + \left(\frac{x}{1-x}\right)^{\alpha/\beta}\right)^\beta. \tag{2.3}$$

If $x \leqslant 1/2$, divide both sides of (2.3) by $(1-x)^\alpha$ and let $u=(x/(1-x))^\alpha$, while when $x>1/2$ multiply both sides by $(1-x)^\alpha/x^{2\alpha}$ and let $u=((1-x)/x)^\alpha$. In either case the condition is

$$1+u+u^2 \geqslant (1+u^{1/\alpha})^\alpha(1+u^{1/\beta})^\beta.$$

The desired result follows from Proposition B. □

We are now ready to extend Theorem A.

THEOREM 2.2. *Suppose that* $0<\alpha,\beta \leqslant 1$ *and* E, F *are Borel subsets of* $[0,1]$. *Then*

$$m(E+F) \geqslant \mu_c(E)^\alpha \mu_c(F)^\beta \tag{2.4}$$

whenever $\alpha+\beta=\log 3/\log 2$ *and* $3(\alpha^{-1}+\beta^{-1}) \leqslant 8$.

Proof. Since μ_c and m are regular measures, it suffices to prove the result for E, F closed and contained in C. Let

$$A_n=\left\{\sum_{i=1}^n \frac{\epsilon_i}{3^i}\,\middle|\, \text{there exists } x\in E \text{ such that } x=\sum_{i=1}^\infty \frac{\epsilon_i}{3^i},\ \epsilon_i=0 \ \text{ or } \ 2\right\},$$

$$A_{n+1}^j=\left\{\sum_{i=1}^n \frac{\epsilon_i}{3^i}\in A_{n+1}\,\middle|\, i_{n+1}=2j\right\} \quad (j=0,1)$$

and let B_n, B_n^j denote the corresponding quantities for $x\in F$.

We can establish by induction that

$$|A_n+B_n| \ \geqslant \ |A_n|^\alpha |B_n|^\beta, \tag{2.5}$$

where as subsequently $|A|=$ card A. By a direct check, this holds for $n=1$. Suppose the result is true for some $n \geqslant 1$. By considering the last digit in the base 3 expansion of $A_{n+1}+B_{n+1}$, we obtain that

$$\begin{aligned}
|A_{n+1}+B_{n+1}| \ &\geqslant \ |A_{n+1}^0+B_{n+1}^0|+|A_{n+1}^1+B_{n+1}^1|+\max\{|A_{n+1}^0+B_{n+1}^1|,|A_{n+1}^1+B_{n+1}^0|\}\\
&\geqslant \ |A_{n+1}^0|^\alpha|B_{n+1}^0|^\beta+|A_{n+1}^1|^\alpha|B_{n+1}^1|^\beta\\
&\qquad\qquad +\max\{|A_{n+1}^0|^\alpha|B_{n+1}^1|^\beta,|A_{n+1}^1|^\alpha|B_{n+1}^0|^\beta\}.
\end{aligned}$$

Here the second step follows from the inductive assumption. To derive (2.5) it suffices to show that

$$\begin{aligned}
&\left(\frac{|A_{n+1}^0|}{|A_{n+1}|}\right)^\alpha\left(\frac{|B_{n+1}^0|}{|B_{n+1}|}\right)^\beta+\left(\frac{|A_{n+1}^1|}{|A_{n+1}|}\right)^\alpha\left(\frac{|B_{n+1}^|)}{|B_{n+1}|}\right)^\beta\\
&\qquad +\max\left\{\left(\frac{|A_{n+1}^0|}{|A_{n+1}|}\right)^\alpha\left(\frac{|B_{n+1}^1|}{|B_{n+1}|}\right)^\beta,\left(\frac{|A_{n+1}^1|}{|A_{n+1}|}\right)^\alpha\left(\frac{|B_{n+1}^0|}{|B_{n+1}|}\right)^\beta\right\}\geqslant 1.
\end{aligned}$$

Since $A^j_{n+1} \subset A_{n+1}$ and $B^j_{n+1} \subset B_{n+1}$, the inductive step now follows from Proposition 2.1.

Define

$$E_n = A_n + [0, 3^{-n}], \quad F_n = B_n + [0, 3^{-n}].$$

Then $E = \bigcap_{n=1}^\infty E_n$, $F = \bigcap_{n=1}^\infty F_n$ and by compactness $E + F = \bigcap_{n=1}^\infty (E_n + F_n)$.

Since $2^{\alpha+\beta} = 3$, we have

$$\begin{aligned} m(E+F) &= \lim_n m(E_n + F_n) \\ &= \lim_n |A_n + B_n| 3^{-n} \\ &\geqslant \lim_n |A_n|^\alpha |B_n|^\beta 3^{-n} \\ &= \lim_n |A_n|^\alpha 2^{-n\alpha} \times \lim_n |B_n|^\beta 2^{-n\beta} \\ &= \mu_c(E)^\alpha \mu_c(F)^\beta. \end{aligned}$$

□

Since all measures under consideration are probability measures, (2.4) entails that for any $\gamma \geqslant \alpha$ and $\delta \geqslant \beta$ we have

$$m(E+F) \geqslant \mu_c(E)^\gamma \mu_c(F)^\delta,$$

which provides an extension of Theorem 2.2. However we can do better. This is is the aim of the next section.

To complete our preliminary overview, we now list several other measure–theoretic results like Theorem 2.2 and the multivariate inequalities in order from which they derive. The derivation is in each case closely similar to that presented in this section. We return to the question of proving the multivariate inequalities in Section 5, where we establish strengthened versions of them.

We use ν_1 to denote the probability measure on $[0,1]$ which is uniformly distributed on all the numbers whose base 4 expansion contains only the digits 0 and 1.

THEOREM 2.3 *Suppose that* $0 \leqslant \alpha,\ \beta \leqslant 1$. *Let* $E,\ F$ *be any Borel subsets of [0,1]. Then we have the following results.*

(i) $m(E+F) \geqslant m(E)^\alpha \mu_c(F)^\beta$ *whenever* $\alpha + \beta \dfrac{\log 2}{\log 3} = 1$;

(ii) $m(E+F) \geqslant m(E)^\alpha \nu_1(F)^\beta$ *whenever* $\alpha + \dfrac{1}{2}\beta = 1$.

PROPOSITION 2.4. *Suppose that* a_i, b_i *are nonnegative numbers satisfying* $\sum_i a_i = 1 = \sum_i b_i$, *where the summation is over the relevant set of values* i *appearing in the inequality concerned. Let* $0 \leqslant \alpha, \beta \leqslant 1$.

(i) If $\alpha + \beta \dfrac{\log 2}{\log 3} = 1$, *then*

$$\max\left\{a_0^\alpha b_0^\beta, a_2^\alpha b_1^\beta\right\} + \max\left\{a_0^\alpha b_1^\beta, a_1^\alpha b_0^\beta\right\} + \max\left\{a_1^\alpha b_1^\beta, a_2^\alpha b_0^\beta\right\} \geqslant 1.$$

(ii) If $\alpha + \dfrac{1}{2}\beta = 1$, *then*

$$\max\left\{a_0^\alpha b_0^\beta, a_3^\alpha b_1^\beta\right\} + \max\left\{a_0^\alpha b_1^\beta, a_1^\alpha b_0^\beta\right\} + \max\left\{a_1^\alpha b_1^\beta, a_2^\alpha b_0^\beta\right\} + \max\left\{a_2^\alpha b_1^\beta, a_3^\alpha b_0^\beta\right\} \geqslant 1.$$

3. Strengthening Theorem 2.2

First we derive a sharpened form of Proposition B.

PROPOSITION 3.1. *Suppose that* α, β *are positive constants satisfying*

(i) $\alpha + \beta \geqslant \log 3/\log 2$;

(ii) $3(\alpha^{-1} + \beta^{-1}) \leqslant 8$.

Then for all $x \geqslant 0$,

$$1 + x + x^2 \geqslant \frac{3}{2^{\alpha+\beta}}(1 + x^{1/\alpha})^\alpha (1 + x^{1/\beta})^\beta. \tag{3.1}$$

Proof. Since (3.1) is invariant under the transformation $x \to 1/x$, it is sufficient to establish the result for $0 \leqslant x \leqslant 1$. Further, for each given value of

$$u = \alpha + \beta, \tag{3.2}$$

we have by [7, Lemma 4] that the right–hand side of (3.1) is maximized by the extremal values of α, β subject to (ii). Hence without loss of generality we may replace (ii) by

$$3(\alpha^{-1} + \beta^{-1}) = 8. \tag{3.3}$$

We shall proceed regarding α, β as functions of u determined by (3.2) and (3.3). If α and β were equal, then (3.3) would imply $\alpha = \beta = 3/4$, so that $u = 3/2$, which contradicts (i) since $\log 3/\log 2 > 3/2$. Hence α, β are distinct and we may choose $\alpha > \beta$.

Set

$$f(u,x) := \frac{3}{2^u}(1 + x^{1/\alpha})^\alpha (1 + x^{1/\beta})^\beta.$$

When $u = \log 3/\log 2$, we have $3/2^u = 1$ and so by Proposition B $f(u,x) \leqslant 1+x+x^2$. We establish our result by showing that $f(u,x)$ is decreasing in u. For convenience of calculation we consider the logarithm of $f(u,x)$. Let

$$g(u,x) := \ln f(u) = \ln 3 - u\ln 2 + \alpha \ln(1 + x^{\alpha^{-1}}) + \beta \ln(1 + x^{\beta^{-1}}).$$

Then

$$g'_u(u,x) = -\ln 2 \; + \; \left[\ln(1 + x^{\alpha^{-1}}) - \frac{x^{\alpha^{-1}} \ln x}{\alpha(1 + x^{\alpha^{-1}})}\right] \frac{d\alpha}{du} + \left[\ln(1 + x^{\beta^{-1}}) - \frac{x^{\beta^{-1}} \ln x}{\beta(1 + x^{\beta^{-1}})}\right] \frac{d\beta}{du}.$$

From (3.2) and (3.3) we get

$$\frac{d\alpha}{du} = \frac{\alpha^2}{\alpha^2 - \beta^2}, \qquad \frac{d\beta}{du} = \frac{\beta^2}{\alpha^2 - \beta^2}.$$

Therefore

$$g'_u(u,x) = -\ln 2 \; + \; \frac{1}{\alpha^2 - \beta^2}\left[\alpha\left(\alpha \ln(1 + x^{\alpha^{-1}}) - \frac{x^{\alpha^{-1}} \ln x}{(1 + x^{\alpha^{-1}})}\right) - \beta\left(\beta \ln(1 + x^{\beta^{-1}}) - \frac{x^{\beta^{-1}} \ln x}{(1 + x^{\beta^{-1}})}\right)\right].$$

Let $h(u,x)$ denote the expression in brackets. Then

$$\begin{aligned} h'_x(u,x) &= \left(\frac{\alpha x^{\alpha^{-1}-1}}{1 + x^{\alpha^{-1}}} - \frac{x^{\alpha^{-1}-1} \ln x + \alpha x^{\alpha^{-1}-1}}{1 + x^{\alpha^{-1}}} - \frac{x^{2\alpha^{-1}-1} \ln x}{(1 + x^{\alpha^{-1}})^2}\right) \\ &\quad - \left(\frac{\beta x^{\beta^{-1}-1}}{1 + x^{\beta^{-1}}} - \frac{x^{\beta^{-1}-1} \ln x + \beta x^{\beta^{-1}-1}}{1 + x^{\beta^{-1}}} - \frac{x^{2\beta^{-1}-1} \ln x}{(1 + x^{\beta^{-1}})^2}\right) \\ &= x^2 \ln x \left(-\frac{x^{\alpha^{-1}}}{(1 + x^{\alpha^{-1}})^2} - + \frac{x^{\beta^{-1}}}{(1 + x^{\alpha^{-1}})^2}\right) \\ &\geqslant 0, \end{aligned}$$

since $y/(1+y)^2$ increases as y increases on $[0,1]$ and $x^{\alpha^{-1}} \geqslant x^{\beta^{-1}}$ for $\alpha > \beta$ and $x \in [0,1]$. Hence

$$g'_u(u,x) \leqslant -\ln 2 + \frac{1}{\alpha^2 - \beta^2} h(u,1) = 0,$$

which completes the proof. □

This result enables us to derive the following further extension of Woodall's inequality.

PROPOSITION 3.2. *Suppose that α, β are positive constants satisfying*

(i) $\alpha, \beta \geqslant \dfrac{\log 3}{\log 2} - 1$*;*

(ii) $\alpha + \beta \geqslant \dfrac{\log 3}{\log 2}$*;*

(iii) $3(\alpha^{-1} + \beta^{-1}) \leqslant 8$*;*

(iv) at least one of α, β *does not exceed unity.*

Then for $a, b \in [0, 1]$ *we have*

$$a^\alpha b^\beta + \max\{a^\alpha(1-b)^\beta, (1-a)^\alpha b^\beta\} + (1-a)^\alpha(1-b)^\beta \geqslant \frac{3}{2^{\alpha+\beta}}. \tag{3.4}$$

Proof. First we remark that (ii) is equivalent to

$$\frac{3}{2^{\alpha+\beta}} \leqslant 1. \tag{3.5}$$

We now observe that (3.4) holds when either a or b is an endpoint of $[0, 1]$. For by symmetry it suffices to consider $a = 0, 1$. We then require

$$b^\beta + (1-b)^\beta \geqslant \frac{3}{2^{\alpha+\beta}}. \tag{3.6}$$

The left–hand side is convex or concave according as $\beta \geqslant 1$ or $\beta \leqslant 1$, so by symmetry it suffices to verify (3.6) for $b = 0, 1/2, 1$. The first and last cases are immediate by (3.5), while for $b = 1/2$, (3.6) becomes

$$2\left(\frac{1}{2}\right)^\beta \geqslant \frac{3}{2^{\alpha+\beta}}$$

which follows from (i). We may therefore suppose that $a, b \in (0, 1)$.

A relation equivalent to (3.4) is obtained by dividing both sides by $(1-a)^\alpha(1-b)^\beta$ and setting

$$x = [a/(1-a)]^\alpha, \qquad y = [b/(1-b)]^\beta.$$

We derive

$$1 + \max(x, y) + xy \geqslant \frac{3}{2^{\alpha+\beta}}(1 + x^{\frac{1}{\alpha}})^\alpha(1 + x^{\frac{1}{\beta}})^\beta, \qquad x, y \geqslant 0.$$

The cases where one of x, y is zero or infinity correspond respectively to one of a, b being zero or unity and have been dealt with.

Referring to condition (iv), let us take for definiteness $\beta \leqslant 1$ and consider

$$F(x, y) := 1 + (x, y) + xy - \frac{3}{2^{\alpha+\beta}}(1 + x^{\frac{1}{\alpha}})^\alpha(1 + y^{\frac{1}{\beta}})^\beta$$

for $0 < x,\ y < \infty$. Since $\beta \leqslant 1$, F is a concave function of y on $(0,x)$ and on (x,∞), so to verify that $F(x,y) \geqslant 0$ it is sufficient to check that this holds for $y = 0, x, \infty$. The first and last cases have already been established, so it remains to show that

$$1 + x + x^2 \geqslant \frac{3}{2^{\alpha+\beta}}(1 + x^{\frac{1}{\alpha}})^{\alpha}(1 + x^{\frac{1}{\beta}})^{\beta} \qquad \text{for } x > 0$$

given that (ii) and (iii) hold. This is the content of Proposition 3.1, so we are done. □

An argument parallel to that of Theorem 2.2 now establishes the following result.

THEOREM 3.3. *Let $E, F \subset [0,1]$ be Borel sets and suppose α, β are real numbers satisfying*

$$\begin{aligned} \alpha + \beta &\geqslant \frac{\log 3}{\log 2}, \\ 3(\alpha^{-1} + \beta^{-1}) &\leqslant 8, \\ \alpha, \beta &\geqslant \frac{\log 3}{\log 2} - 1. \end{aligned}$$

Then

$$m(E + F) \geqslant \mu_c(E)^{\alpha}\mu_c(F)^{\beta}.$$

This does not entail the restriction $\alpha, \beta \leqslant 1$. To see that it really has broadened the domain of values (α, β) to which (2.4) applies, note that $\alpha = \log 3/\log 2 - 1$, $\beta = 2$ satisfy the restrictions on α, β in Theorem 3.3. However, since

$$3\left(\left(\frac{\log 3}{\log 2} - 1\right)^{-1} + 1\right) > 8,$$

we cannot obtain this pair (α, β) by incrementing a pair satisfying Theorem 2.2 as proposed at the end of the preceding section.

An exactly similar development to the foregoing may be used to derive the following.

THEOREM 3.4. *Denote by ν_2 the probability measure spread uniformly on on all numbers on [0,1] in whose base 4 expansion no digit 3 appears. Let $E, F \subset [0,1]$ be Borel sets and suppose α, β are real numbers satisfying*

$$\begin{aligned} \alpha + \beta &\geqslant \frac{\log 3}{\log 2}, \\ 3(\alpha^{-1} + \beta^{-1}) &\leqslant 8, \\ \alpha, \beta &\geqslant \frac{\log 3}{\log 2} - 1. \end{aligned}$$

Then

$$\nu_2(E + F) \geqslant \nu_1(E)^{\alpha}\nu_1(F)^{\beta}.$$

This reflects the fact that a natural measure–preserving isomorphism σ exists between subsets of full measure of the supports of ν_1 and μ_c. The subset of ν_1 of full measure is the set of points $x=\sum_{i=1}^{\infty} a_i 4^{-i}$ with $a_i \in \{0,1\}$ and $a_i=0$ for infinitely many values of i. If $x=\sum_{i=1}^{\infty} a_i 4^{-i}$ with $a_i \in \{0,1\}$ then $\sigma(x)=\sum_{i=1}^{\infty} b_i 3^{-i}$ with $b_i=2a_i$. If E, F are Borel subsets of supp (ν_1), then $E+F$ is Lebesgue measurable and $\nu_2(E+F)=m(\sigma(E+F))$.

4. Canonical univariate inequalities

Our starting point is a collection of results proved in [2] (see also [13]).

PROPOSITION 4.1. *Suppose that* $s,t \geqslant 1$. *Then for* $0 \leqslant x \leqslant 1$ *we have*

(i) $1+x+x^2 \geqslant (1+x^s)^{1/s}(1+x^t+x^{2t})^{1/t}$ *whenever* $\dfrac{\log 2}{\log 3}s^{-1}+t^{-1}=1$;

(ii) $1+x+x^2+x^3 \geqslant (1+x^s)^{1/s}(1+x^t+x^{2t}+x^{3t})^{1/t}$ *whenever* $\dfrac{1}{2}s^{-1}+t^{-1}=1$;

(iii) $1+x+x^2+x^3 \geqslant (1+x^s+x^{2s})^{1/s}(1+x^t+x^{2t})^{1/t}$ *whenever* $s^{-1}+t^{-1}=\dfrac{\log 4}{\log 3}$;

(iv) $1+x+x^2+x^3 \geqslant (1+x^s+x^{2s})^{1/s}(1+x^t+x^{2t}+x^{3t})^{1/t}$ *whenever* $\dfrac{\log 3}{\log 4}s^{-1}+t^{-1}=1$.

To these we add the following result.

PROPOSITION 4.2. *Suppose that* $s,\ t>0$ *and satisfy* $\dfrac{1}{2}s^{-1}+\dfrac{\log 3}{\log 4}t^{-1}=1$. *Then for* $0 \leqslant x \leqslant 1$ *we have*

$$1+x+x^2+x^3 \geqslant (1+x^s)^{1/s}(1+x^t+x^{2t})^{1/t} \tag{4.1}$$

if and only if $3s+8t \leqslant 15$.

Sketch of Proof. First consider necessity. Let $x=1-y$. By a second–order Taylor expansion with respect to y we get that

$$1+x+x^2+x^3=4-6y+4y^2+\mathrm{o}(y^2)$$

and

$$(1+x^s)^{1/s}(1+x^t+x^{2t})^{1/t}=4-6y+\frac{3s+8t+9}{6}y^2+\mathrm{o}(y^2).$$

Hence if (4.1) holds then we must have $\dfrac{3s+8t+9}{6} \leqslant 4$, that is, $3s+8t \leqslant 15$.

For sufficiency, note that by Lemma 4 of [7] we need prove (4.1) only when $3s+8t=15$. From

$$\frac{1}{2}s^{-1}+\frac{\log 3}{\log 4}t^{-1}=1,$$
$$3s+8t=15,$$

we get that

$$s = 5 - \frac{8}{3}t, \qquad t_1 = 1.0126\ldots, \qquad t_2 = 1.4673\ldots .$$

Let

$$F_t(x) = \log(1 + x + x^2 + x^3) - \frac{1}{s(t)}\log(1 + x^{s(t)}) - \frac{1}{t}\log(1 + x^t + x^{2t}).$$

Then (3.1) is equivalent to $F_t(x) \geqslant 0$ when $0 \leqslant x \leqslant 1$. We have $F_t(0) = F_t(1) = 0$ and $F_t'(0) = 1$, $F_t'(1) = 0$. If $F_t'(x)$ has at most one zero in (0,1), then the desired result follows. To show this, we consider $G_t(x) = H_t(x)F_t'(x)$, where $G_t(x)$ and $H_t(x)$ are sums of nonnegative powers of x whose exponents are functions of t. We have $H_t(x) > 0$ for all $x \in [0,1]$. Then $G_t(x)$ and $F_t'(x)$ have the same zeros in(0,1). For $t = t_1$ we consider

$$\Phi(x) = x^{-1}G_{t_1}(x)$$

and when $t = t_2$

$$\Psi(x) = x^{-\frac{1}{2}}G_{t_2}(x).$$

By studying up to the fourth derivative of $\Phi(x)$ and the sixth of $\Psi(x)$, we can show that $G_{t_1}(x)$ and $G_{t_2}(x)$ have the required properties. For example, we can calculate that

$$\Psi(0) = \Psi'(0) = \Psi''(0) = \infty$$

and

$$\Psi(1) = \Psi'(1) = \Psi''(1) = 0.$$

If we can show that $\Psi^{(4)}(x) > 0$ for $x \in (0,1)$, then $\Psi''(x)$ is convex on (0,1) and we can obtain the required result. In fact, if $\Psi''(x)$ is convex on (0,1) then it is either positive on (0,1) or first positive and then negative on (0,1). If $\Psi''(x) > 0$ for $0 < x < 1$ then $\Psi'(x) < 0$ for $0 < x < 1$. Therefore $\Psi(x) > 0$ for $0 < x < 1$. Thus we get that $\Psi(x)$, then $G_{t_2}(x)$ and then $F_{t_2}'(x)$ has no zero on (0,1). If $\Psi''(x)$ is initially positive, $G_{t_2}(x)$ and then $F_{t_2}'(x)$ has only one zero on (0,1).

To show $\Psi^{(4)}(x) > 0$ we first derive

$$\Psi^{(4)}(0) = \infty, \quad \Psi^{(5)}(0) = -\infty, \quad \Psi^{(4)}(1) = 6, \quad \Psi^{(6)}(1) = -6.$$

The last step is checking that $\Psi^{(6)}(x) > 0$ for $0 < x < 1$ by comparing the coefficients of positive and negative terms. Because $\Psi^{(6)}(x)$ has 16 terms and the coefficients (except the first) are polynomials of t_2 of degree 6, the checking process entails tedious calculation and some numerical testing. The details are given in [10]. A simple proof would be welcome.

We now extend these canonical results.

PROPOSITION 4.3. *Let* α, β *be positive numbers. Then the following apply for all* $x \in [0,1]$.

(i) *If* $\beta \leqslant \alpha + \frac{\log 2}{\log 3}\beta$, *then*

$$1 + x + x^2 \geqslant \min\left\{1, \frac{3}{2^{\beta}3^{\alpha}}\right\}(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + x^{2/\alpha})^{\alpha}.$$

(ii) *If* $\beta \leqslant 2\alpha$, *then*

$$1 + x + x^2 + x^3 \geqslant \min\left\{1, \frac{4}{2^{\beta}4^{\alpha}}\right\}(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + x^{2/\alpha} + x^{3/\alpha})^{\alpha}.$$

(iii) *If* $\alpha, \beta \leqslant (\alpha + \beta)\frac{\log 3}{\log 4}$, *then*

$$1 + x + x^2 + x^3 \geqslant \min\left\{1, \frac{4}{3^{\alpha+\beta}}\right\}(1 + x^{1/\beta} + x^{2/\beta})^{\beta}(1 + x^{1/\alpha} + x^{2/\alpha})^{\alpha}.$$

(iv) *If* $\beta \leqslant \alpha + \beta\frac{\log 3}{\log 4}$, *then*

$$1 + x + x^2 + x^3 \geqslant \min\left\{1, \frac{4}{4^{\alpha}3^{\beta}}\right\}(1 + x^{1/\beta} + x^{2/\beta})^{\beta}(1 + x^{1/\alpha} + x^{2/\alpha} + x^{3/\alpha})^{\alpha}.$$

(v) *If* $u_1\beta \leqslant \alpha \leqslant u_2\beta$, *then*

$$1 + x + x^2 + x^3 \geqslant \min\left\{1, \frac{4}{3^{\alpha}2^{\beta}}\right\}(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + x^{2/\alpha})^{\alpha},$$

where u_1, u_2 *are respectively the lesser and greater solutions of*

$$6u^2 - (27\log_3 4 - 16)u + 8\log_3 4 = 0.$$

Proof. To prove (v), first make the substitutions $x \to x^{1/k}$, $s = k/\beta$, $t = k/\alpha$ in (4.1), which then becomes

$$(1 + x^{1/\beta})^{\beta/k}(1 + x^{1/\alpha} + x^{2/\alpha})^{\alpha/k} \leqslant 1 + x^{1/k} + x^{2/k} + x^{3/k}$$

or

$$(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + x^{2/\alpha})^{\alpha} \leqslant (1 + x^{1/k} + x^{2/k} + x^{3/k})^{k}. \tag{4.2}$$

For (4.1) to hold we need $\frac{1}{2}s^{-1} + t^{-1}\log 3/\log 4 = 1$ and $3s + 8t \leqslant 15$, which translate into $k = \frac{1}{2}\beta + \alpha\log 3/\log 4$ and

$$\left(\frac{1}{2}\beta + \frac{\log 3}{\log 4}\alpha\right)\left(\frac{3}{\beta} + \frac{8}{\alpha}\right) \leqslant 15$$

or

$$(6\log 3)u^2 - (27\log 4 - 16\log 3)u + 8\log 4 \leqslant 0,$$

where $u = \alpha/\beta$. Since $\log_3 4 = \log 4/\log 3$, this is simply the condition stated for (v).

For $k \geqslant 1$, the function $f(w) = w^k$ is convex and by Jensen's inequality, (4.2) gives

$$(1 + x^{1/k} + x^{2/k} + x^{3/k})^k \leqslant 4^{k-1}(1 + x + x^2 + x^3), \tag{4.3}$$

while for $k \leqslant 1$ it is concave and by Petrović's inequality (4.2) yields

$$(1 + x^{1/k} + x^{2/k} + x^{3/k})^k \leqslant 1 + x + x^2 + x^3. \tag{4.4}$$

On combining (4.2), (4.3) and (4.4) we get (v), since $4^k = 2^\beta 3^\alpha$.

The other parts are established in a closely similar way, with most interest residing in the conditions. For example, (i) uses the conditions $s, t \geqslant 1$ and $s^{-1}\log 2/\log 3 + t^{-1}$ of the corresponding part of Proposition 4.1. The second condition provides $k = \alpha + \beta\log 2/\log 3$. The first gives $\beta, \alpha \leqslant k$. The latter of these requirements is trivial, leaving the former as the stated condition $\beta \leqslant \alpha + \beta\log 2/\log 3$ in (i). □

For our further consideration we shall also need the following.

PROPOSITION 4.4. *Let α, β be two positive numbers such that $\beta \leqslant 2\alpha$. Then*

$$1 + x + x^2 \geqslant \min\left\{1, 3^{1-\alpha-\frac{1}{2}\beta}\right\}(1 + x^{1/\beta})^\beta(1 + x^{1/\alpha})^\alpha \tag{4.5}$$

and

$$1 + x + x^2 \geqslant \min\left\{1, 4^{1-\alpha-\frac{1}{2}\beta}\right\}(1 + x^{1/\beta})^\beta(1 + x^{1/\alpha})^\alpha \tag{4.6}$$

for all $0 \leqslant x \leqslant 1$.

Proof. Set $k = \alpha + \frac{\beta}{2}$. If $k = 1$, by [7, Lemma 4] for (4.5) we need only prove that

$$1 + x + x^2 \geqslant (1 + x^{1/\beta})^\beta(1 + x^{1/\alpha})^\alpha$$

holds for $\beta = 0$ and $\beta = 1$, that is,

$$1 + x + x^2 \geqslant \max\left\{1 + x, (1 + x)\sqrt{1 + x^2}\right\}.$$

This is immediate. The rest of the proof of (4.5) is similar to that of Proposition 4.3. That (4.6) holds now follows from

$$\min\left\{1, 3^{1-\alpha-\frac{1}{2}\beta}\right\} \geqslant \min\left\{1, 4^{1-\alpha-\frac{1}{2}\beta}\right\}.$$

□

5. Multivariate inequalities

In this section we give first the following extension of Proposition (2.4)(i).

THEOREM 5.1. *Suppose* a_i *(* $i = 0,1,2$ *),* b_i *(* $i = 0,1$ *) are nonnegative numbers with*

$$a_0 + a_1 + a_2 = 1 = b_0 + b_1,$$

and α *,* β *positive numbers satisfying*

$$\beta \leqslant 1 \qquad \text{and} \qquad \beta \leqslant \alpha + \beta \frac{\log 2}{\log 3}.$$

Then

$$\max(a_0^\alpha b_0^\beta, a_2^\alpha b_1^\beta) + \max(a_0^\alpha b_1^\beta, a_1^\alpha b_0^\beta) + \max(a_1^\alpha b_1^\beta, a_2^\alpha b_0^\beta) \geqslant \min\left\{1, \frac{3}{2^\beta 3^\alpha}\right\}. \quad (5.1)$$

Proof. For $b_0 = 0$ or $b_1 = 0$, (5.1) reduces to

$$a_0^\alpha + a_1^\alpha + a_2^\alpha \geqslant \min\left\{1, \frac{3}{2^\beta 3^\alpha}\right\}.$$

For $\alpha \leqslant 1$, this is true since $a_0^\alpha + a_1^\alpha + a_2^\alpha \geqslant 1$, while for $\alpha > 1$ Jensen's inequality provides $a_0^\alpha + a_1^\alpha + a_2^\alpha \geqslant 3^{1-\alpha} \geqslant \frac{3}{2^\beta 3^\alpha}$. Hence we may assume without loss of generality that $0 < b_0, b_1 < 1$.

Set $b_1 = 1 - b_0$. Except where one pair of parentheses contains two equal numbers, the second derivative of the left–hand side of (5.1) exists and is nonpositive, since $\beta \leqslant 1$. The left–hand side of (5.1) is thus piecewise concave in b_0 and to establish (5.1) it suffices to show that inequality subsists when a pair of parentheses contains two equal numbers. By symmetry, any of the three cases is representative, and may as well choose

$$a_0^\alpha b_0^\beta = a_2^\alpha b_1^\beta. \quad (5.2)$$

If $a_2 = 0$, then $a_0 = 0$ and $a_1 = 1$ so that (5.1) reduces to

$$b_0^\beta + b_1^\beta \geqslant \min\left\{1, \frac{3}{2^\beta 3^\alpha}\right\},$$

which is trivial since $\beta \leqslant 1$. So we assume that $a_2 \neq 0$ and accordingly from (5.2) that $a_0 \neq 0$.

Define x and y by

$$x = (a_0/a_2)^\alpha, \qquad y = (a_1/a_2)^\alpha.$$

From (5.2) we derive that

$$x = ((1-b_0)/b_0)^\beta, \qquad b_0 = (1 + x^{1/\beta})^{-1}.$$

Since $a_0 \neq 0$, we may take $x > 0$ and by interchanging a_0 and a_2 if necessary guarantee $0 < x \leqslant 1$. As $b_0 \neq 0$ we have $0 \leqslant y < \infty$. In terms of our new variables, (5.1) becomes

$$x + \max(x^2, y) + \max(xy, 1) \geqslant \min\left\{1, \frac{3}{2^\beta 3^\alpha}\right\}(1 + x^{1/\beta})^\beta(1 + x^{1/\alpha} + y^{1/\alpha})^\alpha \quad (5.3)$$

for $0 < x \leqslant 1$ and $0 \leqslant y < \infty$.

Again we may argue *via* second derivatives in y that it suffices to check (5.3) for $y = 0, x^2, 1/x$. The first case requires

$$1 + x + x^2 \geqslant \min\left\{1, \frac{3}{2^\beta 3^\alpha}\right\}(1 + x^{1/\beta})^\beta(1 + x^{1/\alpha})^\alpha, \quad (5.4)$$

the second

$$1 + x + x^2 \geqslant \min\left\{1, \frac{3}{2^\beta 3^\alpha}\right\}(1 + x^{\frac{1}{\beta}})^\beta(1 + x^{\frac{1}{\alpha}} + x^{\frac{2}{\alpha}})^\alpha \quad (5.5)$$

and the third (on multiplication by x) (5.5) again. Proposition 4.3(i) gives that (5.5) holds under the assumptions made on α and β. Since (5.5) implies (5.4), we are done. $\square$

Proposition 2.4(i) entails the conditions $0 \leqslant \alpha, \beta \leqslant 1$ and $\alpha + \beta \log 2/\log 3 = 1$. If α, β satisfy these conditions, then $\beta \leqslant 1 \leqslant \alpha + \beta \log 2/\log 3$ and so the conditions of Theorem 5.1 are satisfied. This shows that Theorem 5.1 is a generalization of Proposition 2.4(i).

The following result gives a generalization of Proposition 2.4 (ii).

THEOREM 5.2. *Suppose that* α, β *are positive numbers such that* $\beta \leqslant 1$ *and* $\beta \leqslant 2\alpha$, *and that* a, b, c, d, u, v *are nonnegative with* $a + b + c + d = 1 = u + v$. *Then*

$$\max(a^\alpha u^\beta, d^\alpha v^\beta) + \max(b^\alpha u^\beta, a^\alpha v^\beta) + \max(c^\alpha u^\beta, b^\alpha v^\beta) + \max(d^\alpha u^\beta, c^\alpha v^\beta) \geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}. \quad (5.6)$$

Proof. For $u = 0$ or $v = 0$, (5.6) reduces to

$$a^\alpha + b^\alpha + c^\alpha + d^\alpha \geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}.$$

If $\alpha < 1$ then $a^\alpha + b^\alpha + c^\alpha + d^\alpha \geqslant 1$, while for $\alpha > 1$ we have by Jensen's inequality that $a^\alpha + b^\alpha + c^\alpha + d^\alpha \geqslant 4^{1-\alpha} \geqslant 4/2^\beta 4^\alpha$. Hence we may assume without loss of generality that $0 < u$, $v < 1$.

Put $v = 1 - u$. The second derivative of the left–hand side of (5.6) with respect to u exists and is nonpositive except when $\alpha u^\beta = d^\alpha(1-u)^\beta$ or $b^\alpha u^\beta = a^\alpha(1-u)^\beta$

or $c^\alpha u^\beta = b^\alpha(1-u)^\beta$ or $d^\alpha u^\beta = c^\alpha(1-u)^\beta$. By symmetry, any one of these is representative so the task is reduced to proving (5.6) under the hypothesis

$$a^\alpha u^\beta = d^\alpha(1-u)^\beta. \tag{5.7}$$

Suppose $d=0$. Then $a=0$, $b+c=1$ and (5.6) becomes

$$(1-c)^\alpha u^\beta + \max\left\{c^\alpha u^\beta, (1-c)^\alpha(1-u)^\beta\right\} + c^\alpha(1-u)^\beta \geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}. \tag{5.8}$$

This in turn is true if it holds when

$$c^\alpha u^\beta = (1-c)^\alpha(1-u)^\beta.$$

In this situation set $x=[(1-u)/u]^\beta$. Then (5.8) becomes

$$1+x+x^2 \geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}\left(1+x^{1/\alpha}\right)^\alpha\left(1+x^{1/\beta}\right)^\beta, \tag{5.9}$$

which holds by Proposition 4.4. So without loss of generality we may require $d \neq 0$ in (5.7).

Write $x=(a/d)^\alpha$, $y=(b/d)^\alpha$, $z=(c/d)^\alpha$. Since $d\neq 0$, we have $x,y,z<\infty$. Also

$$x=((1-u)/u)^\beta; \quad u=(1+x^{\frac{1}{\beta}})^{-1}.$$

By interchanging a and d if necessary we may suppose without loss of generality that $0<x\leqslant 1$. The required task is to show that

$$\begin{aligned} &x+\max(y,x^2)+\max(z,xy)+\max(1,xz)\\ &\geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}(1+x^{1/\beta})^\beta(1+x^{1/\alpha}+y^{1/\alpha}+z^{1/\alpha})^\alpha. \end{aligned} \tag{5.10}$$

By taking all the terms of (5.10) to the left and differentiating with respect to z, we can reduce the problem to the cases $z=0$, xy, x^{-1}, ∞. The last case has already been dealt with.

The first demands that we consider

$$x+\max(y,x^2)+xy+1 \geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}(1+x^{1/\beta})^\beta(1+x^{1/\alpha}+y^{1/\alpha})^\alpha.$$

Differentiation with respect to y allows us to consider only $y=0$, x^2, ∞, and again the last possibility has been treated. The first subcase is

$$1+x+x^2 \geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}(1+x^{1/\beta})^\beta(1+x^{1/\alpha})^\alpha, \tag{5.11}$$

which holds by Proposition 4.4. The middle subcase $y=x^2$ corresponds to

$$1+x+x^2+x^3 \geqslant \min\left\{1, \frac{4}{2^\beta 4^\alpha}\right\}(1+x^{1/\beta})^\beta(1+x^{1/\alpha}+x^{2/\alpha})^\alpha, \tag{5.12}$$

which is subsumed under Proposition 4.4, since

$$\min\left\{1, \frac{4}{3^{\alpha+\beta}}\right\} \geqslant \min\left\{1, \frac{4}{2^{\beta}4^{\alpha}}\right\}.$$

Now we must consider $z = xy$, which gives

$$\begin{aligned} &x + \max(y, x^2) + xy + \max(1, x^2y) \\ &\geqslant \min\left\{1, \frac{4}{2^{\beta}4^{\alpha}}\right\}(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + y^{1/\alpha} + (xy)^{1/\alpha})^{\alpha}. \end{aligned}$$

Differentiation with respect to y yields the cases $y = 0$, x^2, x^{-2}, ∞, of which the last can be removed. The first case is (5.11) once more, and the second subcase reduces to Proposition 4.3 (ii), as does the third on multiplication by x^2.

Next we take $z = x^{-1}$, and must consider

$$x + (y, x^2) + (x^{-1}, xy) + 1 \geqslant \min\left\{1, \frac{4}{2^{\beta}4^{\alpha}}\right\}(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + y^{1/\alpha} + x^{-1/\alpha})^{\alpha},$$

with the subcases $y = 0$, x^2, x^{-2}, ∞, of which the last is again resolved. When $y = 0$, we must consider

$$1 + x + x^{-1} + x^2 \geqslant \min\left\{1, \frac{4}{2^{\beta}4^{\alpha}}\right\}(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + x^{-1/\alpha})^{\alpha}.$$

Multiplication by x converts this to (5.12), which has already been dealt with. We consider next $y = x^2$ and the inequality

$$1 + x + x^{-1} + x^2 \geqslant \min\left\{1, \frac{4}{2^{\beta}4^{\alpha}}\right\}(1 + x^{1/\beta})^{\beta}(1 + x^{1/\alpha} + x^{2/\alpha} + x^{-1/\alpha})^{\alpha},$$

which is covered by Proposition 4.3(ii) on multiplication by x. When $y = x^{-2}$, we multiply both sides by x^2 to recover the same inequality. □

Suppose that α, β satisfy the conditions $\alpha + \frac{1}{2}\beta = 1$ and $0 \leqslant \alpha, \beta \leqslant 1$ of Proposition 2.4(ii). Then

$$\beta = 1 - \frac{1}{2}\alpha \geqslant \frac{1}{2} \geqslant \frac{1}{2}\alpha,$$

so the requirements of Theorem 5.2 are satisfied. Clearly the reverse implication does not hold in general. Thus Theorem 5.2 is a generalization of Proposition 2.4(ii).

The concluding theorem in this section hangs on Propositions 3.2 and 4.3(v) and so requires the assumptions of both. There is some overlap here. It is convenient to separate out the following from the argument.

LEMMA 5.3. *Suppose* α, β *are positive numbers satisfying* $\alpha \leqslant 1$ *and* $\alpha + \beta \geqslant \log 2/\log 3$. *Then*

$$\alpha/\beta < u_2.$$

Proof. The given conditions imply that

$$\alpha/\beta \leqslant 1/\left(\frac{\log 3}{\log 2} - 1\right) = \frac{L}{2-L},$$

where $L = \log 4/\log 3$. From Proposition 4.3(v), it suffices to show that

$$\left[6u^2 - (27L-16)u + 8L\right]_{u=L/(2-L)} < 0.$$

The left–hand side can be written as

$$\frac{L}{(2-L)^2}(5L-8)(7L-8).$$

It is readily seen that $8/5 > L > 8/7$, whence the required result. □

THEOREM 5.4. *Suppose* α, β *are positive numbers such that*

$$\alpha, \beta \leqslant 1, \tag{5.13}$$

$$\alpha \geqslant \frac{\log 4/3}{\log 3/2}, \tag{5.14}$$

$$\beta \geqslant \frac{\log 3/2}{\log 2}, \tag{5.15}$$

$$\alpha + \beta \geqslant \frac{\log 3}{\log 2}, \tag{5.16}$$

$$3(\alpha^{-1} + \beta^{-1}) \leqslant 8, \tag{5.17}$$

$$u_1\beta \leqslant \alpha. \tag{5.18}$$

Suppose further that a_i, b_i *are nonnegative numbers satisfying* $a_0 + a_1 + a_2 = 1 = b_0 + b_1$. *Then*

$$a_0^\alpha b_0^\beta + \max\left\{a_0^\alpha b_1^\beta, a_1^\alpha b_0^\beta\right\} + \max\left\{a_1^\alpha b_1^\beta, a_2^\alpha b_0^\beta\right\} + a_2^\alpha b_1^\beta \geqslant \frac{4}{2^\beta 3^\alpha}. \tag{5.19}$$

Proof.. As a preliminary, we remark that (5.14), (5.16) entail respectively $(3/2)^\alpha \geqslant 4/3$ and $2^{\alpha+\beta} \geqslant 3$. Multiplication gives

$$3^\alpha 2^\beta \geqslant 4, \tag{5.20}$$

or equivalently

$$\frac{\beta}{2} + \alpha\frac{\log 3}{\log 4} \geqslant 1. \tag{5.21}$$

This enables us to deal with some special cases. For $b_0 = 0$ or $b_1 = 0$, (5.19) reduces to

$$a_0^\alpha + a_1^\alpha + a_2^\alpha \geqslant \frac{4}{3^\alpha 2^\beta}. \tag{5.22}$$

Since $\alpha \leqslant 1$, the left–hand side is at least unity. Now (5.22) is immediate from (5.20). Hence we need only establish (5.19) for $0 < b_0, b_1 < 1$.

Also, if $a_2 = 0$, then (5.19) becomes

$$a_0^\alpha b_0^\beta + \max\left\{a_0^\alpha b_1^\beta, a_1^\alpha b_0^\beta\right\} + a_1^\alpha b_1^\beta \geqslant \frac{4}{3^\alpha 2^\beta} \tag{5.23}$$

with $a_0 + a_1 = 1 = b_0 + b_1$. Since $\frac{\log 4/3}{\log 3/2} > \frac{\log 3}{\log 2} - 1$, (5.13)–(5.17) give that α, β satisfy the conditions of Proposition 3.2 and hence the left–hand side of (5.23) is at least $3/2^{\alpha+\beta}$. But (5.14) entails that $3/2^{\alpha+\beta} \geqslant 4/(3^\alpha 2^\beta)$. Hence (5.19) holds. Similarly there is no problem if $a_0 = 0$.

Put $b_1 = 1 - b_0$. The second derivative of the left–hand side of (5.19) with respect to b_0 exists and is nonpositive except when

$$a_0^\alpha b_1^\beta = a_1^\alpha b_0^\beta \qquad \text{or} \qquad a_1^\alpha b_1^\beta = a_2^\alpha b_0^\beta.$$

By symmetry, either is representative and it suffices to prove (5.19) under the condition

$$a_0^\alpha b_1^\beta = a_1^\alpha b_0^\beta. \tag{5.24}$$

We have seen that we may take $a_0, a_2, b_0, b_1 > 0$, so we can also assume $a_1 > 0$. Put $x = (b_0/b_1)^\beta$, so $x = (a_0/a_1)^\beta$ too, by (5.24). Also put $y = (a_2/a_1)^\alpha$. Then $1/b_1 = 1 + x^{1/\alpha}$ and $1/a_1 = 1 + x^{1/\alpha} + y^{1/\alpha}$. Under (5.24), inequality (5.19) becomes

$$x^2 + x + \max(1, xy) + y \geqslant \frac{4}{3^\alpha 2^\beta}(1 + x^{1/\beta})^\beta (1 + x^{1/\alpha} + y^{1/\alpha})^\alpha.$$

Differentiation with respect to y allows us to reduce this to the cases $y = 0, 1/x, \infty$, the first and last of which have already been dealt with under the banners $a_2 = 0$ and $a_1 = 0$. Thus we need to establish

$$x^2 + x + 1 + 1/x \geqslant \frac{4}{3^\alpha 2^\beta}(1 + x^{1/\beta})^\beta (1 + x^{1/\alpha} + x^{-1/\alpha})^\alpha.$$

Multiplication by x converts this to

$$x^3 + x^2 + x + 1 \geqslant \frac{4}{3^\alpha 2^\beta}(1 + x^{1/\beta})^\beta (1 + x^{1/\alpha} + x^{2/\alpha})^\alpha. \tag{5.25}$$

By (5.13) and (5.16), Lemma 5.3 applies, so that by (5.18) we have

$$u_1\beta \leqslant \alpha < u_2\beta.$$

This ensures that Proposition 4.3(v) applies for $0 < x \leqslant 1$. As (5.25) holds when x is replaced by $1/x$, it therefore applies for all $x > 0$, and we are done. □

6. Measure–theoretic results

The three theorems of the previous section lead immediately to the following measure–theoretic results for sum sets.

THEOREM 6.1. *Suppose* $E, F \subset [0,1]$ *are Borel sets.*
(a) If $\alpha + \frac{\log 2}{\log 3}\beta \geqslant 1$ *and* $\alpha \geqslant 1 - \frac{\log 2}{\log 3}$*, then*

$$m(E+F) \geqslant m(E)^\alpha \mu_c(F)^\beta.$$

(b) If $\alpha + \frac{1}{2}\beta \geqslant 1$ *and* $\alpha \geqslant \frac{1}{2}$*, then*

$$m(E+F) \geqslant m(E)^\alpha \nu_1(F)^\beta.$$

(c) If $\alpha \geqslant \alpha_0$*,* $\beta \geqslant \beta_0$*, where* α_0*,* β_0 *satisfy (5.13)–(5.18), then*

$$m(E+F) \geqslant \nu_1(E)^\alpha \nu_2(F)^\beta.$$

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