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INEQUALITIES FOR THE INCOMPLETE GAMMA AND RELATED FUNCTIONS

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Abstract. In the article, some inequalities of the incomplete gamma function $\int_0^x e^{-t^p}\,dt = \left[\Gamma(1/p) - \Gamma(1/p, x^p)\right]/p$ and the related function $\int_0^x e^{t^p}\,dt$ for $x > 0$, $p > 0$ are established by using Hermite-Hadamard inequality.

1. Introduction

It is well-known that the function $\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ is called the error function or probability integral denoted by $\operatorname{erf}(x)$, and $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$ the complementary error function denoted by $\operatorname{erfc}(x)$. Moreover, the function $E_1(x) = \Gamma(0,x) = \int_x^\infty e^{-t}/t\,dt$ is called the exponential integral.

The study of these functions and inequalities involving them and related functions have a rich literature. A survey of some recent development can be found in references listed in this paper.

Inequalities of these functions have been extended and refined in many different directions using different principles or devices. These results are related to Mills' ratio.

In [2], J. T. Chu presented sharp upper and lower bounds for $\operatorname{erf}(x)$ as follows

$$\left[1 - e^{-rx^2}\right]^{1/2} \leqslant \operatorname{erf}(x) \leqslant \left[1 - e^{-sx^2}\right]^{1/2} \tag{1}$$

are valid for all $x \geqslant 0$ if and only if $0 \leqslant r \leqslant 1$ and $s \geqslant 4/\pi$.

In [5, 7], W. Gautschi provided upper and lower bounds for the general expression

$$I_p(x) = e^{x^p}\int_x^\infty e^{-t^p}\,dt. \tag{2}$$

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He established that the double-inequality

$$\frac{1}{2}\left[(x^p+2)^{1/p}-x\right] < I_p(x) \leqslant c_p\left[(x^p+1/c_p)^{1/p}-x\right] \tag{3}$$

holds for all real numbers $p > 1$ and $x \geqslant 0$, where $c_p = \left[\Gamma(1+1/p)\right]^{p/(p-1)}$.

In 1997, Horst Alzer [1] extended and refined the above results (1) and (3) and determined all real numbers $\alpha = \alpha(p)$ and $\beta = \beta(p)$ such that the inequalities

$$\Gamma(1+1/p)\left[1-\mathrm{e}^{-\beta x^p}\right]^{1/p} < \int_0^x \mathrm{e}^{-t^p}\,\mathrm{d}t < \Gamma(1+1/p)\left[1-\mathrm{e}^{-\alpha x^p}\right]^{1/p} \tag{4}$$

are valid for all $x > 0$ and for a positive real number $p \neq 1$, where

$$\alpha = 1, \quad \beta = \left[\Gamma(1+1/p)\right]^{-p}, \quad \text{if } 0 < p < 1, \tag{5}$$

and

$$\alpha = \left[\Gamma(1+1/p)\right]^{-p}, \quad \beta = 1, \text{ if } p > 1. \tag{6}$$

Meanwhile, he also verified that inequality

$$-\ln(1-\mathrm{e}^{-ax}) \leqslant E_1(x) \leqslant -\ln(1-\mathrm{e}^{-bx}) \tag{7}$$

are valid for all positive real x if and only if $a \geqslant \mathrm{e}^C$ and $0 < b \leqslant 1$, where $C = 0.5772\cdots$ is Euler's constant.

It has been pointed out in [5] that the integral in (2) for $p = 3$ occurs in heat transfer problems, and for $p = 4$ in the study of electrical discharge through gases.

For all $x > 0$, we have, in [3, p. 229], [6, pp. 591–593] and [7, pp. 177–181]

$$\left(x+\frac{x^2}{24}+\frac{x^3}{12}\right)\mathrm{e}^{-3x^2/4} < \mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t \leqslant \frac{\pi^2}{8x}(1-\mathrm{e}^{-x^2}), \quad x > 0. \tag{8}$$

Inequality (8) is called Conte's inequality. It is also related to Mills' ratio.

In [6, pp. 591–593], [7, pp. 291–292] and other references, more related inequalities were given, for instance

$$\int_0^{\pi/2} \mathrm{e}^{-x^2\sin^2 t}\sin t\,\mathrm{d}t \leqslant \frac{\pi^2}{8x^2}(1-\mathrm{e}^{-x^2}), \quad x > 0. \tag{9}$$

Recently, the first author in [4, 9], among other things, improved the upper bounds of inequalities (8) and (9) using Tchebycheff's integral inequality below

$$\int_0^t \mathrm{e}^{x^2}\,\mathrm{d}x < \tfrac{1}{t}(\mathrm{e}^{t^2}-1), \qquad t > 0, \tag{10}$$

$$\int_0^{\pi/2} \mathrm{e}^{-t^2\sin^2 x}\sin x\,\mathrm{d}x \leqslant \tfrac{1-\mathrm{e}^{-t^2}}{t^2}, \qquad t \neq 0. \tag{11}$$

Inequality (10) refines the upper bound of (8), and inequality (11) improves the upper bound of inequality (9).

Moreover, in [4, 9], the following more general results were obtained

$$\int_0^x e^{-t^\alpha} dt \geqslant \frac{1-e^{-x^\alpha}}{x^{\alpha-1}}, \quad x>0, \quad \alpha \geqslant 1, \tag{12}$$

and

$$\int_0^x e^{t^\alpha} dt \leqslant \frac{e^{x^\alpha}-1}{x^{\alpha-1}}, \quad x>0, \quad \alpha \geqslant 1. \tag{13}$$

Note that the integral $\int_x^\infty e^{-t^p} dt$ can be expressed in terms of the incomplete gamma function

$$\int_x^\infty e^{-t^p} dt = \frac{1}{p}\Gamma\Big(\frac{1}{p}, x^p\Big), \tag{14}$$

where

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt. \tag{15}$$

Therefore, we have

$$\int_0^x e^{-t^p} dt = \frac{1}{p}\Big[\Gamma\Big(\frac{1}{p}\Big) - \Gamma\Big(\frac{1}{p}, x^p\Big)\Big]. \tag{16}$$

The main purpose of this paper is to establish new inequalities for the integrals $\int_0^x e^{-t^p} dt$ and $\int_0^x e^{t^p} dt$. In Section 2., we give upper and lower bounds of these integrals by using Hermite-Hadamard inequality. Finally, in Section 3., we compare our bounds with those given above.

2. Main Results

The following Lemma is necessary

LEMMA. *Let* $f : [a,b] \longrightarrow \mathbf{R}$ *be a convex function, then*

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}. \tag{17}$$

Inequality (17) is called the Hermite-Hadamard inequality in [8, p. 10].

THEOREM 1. *Let* p *be a positive number.*
(1) If $0 < p \leqslant 1$, *for all* $x > 0$, *we have*

$$x e^{-(x/2)^p} \leqslant \int_0^x e^{-t^p} dt \leqslant \frac{1}{2} x (1 + e^{-x^p}); \tag{18}$$

(2) If $p > 1$, *for* $0 < x < \sqrt[p]{1-\frac{1}{p}}$, *the inequalities in (18) reverse;*

(3) If $p>1$, for $x>\sqrt[p]{1-\frac{1}{p}}$, we have

$$\frac{1}{2}\sqrt[p]{1-\tfrac{1}{p}}\left(1+\mathrm{e}^{1/p-1}\right)+\left(x-\sqrt[p]{1-\tfrac{1}{p}}\right)\mathrm{e}^{-\left(x+\sqrt[p]{1-1/p}\right)^{p}/2^{p}} \leqslant \int_0^x \mathrm{e}^{-t^p}\,\mathrm{d}t \leqslant \frac{1}{2}\left(x-\sqrt[p]{1-\tfrac{1}{p}}\right)\left(\mathrm{e}^{-x^p}+\mathrm{e}^{1/p-1}\right)+\sqrt[p]{1-\tfrac{1}{p}}\,\mathrm{e}^{(1-p)/2^p p}. \tag{19}$$

Proof. Let $f(t)=\mathrm{e}^{-t^p}$ for $p>0$ and $t>0$. By direct calculation we get

$$f''(t)=pt^{p-2}(pt^p-p+1)\mathrm{e}^{-t^p}. \tag{20}$$

It is easy to see that, if $0<p\leqslant 1$, then $f''(t)>0, f(t)$ is convex on $(0,+\infty)$. If $p>1$, on the interval $\left(\sqrt[p]{1-\frac{1}{p}},+\infty\right)$, $f''(t)>0, f(t)$ is also convex; and on $\left(0,\sqrt[p]{1-\frac{1}{p}}\right)$, $f''(t)<0, f(t)$ is concave. From Hermite-Hadamard inequality (17), if $0<p\leqslant 1$, we obtain

$$x\mathrm{e}^{-(x/2)^p} \leqslant \int_0^x \mathrm{e}^{-t^p}\,\mathrm{d}t \leqslant \frac{(1+\mathrm{e}^{-x^p})x}{2} \tag{21}$$

for all $x>0$. If $p>1$, for $0<x<\sqrt[p]{1-\frac{1}{p}}$, using the inequality (17), we have the reversed double-inequality of (21); for $x>\sqrt[p]{1-\frac{1}{p}}$, we find

$$\begin{aligned}&\frac{1}{2}\sqrt[p]{1-\tfrac{1}{p}}\left(1+\mathrm{e}^{1/p-1}\right)+\left(x-\sqrt[p]{1-\tfrac{1}{p}}\right)\mathrm{e}^{-\left[\left(x+\sqrt[p]{1-\frac{1}{p}}\right)/2\right]^{p}}\\ \leqslant{}& \int_0^x \mathrm{e}^{-t^p}\,\mathrm{d}t=\int_0^{\sqrt[p]{1-\frac{1}{p}}}\mathrm{e}^{-t^p}\,\mathrm{d}t+\int_{\sqrt[p]{1-\frac{1}{p}}}^{x}\mathrm{e}^{-t^p}\,\mathrm{d}t\\ \leqslant{}& \sqrt[p]{1-\tfrac{1}{p}}\,\mathrm{e}^{(1-p)/2^p p}+\frac{1}{2}\left(x-\sqrt[p]{1-\tfrac{1}{p}}\right)\left(\mathrm{e}^{-x^p}+\mathrm{e}^{1/p-1}\right).\end{aligned} \tag{22}$$

This completes the proof of Theorem 1.

THEOREM 2. *Let p be a positive number.*

(1) If $p\geqslant 1$, for all $x>0$, we have

$$x\mathrm{e}^{(x/2)^p} \leqslant \int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t \leqslant \frac{1}{2}x(1+\mathrm{e}^{x^p}); \tag{23}$$

(2) If $p<1$, for all $0<x<\sqrt[p]{\frac{1}{p}-1}$, the inequalities in (23) reverse;

(3) If $p<1$, for all $x>\sqrt[p]{\frac{1}{p}-1}$, we have

$$\frac{1}{2}\sqrt[p]{\tfrac{1}{p}-1}\left(1+\mathrm{e}^{1/p-1}\right)+\left(x-\sqrt[p]{\tfrac{1}{p}-1}\right)\mathrm{e}^{\left(x+\sqrt[p]{1/p-1}\right)^{p}/2^{p}} \leqslant \int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t \leqslant \frac{1}{2}\left(x-\sqrt[p]{\tfrac{1}{p}-1}\right)\left(\mathrm{e}^{x^p}+\mathrm{e}^{1/p-1}\right)+\sqrt[p]{\tfrac{1}{p}-1}\,\mathrm{e}^{(1-p)/2^p p}. \tag{24}$$

Proof. Let $g(t) = \mathrm{e}^{t^p}$ for $p > 0$ and $t > 0$. By straightforward calculation we obtain

$$g''(t) = pt^{p-2}(pt^p + p - 1)\mathrm{e}^{t^p}. \tag{25}$$

If $p \geqslant 1$, then $g''(t) > 0, g(t)$ is convex on $(0, +\infty)$. If $p < 1$, on the interval $\left(\sqrt[p]{\frac{1}{p} - 1}, +\infty\right)$, $g''(t) > 0, g(t)$ is also convex; and on $\left(0, \sqrt[p]{\frac{1}{p} - 1}\right)$, $g''(t) < 0, g(t)$ is concave. From Hermite-Hadamard inequality (17), if $p \geqslant 1$, we obtain

$$x\mathrm{e}^{(x/2)^p} \leqslant \int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t \leqslant \frac{(1 + \mathrm{e}^{x^p})x}{2} \tag{26}$$

for all $x > 0$. If $p < 1$, for $0 < x < \sqrt[p]{\frac{1}{p} - 1}$, using the inequality (17), we have the reversed double-inequality of (26); for $x > \sqrt[p]{\frac{1}{p} - 1}$, we have

$$\begin{aligned}
&\frac{1}{2}\sqrt[p]{\frac{1}{p} - 1}\left(1 + \mathrm{e}^{1/p-1}\right) + \left(x - \sqrt[p]{\frac{1}{p} - 1}\right)\mathrm{e}^{\left[\left(x + \sqrt[p]{\frac{1}{p} - 1}\right)/2\right]^p} \\
\leqslant\ & \int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t = \int_0^{\sqrt[p]{\frac{1}{p}-1}} \mathrm{e}^{t^p}\,\mathrm{d}t + \int_{\sqrt[p]{\frac{1}{p}-1}}^x \mathrm{e}^{t^p}\,\mathrm{d}t \\
\leqslant\ & \sqrt[p]{\frac{1}{p} - 1}\,\mathrm{e}^{(1-p)/2^p p} + \frac{1}{2}\left(x - \sqrt[p]{\frac{1}{p} - 1}\right)\left(\mathrm{e}^{x^p} + \mathrm{e}^{1/p-1}\right).
\end{aligned} \tag{27}$$

The proof of Theorem 2 is completed.

3. Comparisons and Remarks

3.1. We first claim that, if $0 < p < 1$, the upper bounds of inequalities (4) and (18) do not imply each other.

Taking $p = 1/2$ in inequalities (4) and (18), then the upper bounds reduce to $2\left(1 - \mathrm{e}^{-\sqrt{x}}\right)^2$ and $x\left(1 + \mathrm{e}^{-\sqrt{x}}\right)/2$, respectively.

Let $\phi(x) = 2\left(1 - \mathrm{e}^{-\sqrt{x}}\right)^2 - x\left(1 + \mathrm{e}^{-\sqrt{x}}\right)/2, x > 0$. Direct calculation yields

$$\begin{aligned}
\phi(x) &= \frac{1}{2}\mathrm{e}^{-2\sqrt{x}}\left[4(\mathrm{e}^{\sqrt{x}} - 1)^2 - x\mathrm{e}^{\sqrt{x}}(\mathrm{e}^{\sqrt{x}} + 1)\right] \\
&= \frac{1}{2}\mathrm{e}^{-2t}\left[(4 - t^2)\mathrm{e}^{2t} - (t^2 + 8)\mathrm{e}^t + 4\right] \qquad \left(\sqrt{x} = t\right) \qquad (28) \\
&\equiv \frac{1}{2}\mathrm{e}^{-2t}\phi_1(t), \\
\phi_1'(t) &= \left[2(4 - t - t^2)\mathrm{e}^t - (t^2 + 2t + 8)\right]\mathrm{e}^t \\
&\equiv \mathrm{e}^t\phi_2(t), \qquad (29) \\
\phi_2'(t) &= 2[(3 - 3t - t^2)\mathrm{e}^t - t - 1], \qquad (30) \\
\phi_2''(t) &= -2[t(t + 5)\mathrm{e}^t + 1] < 0. \qquad (31)
\end{aligned}$$

Therefore, $\phi_2'(t)$ is decreasing, and since $\phi_2'(0) = 4$, $\lim_{t\to+\infty} \phi_2'(t) = -\infty$, thus $\phi_2'(t)$ has only one zero, $\phi_2(t)$ has one unique maximum. Furthermore, since $\phi_2(0) = 0$, $\lim_{t\to+\infty} \phi_2(t) = -\infty$, hence $\phi_2(t)$ has only one zero, namely $\phi_1'(t)$ has an unique zero, $\phi_1(t)$ has an unique maximum. Since $\phi_1(0) = 0$, $\lim_{t\to+\infty} \phi_1(t) = -\infty$, therefore $\phi_1(t)$ has an unique zero $\theta \in (0,+\infty)$, namely $\phi(x)$ has an unique zero $\theta \in (0,+\infty)$, and $\phi(x) > 0$ on $(0,\theta)$, $\phi(x) < 0$ on $(\theta,+\infty)$. That is, the function $\phi(x)$ does not keep the same sign for $x > 0$. This completes the proof of the above claim.

3.2. The lower bounds of inequalities (4) and (18) do not imply each other.

The difference between the logarithms of the lower bounds in inequalities (4) and (18) is

$$\psi(p;x) = \ln\Gamma\Big(1+\frac{1}{p}\Big) + \frac{1}{p}\ln\big[1-\mathrm{e}^{-(x/\Gamma(1+1/p))^p}\big] - \ln x + \Big(\frac{x}{2}\Big)^p \tag{32}$$

for $0 < p < 1$ and $0 < x < +\infty$. Calculating directly leads to

$$\begin{aligned}
\frac{\partial\psi(p;x)}{\partial x} &= \frac{1}{x\big[\mathrm{e}^{(x/\Gamma)^p}-1\big]}\Big[\Big(\frac{x}{\Gamma}\Big)^p + \Big(p\Big(\frac{x}{2}\Big)^p - 1\Big)\big(\mathrm{e}^{(x/\Gamma)^p}-1\big)\Big] \\
&= \frac{1}{x(\mathrm{e}^t-1)}\Big[\Big(1-\frac{p\Gamma}{2}\Big)t + \Big(\Big(\frac{\Gamma}{2}\Big)^p pt - 1\Big)\mathrm{e}^t + 1\Big] \qquad (33)\\
&\equiv \frac{1}{x(\mathrm{e}^t-1)}\psi_1(p;t),\\
\frac{\partial\psi_1(p;t)}{\partial t} &= 1-\frac{p\Gamma}{2} + \Big[\Big(\frac{\Gamma}{2}\Big)^p pt + \Big(\frac{\Gamma}{2}\Big)^p p - 1\Big]\mathrm{e}^t, \qquad (34)\\
\frac{\partial^2\psi_1(p;t)}{\partial t^2} &= \Big[\Big(\frac{\Gamma}{2}\Big)^p pt + 2p\Big(\frac{\Gamma}{2}\Big)^p - 1\Big]\mathrm{e}^t, \qquad (35)
\end{aligned}$$

where $\Gamma = \Gamma(1+1/p)$ and $(x/\Gamma)^p = t$.

If $0 < t < (2/\Gamma)^p/p - 2$, then $\partial^2\psi_1/\partial t^2 < 0, \partial\psi_1/\partial t$ is decreasing with t; if $t > (2/\Gamma)^p/p - 2, \partial\psi_1/\partial t$ is increasing with t; thus $\partial\psi_1/\partial t$ has one unique minimum at $t_0 = (2/\Gamma)^p/p - 2$. Since

$$\frac{\partial\psi_1(p;t)}{\partial t}\bigg|_{t=0} = p\Big(\frac{\Gamma}{2}\Big)^p\Big(1-\Big(\frac{\Gamma}{2}\Big)^{1-p}\Big), \quad \lim_{t\to+\infty}\frac{\partial\psi_1(p;t)}{\partial t} = +\infty,$$

therefore, when $\Gamma(1+1/p) > 2, \partial\psi_1/\partial t$ has only one zero with t, namely $\psi_1(p;t)$ has an unique minimum. Since $\psi_1(p;0) = 0$, $\lim_{t\to+\infty}\psi_1(p;t) = +\infty$, hence $\psi_1(p;t)$ has only one zero with t, i.e., $\partial\psi(p;x)/\partial x$ has an unique zero with x, henceforth, $\psi(p;x)$ has only one minimum with x.

From $\mathrm{e}^x - 1 \sim x (x \to 0)$, the standard arguments produce $\lim_{x\to 0^+}\psi(p;x) = 0$, $\lim_{x\to+\infty}\psi(p;x) = +\infty$. Therefore $\psi(p;x)$ do not keep the same sign. This implies that the lower bounds in inequalities (4) and (18) do not include each other.

3.3. By the similar arguments, we can prove that the results of Theorem 1 do not be implied by those derived by Mr. H. Alzer in [1] and by Professor W. Gautschi in [5].

REMARK 1. *In the upper and lower bounds of the inequalities in Theorem 1 and 2, the gamma function is not involved.*

REMARK 2. *The results in Theorem 2 extend and refine the Conte's inequality (8) and related inequalities in [3, 4, 6, 7, 9].*

REMARK 3. *Some inequalities for the error function and for the complementary error function are proved in [10].*

REFERENCES

[1] HORST ALZER, *On some inequalities for the incomplete gamma function*, Mathematics of Computation **66** (1997), 771–778.
[2] J. T. CHU, *On bounds for the normal integral*, Biometrika **42** (1955), 263–265.
[3] J. M. CONTE, ET AL., *Solution of problem 5607*, Revue Math. Spec. **74** (1963/64), 227–230.
[4] FENG QI AND ZHENG HUANG, *Inequalities of the complete elliptic integrals*, Tamkang Journal of Mathematics **29** (1998), no. 3, 165–169.
[5] W. GAUTSCHI, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. Phys. **38** (1959), 77–81.
[6] JI-CHANG KUANG, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993, pp. 591–593. (Chinese)
[7] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, 1970, pp. 177–181, 291–292 and 385.
[8] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
[9] FENG QI, LI-HONG CUI AND SEN-LIN XU, *Applications of Tchebycheff's integral inequality*, Publ. Inst. Mat. Beograd. N. Ser. (1999), to appear.
[10] A. LAFORGIA AND S. SISMONDI, *Monotonicity results and inequalities for the gamma and error functions*, Comp. Appl. Math. **23** (1998), 25–33.

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