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INEQUALITIES OF FURUTA AND MOND–PEČARIĆ

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Abstract. Furuta gives extensions of inequalities due to Ky Fan and Mond-Pečarić which are associated with Hölder-McCarthy and Kantorovich type inequalities. In this paper, inspired by Furuta's idea, we shall generalize a theorem by Mond-Pečarić on the converse of Jensen's inequality. It explains us Furuta's generalizations well. As applications, we shall show general difference and ratio inequalities that can be given for several positive operators on a Hilbert space and give the explicit expressions in their estimations.

1. Introduction

Jensen's inequality is one of the most important inequalities for convex functions. Recently one of the authors et al. [16] discuss an inverse type of Jensen's inequality. They show that if f is a nonnegative real valued strictly convex function defined on the interval $[m, M]$ and φ is a measurable function on a probability measure space $(\Omega, \mathscr{F}, \mu)$ with $\varphi(\Omega) \subseteq [m, M]$, then for each $\alpha > 0$ there exists a constant β such that

$$\int f(\varphi) d\mu \leqslant \alpha f\left(\int \varphi d\mu\right) + \beta, \tag{1}$$

and moreover consider the conditions under which the equality in (1) holds, which induces an interesting family of operator means including the logarithmic mean, see also [4].

On the other hand, Furuta [5], [6], [7] and [8] recently discusses operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities and moreover gives both extensions of Ky Fan and Mond-Pečarić generalizations of Kantorovich type inequalities. Very recently a closely related result is shown in Fujii, Izumino, Nakamoto and Seo in [3]. For instance, Furuta shows that if $A_1, \cdots, A_k$ are positive operators on a Hilbert space H and $x_1, x_2, \cdots, x_k \in H$ with $\sum_{j=1}^{k} \|x_j\|^2 = 1$, then for each $p, q > 1$ there exists an $\alpha > 0$ such that

$$\sum_{j=1}^{k} (A_j^p x_j, x_j) \leqslant \alpha \Big(\sum_{j=1}^{k} (A_j x_j, x_j) \Big)^q \tag{2}$$

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under some assumptions. The original idea in (2) of Furuta is to give the estimation by using a function $g(t) = t^q$ which is independent of $f(t) = t^p$. The idea urges us toward an extension of a theorem by Mond-Pečarić [9], [11].

In this paper, inspired by Furuta's idea, we shall present an extension of a theorem by Mond-Pečarić on the converse of Jensen's inequality [9, Theorem 4]. It gives an operator theoretic interpretation to (1), and moreover explains us Furuta's generalizations stated in the next section well. As applications, we shall show general difference and ratio inequalities that can be given for several positive operators on a Hilbert space and give the explicit expressions in their estimations.

2. Preliminaries

Let C be a positive operator on a Hilbert space H. We say that

$$mI \leqslant C \leqslant MI$$

where I is the identity operator if

$$m(x,x) \leqslant (Cx,x) \leqslant M(x,x)$$

for all $x \in H$.

To the real valued function $z(\lambda)$, defined and continuous on $[m,M]$, there is associated in a natural way a self-adjoint operator on H denoted by $z(C)$, such that $z(C) = \int_{m-0}^{M} z(\lambda) dE_\lambda$, in the sense of convergence in the norm of sums of Riemann-Stieltjes type and $(E_\lambda, \lambda \in \mathbf{R})$ is a spectral resolution coresponding to C. For functions of the form $F(t) = f(h(t), g(t))$, we write $f(h(C), g(C))$ instead of the operator $F(C)$.

We shall make use of the following [15, p. 265-273]:

LEMMA. *If $z(\lambda) \geqslant 0$ for $m \leqslant \lambda \leqslant M$, then $z(C) \geqslant 0$, i.e. $z(C)$ is a positive operator.*

The celebrated Kantorovich inequality asserts that if A is a positive operator on H satisfying $mI \leqslant A \leqslant MI$ where $0 < m < M$, then:

$$(Ax,x)(A^{-1}x,x) \leqslant \frac{(M+m)^2}{4mM}, \tag{3}$$

and well known inequality related to the Kantorovich one:

$$(A^2x,x) \leqslant \frac{(M+m)^2}{4mM}(Ax,x)^2 \tag{4}$$

holds for every unit vector x in H.

We use H_n to denote the space of $n \times n$ Hermitian matrices and use $\geqslant$ to denote the positive definite partial order, so that $A \geqslant B$ means that $A - B$ is positive definite, i.e. $X^*(A-B)X \geqslant 0$ for every n-vector X.

If $A \in H_n$, then there exists a unitary matrix U such that

$$A = U^* \Lambda U,$$

where $\Lambda = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ and the λ_i are the eigenvalues of A. Assume now that $f(\lambda_i) \in C, i \in \{1, 2, \dots, n\}$ is well defined. Then $f(A)$ may be defined by

$$f(A) = U^* diag(f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n))U.$$

If $F(t) = F(f(t), g(t))$, we will write $F(f(A), g(A))$ for the operator $F(A)$, while the function $F(A, B)$ denotes the matrix function of two variables when it is well-defined.

We have the matrix version of the Kantorovich inequality:

$$X^* A^{-1} X \leqslant \frac{(M+m)^2}{4mM} (X^*AX)^{-1}, \tag{5}$$

and well known inequality related to the Kantorovich one:

$$X^* A^2 X \leqslant \frac{(M+m)^2}{4mM} (X^*AX)^2 \tag{6}$$

holds for every unit n-vector X.

Now, we state a series of inequalities of Furuta which is extensions of inequalities due to Ky Fan and Mond-Pečarić associated with Hölder-McCarthy and Kantorovich type inequalities. Furuta's original idea is to give the estimation of the expectation of $f(A)$ by using a function which is independent of $f(t)$, and to give an explicit expression in his estimation. More precisely, he observed the following inequality associated with Kantorovich inequality in [8]. Here, we introduce the following constant by Furuta [8] and it will be used in the sequel:

$$C_f(m, M; q) = \frac{mf(M) - Mf(m)}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))} \right)^q,$$

where q is a real number such that $q > 1$ or $q < 0$. It is denoted simply by $C(q)$.

THEOREM A. *Let A be a positive operator on a Hilbert space H satisfying $mI \leqslant A \leqslant MI$ where $0 < m < M$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$. Then the following inequality holds for every unit vector x and for any real number q depending on (i) or (ii) under below;*

$$(f(A)x, x) \leqslant C(q)(Ax, x)^q$$

under any one of the following conditions (i) and (ii) respectively:

(i) $f(M) > f(m)$, $\frac{f(M)}{M} > \frac{f(m)}{m}$ *and* $\frac{f(m)}{m} q \leqslant \frac{f(M) - f(m)}{M - m} \leqslant \frac{f(M)}{M} q$ *holds for any real number* $q > 1$.

(ii) $f(M) < f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ *and* $\frac{f(m)}{m} q \leqslant \frac{f(M) - f(m)}{M - m} \leqslant \frac{f(M)}{M} q$ *holds for any real number* $q < 0$.

REMARK 1. As a special case of Theorem A, we have the following Ky Fan-Furuta inequality on a complementary inequality of Hölder-McCarthy and Kantorovich inequalities: If we put $f(t) = t^p$ and $q = p$, then the constant $C(p)$ coincides with the constant defined by Ky Fan [1] and it follows that for any real number $p > 1$ or $p < 0$ the following inequality

$$(A^p x, x) \leqslant C(p)(Ax, x)^p$$

holds for every unit vector x in H because the conditions in Theorem A are automatically satisfied in this case.

The following theorem is an extension of the Ky Fan-Mond-Pečarić generalizations of Hölder-McCarthy and Kantorovich inequalities in [6] and [7]:

THEOREM B. *Let* A_j *be positive operators on a Hilbert space* H *satisfying* $mI \leqslant A_j \leqslant MI$ $(j = 1, 2, \cdots, k)$ *where* $0 < m < M$. *Let* $f(t)$ *be a real valued continuous convex function on* $[m, M]$ *and* $x_1, x_2, \cdots, x_k \in H$ *with* $\sum_{j=1}^k \|x_j\|^2 = 1$. *Then the following inequality holds:*

$$\sum_{j=1}^k (f(A_j)x_j, x_j) \leqslant C(q) \Big(\sum_{j=1}^k (A_j x_j, x_j) \Big)^q$$

under any one of the conditions (i) and (ii) in Theorem A.

REMARK 2. Furthermore, Furuta extends Theorem B as follows: Under the same situation as in Theorem B, the inequality

$$\sum_{j=1}^k U_j f(A_j) U_j^* \leqslant C(q) \Big(\sum_{j=1}^k U_j A_j U_j^* \Big)^q$$

holds for contractions U_j $(j = 1, 2, \cdots, k)$ with $\sum_{j=1}^k U_j U_j^* = 1$. Also, it can be extended to unital positive linear maps.

On the other hand, Mond and Pečarić showed the following inequality which is a general one for positive operators including real valued convex functions (cf.[9, Theorem 4]):

THEOREM C. *Let* A *be a positive operator on a Hilbert space* H *satisfying* $mI \leqslant A \leqslant MI$ *where* $0 < m < M$. *Let* $f(t)$ *be a real valued continuous convex function* *on $[m,M]$ and J an interval including $f[m,M]$. If $F(u,v)$ is a real valued function defined on $J\times J$, non-decreasing in u, then*

$$F\left[(f(A)x,x),f((Ax,x))\right]\leqslant \max_{m\leqslant t\leqslant M} F\left[f(m)+\frac{f(M)-f(m)}{M-m}(t-m),f(t)\right]$$

for every unit vector x in H.

Unfortunately, Theorem C of Mond-Pečarić is not able to cover Theorem A and Theorem B of Furuta. However, our general setting constructed in the next section is available for us to understand theorems of Furuta.

3. A general theorem

We first show Jensen's type inequality in multiple positive operator cases and extend Theorem C in the preceding section, which is based on the idea due to Furuta [5], [6], [7] and [8].

THEOREM 1. *Let A_j be positive operators on a Hilbert space H satisfying $mI\leqslant A_j\leqslant MI$ $(j=1,2,\ldots,k)$, where $0<m<M$. Let $f(t)$ be a real valued continuous convex function on $[m,M]$ and also let $x_1,x_2,\ldots,x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k}\|x_j\|^2=1$. Then the following inequalities hold*

$$f\Big(\sum_{j=1}^{k}(A_jx_j,x_j)\Big)\leqslant\sum_{j=1}^{k}(f(A_j)x_j,x_j), \tag{7}$$

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j)\leqslant f(m)+\frac{f(M)-f(m)}{M-m}\left(\sum_{j=1}^{k}\Big(A_jx_j,x_j\Big)-m\right). \tag{8}$$

Proof. The inequality (7) was given in [10, Theorem 1], the inequality (8) in [10, Theorem 2] by applying Lemma and the operational calculus.

THEOREM 2. *Assume that the conditions of Theorem 1 hold and let $g(t)$ be a real valued continuous function on $[m,M]$. Let U and V be two intervals such that $U\supset f[m,M]$ and $V\supset g[m,M]$. If $F(u,v)$ is a real valued function defined on $U\times V$, non-decreasing in u, then the following inequality holds*

$$\begin{aligned}
&F\Big[\sum_{j=1}^{k}(f(A_j)x_j,x_j),g\Big(\sum_{j=1}^{k}(A_jx_j,x_j)\Big)\Big]\\
&\quad\leqslant\max_{m\leqslant t\leqslant M}F\Big[f(m)+\frac{f(M)-f(m)}{M-m}(t-m),g(t)\Big]\\
&\quad=\max_{0\leqslant\theta\leqslant1}F\left[\theta f(m)+(1-\theta)f(M),g(\theta m+(1-\theta)M)\right].
\end{aligned}\tag{9}$$

Proof. Let us take $t_0 = \sum_{j=1}^{k}(A_j x_j, x_j)$ in (8). The hypothesis ensures inequality $m = \sum_{j=1}^{k}(m x_j, x_j) \leqslant \sum_{j=1}^{k}(A_j x_j, x_j) \leqslant \sum_{j=1}^{k}(M x_j, x_j) = M$, i.e. $m \leqslant t_0 \leqslant M$. Using the non-decreasing character of $F(\cdot, v)$, we have

$$\begin{aligned} F\Big[\sum_{j=1}^{k}(f\,(A_j)\,x_j, x_j)\,, g\Big(\sum_{j=1}^{k}(A_j x_j, x_j)\Big)\Big] & \\ \leqslant F\Big[f\,(m) + \frac{f\,(M) - f\,(m)}{M-m}(t_0 - m), g(t_0)\Big] & \\ \leqslant \max_{m \leqslant t \leqslant M} F\Big[f\,(m) + \frac{f\,(M) - f\,(m)}{M-m}(t - m), g(t)\Big]. & \end{aligned}$$

The second form on the right side of inequality (9) follows at once from the change of variable $\theta = (M-t)/(M-m)$, so $t = \theta m + (1-\theta)M$ with $0 \leqslant \theta \leqslant 1$.

THEOREM 3. *Assume that the conditions of Theorem 2 hold except that $F(u,v)$ is non-increasing in u. Then the following inequality holds*

$$\begin{aligned} F\Big[\sum_{j=1}^{k}(f\,(A_j)\,x_j, x_j)\,, g\Big(\sum_{j=1}^{k}(A_j x_j, x_j)\Big)\Big] & \\ \geqslant \min_{m \leqslant t \leqslant M} F\Big[f\,(m) + \frac{f\,(M) - f\,(m)}{M-m}(t - m), g(t)\Big] & \\ = \min_{0 \leqslant \theta \leqslant 1} F\left[\theta f\,(m) + (1-\theta) f\,(M), g\left(\theta m + (1-\theta)M\right)\right]. & \qquad (10) \end{aligned}$$

Proof. We have this Theorem by replacing F by $-F$ in Theorem 2.

REMARK 3. We remark that Theorems 2 and 3 in the case $g(t) = f\,(t)$ are proved by Mond-Pečarić [9, Theorems 3,4].

By a similar method as in Theorem 2, we obtain the following result which is an extension of Mond-Pečarić in [11, Theorem 7]:

COROLLARY 1. *Assume that the conditions of Theorem 2 hold except that $F(u,v)$ is operator monotone in u. Let U_j $(j = 1, 2, \cdots, k)$ be contractions with $\sum_{j=1}^{k} U_j U_j^* = 1$, then the following inequality holds*

$$\begin{aligned} F\Big[\sum_{j=1}^{k} U_j f\,(A_j) U_j^*, g\big(\sum_{j=1}^{k} U_j A_j U_j^*\big)\Big] & \\ \leqslant \Big\{\max_{m \leqslant t \leqslant M} F\Big[f\,(m) + \frac{f\,(M) - f\,(m)}{M-m}(t - m), g(t)\Big]\Big\} I & \\ = \Big\{\max_{0 \leqslant \theta \leqslant 1} F\left[\theta f\,(m) + (1-\theta) f\,(M), g\left(\theta m + (1-\theta)M\right)\right]\Big\} I. & \end{aligned}$$

4. Converses of Jensen's inequality

As a simple application of Theorem 2, we discuss an extension of [16, Theorem 1], which give us a unified view to several inequalities due to Ky Fan, Furuta, Mond and Pečarić. Moreover we shall consider the conditions under which the equality holds. For convenience, we define $\mu = \frac{f(M)-f(m)}{M-m}$ for a real valued function f on the interval $[m,M]$.

THEOREM 4. *Assume that the conditions of Theorem 1 hold and let $g(t)$ be a real valued continuous function on $[m,M]$. Then for any real number $\alpha \in \mathbf{R}$ the following inequality*

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \leqslant \alpha g\Big(\sum_{j=1}^{k}(A_jx_j,x_j)\Big)+\beta \tag{11}$$

holds for $\beta = \max_{m\leqslant t\leqslant M}\{f(m)+\mu(t-m)-\alpha g(t)\}$.

Moreover, suppose that $\beta = f(m)+\mu\left(\sum_{j=1}^{k}(A_jx_j,x_j)-m\right)-\alpha g\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right)$ for some vectors x_j in H such that $\sum_{j=1}^{k}\|x_j\|^2=1$. Then the equality is attained in (11) if and only if there exist orthogonal vectors y_j and z_j such that

$$x_j = y_j+z_j, \quad A_jy_j = my_j \quad \textit{and} \quad A_jz_j = Mz_j. \tag{12}$$

Proof. Put $t_0 = \sum_{j=1}^{k}(A_jx_j,x_j)$, then the hypothesis ensures the inequality $m \leqslant t_0 \leqslant M$. Also, put $F(u,v) = u-\alpha v$, $u = \sum_{j=1}^{k}(f(A_j)x_j,x_j)$ and $v = g(t_0)$. Then it follows from Theorem 2 that

$$\begin{aligned}\sum_{j=1}^{k}(f(A_j)x_j,x_j)-\alpha g\Big(\sum_{j=1}^{k}(A_jx_j,x_j)\Big) &\leqslant \max_{m\leqslant t\leqslant M} F[f(m)+\mu(t-m),g(t)] \\ &= \max_{m\leqslant t\leqslant M}\{f(m)+\mu(t-m)-\alpha g(t)\},\end{aligned}$$

which gives the desired inequality.

We next investigate conditions under which the equality holds. Suppose that the equality $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = \alpha g(t_0)+\beta$ holds. By definition of β, notice that the equality $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = \alpha g(t_0)+\beta$ holds if and only if the equality $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = f(m)+\mu(t_0-m)$ holds. Let $E_j(t)$ be the spectral resolution of the identity of A_j, that is, $A_j = \int_{m-0}^{M} t dE_j(t)$. Put $P_j = E_j(M)-E_j(M-0)$, $Q_j = E_j(M-0)-E_j(m)$ and $R_j = E_j(m)-E_j(m-0)$. Then $(A_jP_jx_j,x_j) = M(P_jx_j,x_j)$ and $(A_jR_jx_j,x_j) = m(R_jx_j,x_j)$. Note also that

$$\begin{aligned}(f(A_j)P_jx_j,x_j) &= \int_{m-0}^{M} f(t)d(E_j(t)P_jx_j,x_j) = f(M)(P_jx_j,x_j) \\ &= ((f(m)+\mu(M-m))P_jx_j,x_j)\end{aligned}$$

and

$$\begin{aligned}(f(A_j)R_jx_j,x_j) &= \int_{m-0}^{M} f(t)d(E_j(t)R_jx_j,x_j) = f(m)(R_jx_j,x_j)\\ &= ((f(m)+\mu(m-m))R_jx_j,x_j).\end{aligned}$$

Since $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = f(m)+\mu(t_0-m)$, it follows that $\sum_{j=1}^{k}((f(m)+\mu(A_j-m)-f(A_j))Q_jx_j,x_j) = 0$ and hence $Q_jx_j = 0$ for any j because $f(m)+\mu(s-m)-f(s) > 0$ for $s \in (m,M)$. Thus we obtain the desired decomposition of x_j setting $y_j = R_jx_j$ and $z_j = P_jx_j$.

Assume conversely (12). Then it follows that

$$\begin{aligned}f(m)+\mu\left(\sum_{j=1}^{k}(A_jx_j,x_j)-m\right) &= f(m)\sum_{j=1}^{k}(\|y_j\|^2+\|z_j\|^2)\\ &\quad +\mu\left(\sum_{j=1}^{k} m\|y_j\|^2+M\|z_j\|^2-m\right)\\ &= f(m)\sum_{j=1}^{k}\|y_j\|^2+f(M)\sum_{j=1}^{k}\|z_j\|^2\\ &= \sum_{j=1}^{k}(f(A_j)x_j,x_j),\end{aligned}$$

which is the desired equality.

Putting $g=f$, we have a multiple operator version in [16, Theorem 1]:

THEOREM 5. *Assume that the conditions of Theorem 1 hold and moreover let $f(t)$ be a nonnegative real valued continuous strictly convex twice differentiable function on $[m,M]$. Then for any positive real number $\alpha(>0)\in\mathbf{R}$ the following inequality*

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \leqslant \alpha f\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right)+\beta \tag{13}$$

holds for $\beta = -\alpha f(t_0)+f(m)+\mu(t_0-m)$ and

$$t_0 = \begin{cases} M & \text{if } M \leqslant f'^{-1}(\frac{\mu}{\alpha})\\ m & \text{if } f'^{-1}(\frac{\mu}{\alpha}) \leqslant m\\ f'^{-1}(\frac{\mu}{\alpha}) & \text{otherwise.}\end{cases}$$

The equality is attained in (13) if and only if there exist orthogonal vectors y_j and z_j such that $x_j = y_j+z_j$, $A_jy_j = my_j$, $A_jz_j = Mz_j$ and $t_0 = m\sum_{j=1}^{k}\|y_j\|^2+M\sum_{j=1}^{k}\|z_j\|^2$.

Proof. By virtue of Theorem 4, it is sufficient to see that $\beta = -\alpha f(t_0) + f(m) + \mu(t_0 - m)$. Put $h(t) = f(m) + \mu(t-m) - \alpha f(t)$. Since $f(t)$ is strictly convex, we put $t_1 = f'^{-1}(\frac{\mu}{\alpha})$. Then we have $h'(t) = 0$ if and only if $t = t_1$. If $m \leqslant t_1 \leqslant M$, then $h''(t) = -\alpha f''(t) < 0$ and so $\beta = \max_{m\leqslant t\leqslant M} h(t) = h(t_1)$. If $M \leqslant t_1$, then $h(t)$ is increasing on $[m,M]$ and hence the maximum value on $[m,M]$ of $h(t)$ is attained for $t_0 = M$. Similarly, we have $t_0 = m$ if $t_1 \leqslant m$.

Next, since the graph of $\alpha f(t)+\beta$ touches the line of $f(m)+\mu(t-m)$ at the point t_0, it follows that the equality $\sum_{j=1}^k (f(A_j)x_j, x_j) = \alpha f(\sum_{j=1}^k (A_jx_j, x_j)) + \beta$ holds if and only if two equalities $t_1 = \sum_{j=1}^k (A_jx_j, x_j)$ and $\sum_{j=1}^k (f(A_j)x_j, x_j) = f(m) + \mu(t_1 - m)$ hold. Therefore we obtain Theorem 5 by the same proof as Theorem 4.

Moreover, we have two corollaries; Corollary 2 (resp. Corollary 3) follows from Thereom 4 (resp. Corollary 1) and they extend the results by Furuta [6], [7] on Hölder-McCarthy and Kantorovich type inequalities.

COROLLARY 2. *Assume that the conditions of Theorem 1 hold. Suppose that either of the following conditions holds*

(I) $f(m) < f(M)$, $\dfrac{f(m)}{m} < \dfrac{f(M)}{M}$, $q > 1$ *any real number*

or

(II) $f(m) > f(M)$, $\dfrac{f(m)}{m} > \dfrac{f(M)}{M}$, $q < 0$ *any real number.*

Then for any positive real number $\alpha(>0) \in \mathbf{R}$ *the following inequality*

$$\sum_{j=1}^k (f(A_j)x_j, x_j) \leqslant \alpha \left(\sum_{j=1}^k (A_jx_j, x_j)\right)^q + \beta \tag{14}$$

holds for

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{\mu}{\alpha q}\right)^{\frac{q}{q-1}} + \dfrac{Mf(m) - mf(M)}{M-m} & \text{if} \quad \alpha m^{q-1}q \leqslant \mu \leqslant \alpha M^{q-1}q, \\ \max\{f(M) - \alpha M^q, f(m) - \alpha m^q\} & \text{otherwise.} \end{cases}$$

COROLLARY 3. *If the conditions of Corollary 1 are satisfied, then for each* $\alpha \in \mathbf{R}$

$$\sum_{j=1}^k U_j f(A_j)U_j^* \leqslant \alpha g\Big(\sum_{j=1}^k U_jA_jU_j^*\Big) + \beta I$$

holds for $\beta = \max_{m\leqslant t\leqslant M}\{f(m) + \mu(t-m) - \alpha g(t)\}$.

5. Applications I

In this section, as applications of our general theorem, we shall show general ratio inequalities that can be given for several positive operators and give the explicit expressions in their estimations.

5.1. Application to ratio operators.

THEOREM 6. *Assume that the condition of Theorem 1 hold and let $g(t)$ be a real valued continuous function on $[m,M]$. Suppose that either of the following conditions holds*
(i) $g(t) > 0$ *for all* $t \in [m,M]$ *and* $f(m) > 0$, $f(M) > 0$
or
(ii) $g(t) < 0$ *for all* $t \in [m,M]$ *and* $f(m) < 0$, $f(M) < 0$.
Then the following inequality

$$\sum_{j=1}^{k} (f\ (A_j)\, x_j, x_j) \leqslant \lambda\ g\Big(\sum_{j=1}^{k} (A_j x_j, x_j)\Big) \tag{15}$$

holds for

$$\lambda = \max_{m \leqslant t \leqslant M} \left\{ \frac{1}{g(t)} \left(f(m) + \mu(t-m)\right) \right\} \tag{16}$$

in case (i), or

$$\lambda = \min_{m \leqslant t \leqslant M} \left\{ \frac{1}{g(t)} \left(f(m) + \mu(t-m)\right) \right\} \tag{17}$$

in case (ii), where $\mu = \frac{f(M)-f(m)}{M-m}$.
(iii) Suppose that moreover $g(t)$ *is the strictly convex twice differentiable function on* $[m,M]$ *under (i) or (ii), then the constant* λ *satisfies the conditon* $\lambda \geqslant \max\{f(m)/g(m), f(M)/g(M)\}$ *in case (i), or* $0 < \lambda \leqslant \min\{f(m)/g(m), f(M)/g(M)\}$ *in case (ii), where the strictly inequalities hold if* $[\mu g(m) - f(m)g'(m)]\,[\mu g(M) - f(M)g'(M)] < 0$.

More precisely, a value of $\lambda \equiv \lambda(m,M,f,g)$ *for* (15) *may be determined as follows:*
If $[\mu g(m) - f(m)g'(m)]\,[\mu g(M) - f(M)g'(M)] \leqslant 0$, *then if* $\mu = 0$, *let* $t = \bar{t}$ *be the unique solution of the equation* $g'(t) = 0$ $(m < t < M)$; *then* $\lambda = f(m)/g(\bar{t})$ *suffices for* (15), *but if* $\mu \neq 0$, *let* $t = \bar{t}$ *be the unique solution in* $[m,M]$ *of the equation*

$$\mu g(t) - g'(t)\left(f(m) + \mu(t-m)\right) = 0 \tag{18}$$

then $\lambda = \mu/g'(\bar{t})$ *suffices for (15).*
If $[\mu g(m) - f(m)g'(m)]\,[\mu g(M) - f(M)g'(M)] > 0$, *then*

$$\lambda = \max\{f(m)/g(m), f(M)/g(M)\} \tag{19}$$

suffices for (15) in case (i), or

$$\lambda = \min\{f(m)/g(m), f(M)/g(M)\} \tag{20}$$

suffices for (15) in case (ii).

Proof. For case $g(t) > 0$ on $[m,M]$ apply Theorem 2 and for case $g(t) < 0$ on $[m,M]$ apply Theorem 3 both with $F(u,v) = u/v$. We proceed only with case (i) since the proof in case (ii) is essentially the same.
The inequality (9) becomes

$$\sum_{j=1}^{k} (f\,(A_j)\,x_j, x_j) \leqslant \max_{m\leqslant t\leqslant M} h(t;m,M,f,g)\cdot g\Big(\sum_{j=1}^{k}(A_jx_j,x_j)\Big), \tag{21}$$

where

$$h(t) \equiv h(t;m,M,f,g) = \frac{f\,(m)+\mu(t-m)}{g(t)}. \tag{22}$$

Now $h'(t) = H(t)/g(t)^2$, where

$$H(t) = \mu g(t) - (f\,(m)+\mu(t-m))\,g'(t). \tag{23}$$

Because $f\,(m) > 0$ and $f\,(M) > 0$, we have $f\,(m)+\mu(t-m) = \frac{f\,(m)(M-t)+f\,(M)(t-m)}{M-m} > 0$ for all $t \in [m,M]$. Let $g(t)$ be the strictly convex twice differentiable function on $[m,M]$, i.e. $g''(t) > 0$ in (iii). It follows that $H'(t) = -(f\,(m)+\mu(t-m))\,g''(t) < 0$, so that H is a decreasing function on $[m,M]$.
Furthermore, if $H(m)H(M) = [\mu g(m) - f\,(m)g'(m)]\,[\mu g(M) - f\,(M)g'(M)] \leqslant 0$, then the equation $H(t) = 0$ has exactly one solution $\bar{t} \in [m,M]$. Hence, the maximum value on $[m,M]$ of the function $h(t)$ is attained for $t = \bar{t}$, since

$$h''(\bar{t}) = \left[H'(\bar{t})g(\bar{t}) - 2H(\bar{t})g'(\bar{t})\right]/g(\bar{t})^3 = H'(\bar{t})/g(\bar{t})^2 < 0.$$

If $H(m)H(M) > 0$ then because $H(t)$ is a decreasing function, we have that either $H(t) > 0$ or $H(t) < 0$ on [m,M], i.e. either $h'(t) > 0$ or $h'(t) < 0$ on $[m,M]$. Hence $h(t)$ is a monotone function on $[m,M]$ and it follows that $\max_{m\leqslant t\leqslant M} h(t) = \max\left\{\frac{f\,(m)}{g(m)}, \frac{f\,(M)}{g(M)}\right\}$. Thus the proof of (15) for λ determined by (16) in case (i) is complete.
Now we remark that if $H(m)H(M) < 0$ then $\bar{t} \in (m,M)$ and $\lambda > \max\left\{\frac{f\,(m)}{g(m)}, \frac{f\,(M)}{g(M)}\right\}$ in case (i), or $\lambda < \min\left\{\frac{f\,(m)}{g(m)}, \frac{f\,(M)}{g(M)}\right\}$ in case (ii). The inequality $h(t) > 0$ on $[m,M]$ ensures the inequality $\lambda > 0$.

REMARK 4. We remark that inequality (16) is proved directly in [6, Lemma 2.1].

COROLLARY 4. *Assume that the conditions of Theorem 1 hold and moreover let $f\,(t)$ be a strictly convex twice differentiable function on $[m,M]$. Suppose that either of the following conditions holds*
(i) $f\,(t) > 0$ *for all* $t \in [m,M]$
or

(ii) $f(t) < 0$ *for all* $t \in [m,M]$.
Then the following inequality

$$\sum_{j=1}^{k} (f\,(A_j)\,x_j, x_j) \leqslant \lambda f\Big(\sum_{j=1}^{k} (A_j x_j, x_j)\Big) \tag{24}$$

holds for $\lambda > 1$ *in case* (i)*, or* $0 < \lambda < 1$ *in case* (ii).

More precisely, a value of $\lambda \equiv \lambda(m,M,f,g)$ *for (24) may be determined as follows:*
If $\mu = 0$*, let* $t = \bar{t}$ *be the unique solution of the equation* $f'(t) = 0$ $(m < \bar{t} < M)$*; then* $\lambda = f(m)/f(\bar{t})$ *suffices for (24). If* $\mu \neq 0$*, let* $t = \bar{t}$ *be the unique solution in* (m,M) *of the equation* $\mu f(t) - f'(t)\,(f(m) + \mu(t-m)) = 0$*; then* $\lambda = \mu/f'(\bar{t})$ *suffices for (24).*

Proof. We have this Corollary by replacing g by f in Theorem 6. The condition $H(m)H(M) < 0$ automatically holds in the case (i) or (ii). In fact, since f is a strictly convex differentiable function on $[m,M]$, it follows that $f(x) - f(y) > (x-y)f'_{+}(y)$ for $x \in O(y)$ and also (i) or (ii) holds. Therefore we obtain $H(m)H(M) = [\mu f(m) - f(m)f'(m)]\,[\mu f(M) - f(M)f'(M)] = f(m)f(M)(\mu - f'(m))(\mu - f'(M)) < 0$. Hence the inequality $\lambda > 1$ or $0 < \lambda < 1$ holds, because $\bar{t} \in (m,M)$ and $h(t) > 0$ on $[m,M]$.

REMARK 5. We remark that Corollary 4 is proved directly in [9, Corollary 1].

5.2. Application to power function.

COROLLARY 5. *Assume that the conditions of Theorem 1 hold. Suppose that either of the following conditions holds*

(I) $f(m) < f(M)$, $\dfrac{f(m)}{m} < \dfrac{f(M)}{M}$, $q > 1$ *any real number*
or
(II) $f(m) > f(M)$, $\dfrac{f(m)}{m} > \dfrac{f(M)}{M}$, $q < 0$ *any real number.*
Then the following inequality

$$\sum_{j=1}^{k} (f\,(A_j)\,x_j, x_j) \leqslant \lambda \,\left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^q \tag{25}$$

holds for

$$\lambda = \begin{cases} \frac{mf(M)-Mf(m)}{(q-1)(M-m)} \left(\frac{(q-1)(f(M)-f(m))}{q(mf(M)-Mf(m))}\right)^q & \text{if } \frac{f(m)}{m}q \leqslant \mu \leqslant \frac{f(M)}{M}q \\ \max\left\{\frac{f(m)}{m^q}, \frac{f(M)}{M^q}\right\} & \text{if } \mu < \frac{f(m)}{m}q \text{ or } \mu > \frac{f(M)}{M}q \end{cases}$$

$$\leqslant \frac{mf(M) - Mf(m)}{(q-1)(M-m)} \left(\frac{(q-1)\,(f(M) - f(m))}{q\,(mf(M) - Mf(m))}\right)^q, \tag{26}$$

where $\mu = (f(M) - f(m))/(M-m)$.

Proof. In Theorem 6 put $g(t) = t^q$ for $t > 0$ and for a real number q such that $q \notin [0,1]$, then we have $g''(t) > 0$. The condition (i) in Theorem 6 is weakened here as we can determine explicitely the solution of the equation (18): $\bar{t} = \frac{q}{q-1}\frac{mf(M) - Mf(m)}{f(M) - f(m)}$. It follows from the conditions (I) or (II) that $h''(\bar{t}) = -q(q-1)(f(m) + \mu(\bar{t} - m))/\bar{t}^{q+2} = (1-q)\mu\bar{t}^{-(q+1)} < 0$ and $\bar{t} > 0$ where $h(t)$ is defined by (22).

The inequality (26) follows from (16). Indeed, if $m \leqslant \bar{t} \leqslant M$, i.e. $H(m)H(M) \leqslant 0$ where $H(t)$ is defined by (23), we have the condition $qf(m)/m \leqslant \mu \leqslant qf(M)/M$. In this case $\lambda = h(\bar{t}) = \frac{mf(M) - Mf(m)}{(q-1)(M-m)}\left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))}\right)^q$. If $\bar{t} \notin [m,M]$, i.e. $H(m)H(M) > 0$ then (19) becomes $\lambda = \max\{f(m)/m^q, f(M)/M^q\}$.

COROLLARY 6. *Let A_j be positive operators on a Hilbert space H satisfying $mI \leqslant A_j \leqslant MI$ $(j = 1,2,\ldots,k)$, where $0 < m < M$. Let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^k \|x_j\|^2 = 1$. Then for any real number $p > 1$ and $q > 1$ or $p < 0$ and $q < 0$ the following inequality*

$$\sum_{j=1}^{k}\left(A_j^p x_j, x_j\right) \leqslant \lambda\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right)^q \tag{27}$$

holds for

$$\begin{aligned}\lambda &= \begin{cases}\dfrac{(mM^p - Mm^p)}{(q-1)(M-m)}\left(\dfrac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)}\right)^q & \text{if } m^{p-1}q \leqslant \bar{\mu} \leqslant M^{p-1}q\\ \max\left\{\dfrac{m^p}{m^q}, \dfrac{M^p}{M^q}\right\} & \text{if } \bar{\mu} < m^{p-1}q \text{ or } \bar{\mu} > M^{p-1}q\end{cases}\\ &\leqslant \frac{(mM^p - Mm^p)}{(q-1)(M-m)}\left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)}\right)^q,\end{aligned} \tag{28}$$

where $\bar{\mu} = (M^p - m^p)/(M-m)$.

Proof. We obtain the inequality (27) if we put $f(t) = t^p$ for a real number q such that $q \notin [0,1]$ in Corollary 5. In fact, the condition (I) in Corollary 5 holds if $p > 1$ and $q > 1$ and also the condition (II) holds if $p < 0$ and $q < 0$.

COROLLARY 7. *Let A_j be positive operators on a Hilbert space H satisfying $mI \leqslant A_j \leqslant MI$ $(j = 1,2,\ldots,k)$, where $0 < m < M$. Let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^k \|x_j\|^2 = 1$. Then for any real number $p \notin [0,1]$ the following inequality holds*

$$\sum_{j=1}^{k}\left(A_j^p x_j, x_j\right) \leqslant \frac{(mM^p - Mm^p)}{(p-1)(M-m)}\left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)}\right)^p\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right)^p. \tag{29}$$

Proof. Put $q = p \notin [0,1]$ in Corollary 6 or $f(t) = t^p$ in Corollary 4.

COROLLARY 8. *Let* A_j *be positive operators on a Hilbert space* H *satisfying* $mI \leqslant A_j \leqslant MI$ $(j = 1, 2, \ldots, k)$*, where* $0 < m < M$*. Let* $x_1, x_2, \ldots, x_k$ *be any finite number of vectors in* H *such that* $\sum_{j=1}^k \|x_j\|^2 = 1$*. Then for any real number* $p > 0$ *the following inequality*

$$\left(\sum_{j=1}^{k} (A_j x_j, x_j) \right)^p \left(\sum_{j=1}^{k} (A_j^{-1} x_j, x_j) \right) \leqslant \lambda_1 \tag{30}$$

holds for

$$\begin{aligned} \lambda_1 &= \begin{cases} \dfrac{p^p}{(p+1)^{p+1}} \dfrac{(m+M)^{p+1}}{mM} & \text{if } \dfrac{m}{M} \leqslant p \leqslant \dfrac{M}{m} \\ \max\left\{m^{p-1}, M^{p-1}\right\} & \text{if } p > \dfrac{M}{m} \text{ or } p < \dfrac{m}{M} \end{cases} \\ &\leqslant \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{mM} \end{aligned} \tag{31}$$

and

$$\sum_{j=1}^{k} (A_j^2 x_j, x_j) \leqslant \lambda_2 \left(\sum_{j=1}^{k} (A_j x_j, x_j) \right)^{p+1} \tag{32}$$

for

$$\begin{aligned} \lambda_2 &= \begin{cases} \dfrac{p^p}{(p+1)^{p+1}} \dfrac{(m+M)^{p+1}}{(mM)^p} & \text{if } \dfrac{m}{M} \leqslant p \leqslant \dfrac{M}{m} \\ \max\left\{m^{1-p}, M^{1-p}\right\} & \text{if } p > \dfrac{M}{m} \text{ or } p < \dfrac{m}{M} \end{cases} \\ &\leqslant \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{(mM)^p}. \end{aligned} \tag{33}$$

Proof. The inequality (30) for λ_1 from (31) follows from Corollary 6 if we put $p = -1$ and replace q by $-p$ for $p > 0$. Also we obtain the inequality (32) for λ_2 from (33) if we put $p = 2$ and replace q by $p + 1$ for $p > 0$.

REMARK 6. We remark that we have the inequalities (3) and (4) if we put $k = 1, p = 1$ in (30) and (32) respectively.

REMARK 7. The Corollary 5 is discussed in somewhat different style by Furuta [6, Theorem 1.1 i.e. Theorem B here]. This Corollary and the Corollaries 6, 8 [6, Corollaries 1.1,1.2] are proved only with appropriate conditions of type $[\mu g(m) - f(m)g'(m)]\,[\mu g(M) - f(M)g'(M)] < 0$.

5.3. Application to exponential function.

COROLLARY 9. *Assume that the conditions of Theorem 1 hold. Suppose that either of the following conditions holds*
(I) $f(m) < f(M)$, $\alpha > 0$ *any real number*
or
(II) $f(m) > f(M)$, $\alpha < 0$ *any real number*.
Then the following inequality

$$\sum_{j=1}^{k} (f(A_j)x_j, x_j) \leqslant \lambda \ \exp\left(\alpha \sum_{j=1}^{k} (A_jx_j, x_j)\right) \tag{34}$$

holds for

$$\begin{aligned} \lambda \ &= \ \begin{cases} \dfrac{\mu}{\alpha e} \exp\left(\dfrac{\alpha\,(Mf(m) - mf(M))}{f(M) - f(m)}\right) & \text{if} \ \ \alpha f(m) \leqslant \mu \leqslant \alpha f(M) \\ \max\left\{\dfrac{f(m)}{e^{\alpha m}}, \dfrac{f(M)}{e^{\alpha M}}\right\} & \text{if} \ \ \mu < \alpha f(m) \ \text{ or } \ \mu > \alpha f(M) \end{cases} \\ &\leqslant \ \frac{\mu}{\alpha e} \exp\left(\frac{\alpha\,(Mf(m) - mf(M))}{f(M) - f(m)}\right), \end{aligned} \tag{35}$$

where $\mu = (f(M) - f(m))/(M - m)$.

Proof. In Theorem 6 put $g(t) = e^{\alpha t}$ for a real number $\alpha \neq 0$. The condition (i) from Theorem 6 are weekened here as we can determine explicitely the solution of the equation (18): $\bar{t} = \frac{1}{\alpha} + m - f(m)\frac{1}{\mu}$. It follows from the conditions (I) or (II) that $h''(\bar{t}) = -\alpha\mu/e^{\alpha\bar{t}} < 0$ where $h(t)$ is defined by (22).

The inequality (35) follows from (16). Indeed, if $m \leqslant \bar{t} \leqslant M$, i.e. $H(m)H(M) \leqslant 0$ where $H(t)$ is defined by (23), then we have $\alpha f(m) \leqslant \mu \leqslant \alpha f(M)$. In this case $\lambda = h(\bar{t}) = \frac{\mu}{\alpha e} \exp\left(\frac{\alpha(Mf(m) - mf(M))}{f(M) - f(m)}\right)$. If $\bar{t} \notin [m, M]$, i.e. $H(m)H(M) > 0$ then (19) becomes $\lambda = \max\{f(m)/e^{\alpha m}, f(M)/e^{\alpha M}\}$.

COROLLARY 10. *Let* A_j *be positive operators on a Hilbert space* H *satisfying* $mI \leqslant A_j \leqslant MI$ $(j = 1, 2, \ldots, k)$, *where* $0 < m < M$. *Let* $x_1, x_2, \ldots, x_k$ *be any finite number of vectors in* H *such that* $\sum_{j=1}^{k} \|x_j\|^2 = 1$. *Then for any real number* $\alpha > 0$ *and* $\beta > 0$ *or* $\alpha < 0$ *and* $\beta < 0$ *the following inequality*

$$\sum_{j=1}^{k} \left(e^{\beta A_j}x_j, x_j\right) \leqslant \lambda \ \exp\left(\alpha \sum_{j=1}^{k} (A_jx_j, x_j)\right) \tag{36}$$

holds for

$$\lambda = \begin{cases} \dfrac{\bar{\mu}}{\alpha e}\exp\left(\dfrac{\alpha\left(Me^{\beta m}-me^{\beta M}\right)}{e^{\beta M}-e^{\beta m}}\right) & \text{if } \ \alpha e^{\beta m} \leqslant \bar{\mu} \leqslant \alpha e^{\beta M} \\ \max\left\{\dfrac{e^{\beta m}}{e^{\alpha m}}, \dfrac{e^{\beta M}}{e^{\alpha M}}\right\} & \text{if } \ \bar{\mu} < \alpha e^{\beta m} \ \text{ or } \ \bar{\mu} > \alpha e^{\beta M} \end{cases}$$

$$\leqslant \frac{\bar{\mu}}{\alpha e}\exp\left(\frac{\alpha\left(Me^{\beta m}-me^{\beta M}\right)}{e^{\beta M}-e^{\beta m}}\right), \tag{37}$$

where $\bar{\mu} = \left(e^{\beta M}-e^{\beta m}\right)/(M-m)$.

Proof. We obtain the inequality (36) if we put $f(t) = e^{\beta t}$ for a real number $\beta \neq 0$ in Corollary 9.

COROLLARY 11. *Assume that the conditions of Corollary 10 hold. Then for any real number* $\alpha \neq 0$ *the following inequality holds*

$$\sum_{j=1}^{k}\left(e^{\alpha A_j}x_j, x_j\right)$$

$$\leqslant \frac{e^{\alpha M}-e^{\alpha m}}{\alpha e(M-m)}\exp\left(\frac{\alpha\left(Me^{\alpha m}-me^{\alpha M}\right)}{e^{\alpha M}-e^{\alpha m}}\right)\exp\left(\alpha\sum_{j=1}^{k}(A_jx_j, x_j)\right). \tag{38}$$

Proof. Put $\alpha = \beta \neq 0$ in Corollary 10. As $f(t) = e^{\alpha t}$ is a real valued continuous convex function for any real number $\alpha \neq 0$, then we have

$$\alpha e^{\alpha m} \leqslant \left(e^{\alpha M}-e^{\alpha m}\right)/(M-m) \leqslant \alpha e^{\alpha M} \quad \text{for any real number } \ \alpha \neq 0,$$

which is just the first condition in the inequality (37).

REMARK 8. We have the inequalities

$$\left(e^{-A}x, x\right)e^{(Ax,x)} \leqslant \frac{e^M-e^m}{e^Me^m(M-m)}\exp\left(\frac{(M-1)e^M-(m-1)e^m}{e^M-e^m}\right)$$

if we put $k = 1, \alpha = -1$ in (38) and

$$\left(e^{A}x, x\right)e^{-(Ax,x)} \leqslant \frac{e^M-e^m}{M-m}\exp\left(\frac{(M+1)e^m-(m+1)e^M}{e^M-e^m}\right)$$

for $k = 1, \alpha = 1$. Here A is a positive operator on a Hilbert space H satisfying $mI \leqslant A \leqslant MI$ where $0 < m < M$ and x is any unit vector in H.

REMARK 9. The Corollaries 9-11 are proved in [6, Theorem 5.1, Corollaries 5.2-5.5], but only with appropriate conditions $\alpha f(m) \leqslant \mu \leqslant \alpha f(M)$ or $\alpha e^{\beta m} \leqslant \left(e^{\beta M}-e^{\beta m}\right)/(M-m) \leqslant \alpha e^{\beta M}$.

6. Applications II

In this section, as applications of our general theorem, we shall show general difference inequalities that can be given for several positive operators and give the explicit expressions in their estimations .

6.1. Application to difference operators.

THEOREM 7. *Assume that the conditions of Theorem 1 hold and let $g(t)$ be a real valued continuous function on $[m,M]$. Then the following inequality*

$$\sum_{j=1}^{k} (f\ (A_j)\, x_j, x_j) \leqslant \lambda \ + \ g\left(\sum_{j=1}^{k} (A_j x_j, x_j)\right) \tag{39}$$

holds for

$$\lambda = \max_{m \leqslant t \leqslant M} \{(f\,(m) + \mu(t-m)) - g(t)\}\,, \tag{40}$$

where $\mu = (f\,(M) - f\,(m))/(M-m)$.
If $g(t)$ is the differentiable function and $g'(t)$ is strictly increasing on $[m,M]$, then the constant λ satisfies the condition $f\,(m) - g(m) \leqslant \lambda \leqslant f\,(m) - g(m) + [\mu - g'(m)]\,(M - m)$.

More precisely, a value of $\lambda \equiv \lambda(m,M,f,g)$ for (39) may be determined as follows:
If $g'(m) \leqslant \mu \leqslant g'(M)$, then let $t = \bar{t}$ be the unique solution in $[m,M]$ of the equation

$$g'(t) = \mu \tag{41}$$

then $\lambda = f\,(m) - g(\bar{t}) + \mu(\bar{t} - m)$ suffices for (39).
If $\mu < g'(m)$ or $\mu > g'(M)$, then

$$\lambda = \max\,\{f\,(m) - g(m), f\,(M) - g(M)\} \tag{42}$$

suffices for (39).

Proof. We put $F(u,v) = u - v$ in Theorem 2. The inequality (9) becomes

$$\sum_{j=1}^{k} (f\ (A_j)\, x_j, x_j) - g\left(\sum_{j=1}^{k} (A_j x_j, x_j)\right) \leqslant \max_{m \leqslant t \leqslant M} h(t; m, M, f, g), \tag{43}$$

where

$$h(t) \equiv h(t; m, M, f, g) = f\,(m) + \mu(t-m) - g(t). \tag{44}$$

Let $g(t)$ be the differentiable function and $g'(t)$ is strictly increasing on $[m,M]$, then it follows that $h'(t) = \mu - g'(t)$ is strictly decreasing on $[m,M]$. If $g'(m) \leqslant \mu \leqslant g'(M)$, then the equation (41) has exactly one solution $\bar{t} \in [m,M]$ and the maximum value on $[m,M]$ of the function $h(t)$ is attained for $t = \bar{t}$. If $\mu < g'(m)$ or $\mu > g'(M)$, i.e.

$[\mu - g'(m)]\,[\mu - g'(M)] > 0$ then because $h'(t)$ is a decreasing function, we have that either $h'(t) > 0$ or $h'(t) < 0$ on [m,M]. Hence $h(t)$ is a monotone function on $[m,M]$ and it follows that $\max_{m\leqslant t\leqslant M} h(t) = \max\{f(m) - g(m), f(M) - g(M)\}$. Thus the proof of (39) for λ determined by (40) is complete.
Now we remark that since $g'(t)$ is an increasing function, i.e. $g(t)$ is a strictly convex function, it follows that

$$g(m) - g(\bar{t}) \leqslant g'(m)(m - \bar{t}) \quad \text{if} \quad m \leqslant \bar{t} \leqslant M.$$

Then we have

$$\lambda = f(m) - g(\bar{t}) + \mu(\bar{t} - m) = f(m) - g(m) + [g(m) - g(\bar{t}) + \mu(\bar{t} - m)]$$
$$\leqslant f(m) - g(m) + [-g'(m) + \mu](\bar{t} - m) \leqslant f(m) - g(m) + [-g'(m) + \mu](M - m).$$

Hence we have the upper bound for λ. The lower bound is evident.

COROLLARY 12. *Assume that the conditions of Theorem 1 hold and moreover let $f(t)$ be a differentiable function and f' is strictly increasing on $[m,M]$. Then the following inequality*

$$\sum_{j=1}^{k} (f(A_j)x_j, x_j) \leqslant \lambda + f\left(\sum_{j=1}^{k} (A_j x_j, x_j)\right) \tag{45}$$

holds for λ satisfying $0 < \lambda < (M-m)[\mu - f'(m)]$, *where* $\mu = (f(M)-f(m))/(M-m)$.

More precisely, a value of $\lambda \equiv \lambda(m,M,f,g)$ for (45) may be determined as follows: let $t = \bar{t}$ be the unique solution of the equation $f'(t) = \mu$ in (m,M). Then $\lambda = f(m) - f(\bar{t}) + \mu(\bar{t} - m)$ suffices for (45).

Proof. We have this Corollary by replacing g by f in Theorem 7. Since f is a strictly convex differentiable function on $[m,M]$, it follows $f(x) - f(y) > (x-y)f'_+(y)$ for $x \in O(y)$ and we obtain $f'(m) < \mu < g'(M)$. Bounds for λ are evident.

REMARK 10. We remark that Corollary 12 is proved directly in [9, Corollary 2].

6.2. Application to power function.

COROLLARY 13. *Assume that the conditions of Theorem 1 hold. Suppose that either of the following conditions holds*
(I) $f(m) < f(M)$, $q > 1$ *any real number*
or
(II) $f(m) > f(M)$, $q < 0$ *any real number.*
Then the following inequality

$$\sum_{j=1}^{k} (f(A_j)x_j, x_j) \leqslant \lambda + \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^q \tag{46}$$

holds for

$$\lambda = \begin{cases} \dfrac{Mf(m)-mf(M)}{M-m} + (q-1)\left(\frac{1}{q}\mu\right)^{q/(q-1)} & \text{if } qm^{q-1} \leqslant \mu \leqslant qM^{q-1} \\ \max\{f(m)-m^q, f(M)-M^q\} & \text{if } \mu < qm^{q-1} \text{ or } \mu > qM^{q-1} \end{cases}$$

$$\leqslant \frac{Mf(m)-mf(M)}{M-m} + (q-1)\left(\frac{1}{q}\mu\right)^{q/(q-1)}, \tag{47}$$

where $\mu = (f(M)-f(m))/(M-m)$.

Proof. In Theorem 7 put $g(t) = t^q$ for $t > 0$ and for a real number q such that $q \notin [0,1]$ when $g'(t)$ is strictly increasing. We can determine explicitely the solution of the equation (41): $\bar{t} = \left(\frac{1}{q}\mu\right)^{1/(q-1)}$.

The inequality (47) follows from (40). Indeed, if $m \leqslant \bar{t} \leqslant M$, we have the condition $qm^{q-1} \leqslant \mu \leqslant qM^{q-1}$. In this case $\lambda = \frac{Mf(m)-mf(M)}{M-m} + (q-1)\left(\frac{1}{q}\mu\right)^{q/(q-1)}$. If $\bar{t} \notin [m,M]$ then because $h(t)$ is an increasing function (respectively decreasing) for $t < \bar{t}$ if the condition (I) holds and for $t > \bar{t}$ if the condition (II) holds (respectively for $t > \bar{t}$ if the condition (I) holds and for $t < \bar{t}$ if the condition (I) holds), we have $\lambda = \max\{f(m)-m^q, f(M)-M^q\}$.

COROLLARY 14. *Let A_j be positive operators on a Hilbert space H satisfying $mI \leqslant A_j \leqslant MI$ $(j = 1,2,\ldots,k)$, where $0 < m < M$. Let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then for any real number $p > 1$ and $q > 1$ or $p < 0$ and $q < 0$ the following inequality*

$$\sum_{j=1}^{k} \left(A_j^p x_j, x_j\right) \leqslant \lambda + \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^q \tag{48}$$

holds for

$$\lambda = \begin{cases} \dfrac{Mm^p - mM^p}{M-m} + (q-1)\left(\frac{1}{q}\bar{\mu}\right)^{q/(q-1)} & \text{if } qm^{q-1} \leqslant \bar{\mu} \leqslant qM^{q-1} \\ \max\{m^p - m^q, M^p - M^q\} & \text{if } \bar{\mu} < qm^{p-1} \text{ or } \bar{\mu} > qM^{p-1} \end{cases}$$

$$\leqslant \frac{Mm^p - mM^p}{M-m} + (q-1)\left(\frac{1}{q}\bar{\mu}\right)^{q/(q-1)}, \tag{49}$$

where $\bar{\mu} = \frac{M^p - m^p}{M-m}$.

Proof. We obtain the inequality (48) if we put $f(t) = t^p$ for a real number q such that $q \notin [0,1]$ in Corollary 13. In fact, the condition (I) in Corollary 13 holds if $p > 1$ and $q > 1$ and also the condition (II) holds if $p < 0$ and $q < 0$.

COROLLARY 15. *Let A_j be positive operators on a Hilbert space H satisfying $mI \leqslant A_j \leqslant MI$ $(j = 1,2,\dots,k)$, where $0 < m < M$. Let $x_1,x_2,\dots,x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then for any real number $p \notin [0,1]$ the following inequality*

$$\sum_{j=1}^{k} \left(A_j^p x_j, x_j\right) \leqslant \lambda + \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^p \tag{50}$$

holds for

$$\lambda = \frac{Mm^p - mM^p}{M-m} + (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}}.$$

Proof. Put $q = p \notin [0,1]$ in Corollary 14.

REMARK 11. The estimation of Corollary 15 is better than one of [2, Theorem 2]:

$$(A^k x, x) - (Ax,x)^k \leqslant \frac{1}{4}((k-1)M^k - kM^{k-1}m + m^k) \tag{51}$$

for every natural number k. Actually it can be checked that the estimation of Corollary 15 is more precise than (51), that is,

$$\left(\frac{1}{p}\right)^{\frac{1}{p-1}} \left(1-\frac{1}{p}\right)\left(\frac{M^p - m^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{Mm^p - M^p m}{M-m} \leqslant \frac{1}{4}((p-1)M^p - pM^{p-1}m + m^p) \tag{52}$$

for every integer $p \geqslant 2$. Here we give a brief proof: We see that both sides of (52) coincide $\frac{1}{4}(M-m)^2$ in case $p = 2$. Multiplying (52) by m^{-p} and putting $x = \frac{M}{m} > 1$, then

$$\left(\frac{1}{p}\right)^{\frac{1}{p-1}} \left(1-\frac{1}{p}\right)\left(\frac{x^p - 1}{x-1}\right)^{\frac{p}{p-1}} + \frac{x - x^p}{x-1} \leqslant \frac{1}{4}((p-1)x^p - px^{p-1} + 1).$$

Therefore it suffices to see that for every integer $p \geqslant 3$

$$\frac{(p-1)x^{p+1} + (5-2p)x^p + px^{p-1} - 3x - 1}{4(x-1)} \geqslant \left(\frac{1}{p}\right)^{\frac{1}{p-1}} \frac{p-1}{p} \left(\frac{x^p-1}{x-1}\right)^{\frac{p}{p-1}}$$

if $x > 1$. To prove it, put $F(x) =$

$$\ln \frac{(p-1)x^{p+1} + (5-2p)x^p + px^{p-1} - 3x - 1}{4(x-1)} - \ln \left(\frac{1}{p}\right)^{\frac{1}{p-1}} \frac{p-1}{p} \left(\frac{x^p-1}{x-1}\right)^{\frac{p}{p-1}}.$$

Then $\lim_{x\to 1} F(x) = 0$ and by differentiating $F(x)$, we have

$$\begin{aligned} F'(x) = &\frac{(p-1)(p+1)x^p + (5-2p)px^{p-1} + p(p-1)x^{p-2} - 3}{(p-1)x^{p+1} + (5-2p)x^p + px^{p-1} - 3x - 1} \\ &+ \frac{(1-p^2)x^p + p^2x^{p-1} - 1}{(p-1)(x-1)(x^p-1)}. \end{aligned}$$

Moreover the numerator of $F'(x)$ becomes

$$(x-1)^5((2p^2-6p+4)x^{2p-5}+(6p^2-23p+20)x^{2p-6}+(12p^2-56p+60)x^{2p-7} + \cdots + \tfrac{1}{6}(2p^4-7p^3+7p^2-2p)x^{p-3}+\tfrac{1}{6}(2p^4-12p^3+22p^2-12p)x^{p-4} + \cdots + (12p-20)x+3p-4).$$

Since the coefficients of x are positive, it follows that $F'(x) > 0$ if $x > 1$. Therefore we have (52).

COROLLARY 16. *Let A_j be positive operator on a Hilbert space H satisfying $mI \leqslant A_j \leqslant MI$ $(j = 1, 2, \ldots, k)$, where $0 < m < M$. Let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then for any real number $p > 0$ the following inequality*

$$\left(\sum_{j=1}^{k}\left(A_j^{-1}x_j, x_j\right)\right) - \left(\sum_{j=1}^{k}\left(A_j x_j, x_j\right)\right)^{-p} \leqslant \lambda_1 \tag{53}$$

holds for

$$\lambda_1 = \begin{cases} \frac{M+m}{mM} - \frac{p+1}{(pMm)^{p/(p+1)}} & \text{if } \frac{m^p}{M} < p < \frac{M^p}{m} \\ \max\left\{\frac{1}{m}-\frac{1}{m^p}, \frac{1}{M}-\frac{1}{M^p}\right\} & \text{if } p \leqslant \frac{m^p}{M} \text{ or } p \geqslant \frac{M^p}{m} \end{cases}$$

$$\leqslant \frac{M+m}{mM} - \frac{p+1}{(pMm)^{p/(p+1)}} \tag{54}$$

and

$$\left(\sum_{j=1}^{k}\left(A_j^2 x_j, x_j\right)\right) - \left(\sum_{j=1}^{k}\left(A_j x_j, x_j\right)\right)^{p+1} \leqslant \lambda_2 \tag{55}$$

for

$$\lambda_2 = \begin{cases} p\left(\frac{M+m}{p+1}\right)^{(p+1)/p} - mM & \text{if } \frac{M+m}{M^p} < p < \frac{M+m}{m^p} \\ \max\left\{m^2 - m^{1+p}, M^2 - M^{1+p}\right\} & \text{if } p \leqslant \frac{M+m}{M^p} \text{ or } p \geqslant \frac{M+m}{m^p} \end{cases}$$

$$\leqslant p\left(\frac{M+m}{p+1}\right)^{(p+1)/p} - mM. \tag{56}$$

Proof. The inequality (53) for λ_1 from (54) follows from Corollary 14 if we put $p = -1$ and replace q by $-p$ for $p > 0$. Also we obtain the inequality (55) for λ_2 from (56) if we put $p = 2$ and replace q by $p+1$ for $p > 0$.

REMARK 12. We remark that we have the inequalities $\left(A^{-1}x, x\right) - (Ax, x)^{-1} \leqslant \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}$ and $\left(A^2x, x\right) - (Ax, x)^2 \leqslant \left(\frac{M-m}{2}\right)^2$ if we put $k = 1, p = 1$ in (53) and (55) respectively.

6.3. Some other inequalities for power function.

THEOREM 8. *Assume that the conditions of Theorem 1 hold and $f(t) > 0$ for all $t \in [m, M]$. Suppose that either of the following conditions holds*
(i) $f(m) > f(M)$, $q < 0$ *any real number*
or
(ii) $f(m) < (M)$, $0 < q < 1$ *any real number.*
Then the following inequality

$$\sum_{j=1}^{k} (A_j x_j, x_j) - \left(\sum_{j=1}^{k} (f(A_j) x_j, x_j) \right)^q \leqslant \lambda_1 \tag{57}$$

holds for

$$\lambda_1 = \begin{cases} -\frac{\nu}{\mu} + \frac{1}{\mu} f_{max} - f_{max}^q & \text{if } \mu < \frac{1}{q} f_{max}^{1-q} \\ -\frac{\nu}{\mu} + (q-1)(\mu q)^{q/(1-q)} & \text{if } \frac{1}{q} f_{max}^{1-q} \leqslant \mu \leqslant \frac{1}{q} f_{min}^{1-q} \\ -\frac{\nu}{\mu} + \frac{1}{\mu} f_{min} - f_{min}^q & \text{if } \mu > \frac{1}{q} f_{min}^{1-q} \end{cases}$$

$$\leqslant -\frac{\nu}{\mu} + (q-1)(\mu q)^{q/(1-q)} \tag{58}$$

in case (i), and

$$\left(\sum_{j=1}^{k} (f(A_j) x_j, x_j) \right)^q - \sum_{j=1}^{k} (A_j x_j, x_j) \leqslant \lambda_2 \tag{59}$$

for

$$\lambda_2 = \begin{cases} \frac{\nu}{\mu} - \frac{1}{\mu} f_{min} + f_{min}^q & \text{if } \mu < \frac{1}{q} f_{min}^{1-q} \\ \frac{\nu}{\mu} + (q-1)(\mu q)^{q/(1-q)} & \text{if } \frac{1}{q} f_{min}^{1-q} \leqslant \mu \leqslant \frac{1}{q} f_{max}^{1-q} \\ \frac{\nu}{\mu} - \frac{1}{\mu} f_{max} + f_{max}^q & \text{if } \mu > \frac{1}{q} f_{max}^{1-q} \end{cases}$$

$$\leqslant -\frac{\nu}{\mu} + (q-1)(\mu q)^{q/(1-q)} \tag{60}$$

in case (ii), where $\mu = \frac{f(M)-f(m)}{M-m}$, $\nu = \frac{Mf(m)-mf(M)}{M-m}$, $f_{min} = \min_{m \leqslant t \leqslant M} f(t)$ *and* $f_{max} = \max_{m \leqslant t \leqslant M} f(t)$.

Proof. Let (i) be satisfied. Since the inequality (8) becomes

$$\sum_{j=1}^{k} (A_j x_j, x_j) \leqslant -\frac{\nu}{\mu} + \frac{1}{\mu} \sum_{j=1}^{k} (f(A_j) x_j, x_j),$$

it folows that

$$\sum_{j=1}^{k}(A_jx_j,x_j)-\left(\sum_{j=1}^{k}(f\ (A_j)\,x_j,x_j)\right)^q$$

$$\leqslant -\frac{\nu}{\mu}+\frac{1}{\mu}\sum_{j=1}^{k}(f\ (A_j)\,x_j,x_j)-\left(\sum_{j=1}^{k}(f\ (A_j)\,x_j,x_j)\right)^q. \tag{61}$$

Let us take $t=\sum_{j=1}^{k}(f\ (A_j)\,x_j,x_j)$ in the right side of this inequality. Because $0<f_{min}\leqslant f(t)\leqslant f_{max}$, by the Lemma, operators $f_{max}I-f\ (A_j)$ and $f\ (A_j)-f_{min}I$ are positive for all $j\in\{1,2,\dots,k\}$, and hence

$$f_{min}\,(x_j,x_j)\leqslant (f\ (A_j)\,x_j,x_j)\leqslant f_{max}\,(x_j,x_j)\,.$$

Then we have $t\in[f_{min},f_{max}]$ by summing over j. The inequality (61) becomes

$$\sum_{j=1}^{k}(A_jx_j,x_j)-\left(\sum_{j=1}^{k}(f\ (A_j)\,x_j,x_j)\right)^q\leqslant \max_{t\in[f_{min},f_{max}]}h(t;m,M,f,q)$$

where

$$h(t)\equiv h(t;m,M,f,q)=-\frac{\nu}{\mu}+\frac{1}{\mu}t-t^q.$$

By the differential calculus of positive number t, the maximum value of $h(t)$ is attained for $\bar{t}=(\mu q)^{1/(1-q)}>0$ if $q<0$. Furthemore, if $\bar{t}\in[f_{min},f_{max}]$ then

$$\frac{1}{q}f_{max}^{\,1-q}\leqslant\mu\leqslant\frac{1}{q}f_{min}^{\,1-q}$$

and

$$\max_{t\in[f_{min},f_{max}]}h(t)=-\frac{\nu}{\mu}+(q-1)(\mu q)^{q/(1-q)}.$$

If $\bar{t}\notin[f_{min},f_{max}]$ then because $h(t)$ is an increasing function (respectively decreasing) for $t<\bar{t}$ (respectively for $t>\bar{t}$) then we have $\max_{t\in[f_{min},f_{max}]}h(t)=h(f_{max})$ (respectively $\max_{t\in[f_{min},f_{max}]}h(t)=h(f_{min})$) if $\bar{t}<f_{min}$ (respectively $\bar{t}>f_{max}$).
Then (57) is proved. For case (ii) we have $\lambda_2=-\min_{t\in[f_{min},f_{max}]}h(t)$.

COROLLARY 17. *Assume that the conditions of Theorem 8 hold. Then for any real number $p>0$ the following inequality*

$$\sum_{j=1}^{k}(A_jx_j,x_j)-\left(\sum_{j=1}^{k}\left(A_j^{-1}x_j,x_j\right)\right)^{-p}\leqslant\lambda_1 \tag{62}$$

holds for

$$\lambda_1 = \begin{cases} M - M^p & \text{if } p < \dfrac{m}{M^p} \\ M + m - (1+p)\left(\dfrac{mM}{p}\right)^{p/(1+p)} & \text{if } \dfrac{m}{M^p} \leqslant p \leqslant \dfrac{M}{m^p} \\ m - m^p & \text{if } p > \dfrac{M}{m^p} \end{cases}$$

$$\leqslant M + m - (1+p)\left(\frac{mM}{p}\right)^{p/(1+p)} \tag{63}$$

and for any real number $p > 1$ *the following inequality*

$$\left(\sum_{j=1}^{k}\left(A_j^2 x_j, x_j\right)\right)^{1/p} - \sum_{j=1}^{k}(A_j x_j, x_j) \leqslant \lambda_2 \tag{64}$$

holds for

$$\lambda_2 = \begin{cases} m^{2/p} - m & \text{if } \left(\dfrac{M+m}{p}\right)^{p/(p-1)} < m^2 \\ -\dfrac{Mm}{M+m} + \dfrac{p-1}{p}\left(\dfrac{M+m}{p}\right)^{1/(p-1)} & \text{if } m^2 \leqslant \left(\dfrac{M+m}{p}\right)^{p/(p-1)} \leqslant M^2 \\ M^{2/p} - M & \text{if } \left(\dfrac{M+m}{p}\right)^{p/(p-1)} > M^2 \end{cases}$$

$$\leqslant -\frac{Mm}{M+m} + \frac{p-1}{p}\left(\frac{M+m}{p}\right)^{1/(p-1)} \tag{65}$$

Proof. The inequality (62) for λ_1 from (63) follows from Theorem 8 if we put $f(t) = t^{-1}$ and replace q by $-p$ for $p > 0$. Also we obtain the inequality (64) for λ_2 from (65) if we put $f(t) = t^2$ and replace q by $1/p$ for $p > 1$.

REMARK 13. We have the inequalities $(Ax,x) - \left(A^{-1}x,x\right)^{-1} \leqslant \left(\sqrt{M} - \sqrt{m}\right)^2$ and $\sqrt{(A^2x,x)} - (Ax,x) \leqslant \frac{(M-m)^2}{4(M+m)}$ if we put $k = 1, p = 1$ in (62) and $k = 1, p = 2$ in (64) respectively.

REMARK 14. The inequalities (57) and (59) which we have got by the inequality (8) might be generalized so that they can be replaced by $\alpha \cdot \sum_{j=1}^{k}(A_j x_j, x_j)$ like in the starting inequality $\sum_{j=1}^{k}(A_j x_j, x_j)$, where α is any real number. The following inequality can be used for that.

COROLLARY 18. *Assume that the conditions of Theorem 1 hold. Then for any real number* α *the following inequality*

$$\sum_{j=1}^{k} (f\ (A_j)\, x_j, x_j) + \alpha \cdot \sum_{j=1}^{k} (A_j x_j, x_j) \leqslant \beta \tag{66}$$

holds for

$$\beta = \begin{cases} f(m) + \alpha m & \text{if } \ \alpha < -\mu \\ \dfrac{f(m)M - f(M)m}{M-m} & \text{if } \ \alpha = -\mu \\ f(M) + \alpha M & \text{if } \ \alpha > -\mu \end{cases} \tag{67}$$

Proof. We put $g(t) = -\alpha t$ in Theorem 7. Then the constant λ is the maximum value on $[m,M]$ of $h(t) = (\alpha + \mu)t + f(m) - \mu m$ (see (44)). The function $h(t)$ is increasing if $\alpha+\mu > 0$, that is $\max_{m\leqslant t\leqslant M} h(t) = h(M)$; $h(t)$ is decreasing if $\alpha+\mu < 0$, that is $\max_{m\leqslant t\leqslant M} h(t) = h(m)$. If $\alpha+\mu = 0$, that is $\max_{m\leqslant t\leqslant M} h(t) = f(m) - \mu m$. In this case the inequality (66) is in fact the inequality (8).

REMARK 15. We remark that we have the inequality

$$\sum_{j=1}^{k} \left(A_j^{-1} x_j, x_j\right) + \frac{1}{Mm} \sum_{j=1}^{k} (A_j x_j, x_j) \leqslant \frac{M+m}{Mm}$$

if we put in Corollary 18 $f(t) = t^{-1}$ and the inequality

$$\sum_{j=1}^{k} \left(A_j^2 x_j, x_j\right) - (M+m) \sum_{j=1}^{k} (A_j x_j, x_j) \leqslant \frac{M+m}{-Mm}$$

if we put $f(t) = t^2$.

REMARK 16. We remark that two inequalities in Remark 15 are proven differently in [5, Proof of Theorem 1. (i), (ii) and Lemma 1] so that is not used that these functions are convex.

6.4. Application to exponential function.

COROLLARY 19. *Assume that the conditions of Theorem 1 hold. Suppose that either of the following conditions holds*
(I) $f(m) < f(M)$, $\alpha > 0$ *any real number*
or
(II) $f(m) > f(M)$, $\alpha < 0$ *any real number.*
Then the following inequality

$$\sum_{j=1}^{k} (f\ (A_j)\, x_j, x_j) \leqslant \lambda + \exp\Big(\alpha \sum_{j=1}^{k} (A_j x_j, x_j)\Big) \tag{68}$$

holds for

$$\lambda = \begin{cases} \dfrac{Mf(m)-mf(M)}{M-m}+\dfrac{\mu}{\alpha}\left(\ln\dfrac{\mu}{\alpha}-1\right) & \text{if } \alpha e^{\alpha m}\leqslant \mu \leqslant \alpha e^{\alpha M} \\ \max\left\{f(m)-e^{\alpha m}, f(M)-e^{\alpha M}\right\} & \text{if } \mu<\alpha e^{\alpha m} \text{ or } \mu>\alpha e^{\alpha M} \end{cases}$$
$$\leqslant \frac{Mf(m)-mf(M)}{M-m}+\frac{\mu}{\alpha}\left(\ln\frac{\mu}{\alpha}-1\right). \tag{69}$$

Proof. In Theorem 7 put $g(t)=e^{\alpha t}$ for a real number $\alpha\neq 0$. The conditions (i) in Theorem 7 are weakened here as we can determine explicitely the solution of the equation $h'(t)=0$ where $h(t)=f(m)+\mu(t-m)-e^{\alpha t}$, $\bar{t}=\frac{1}{\alpha}\ln\frac{\mu}{\alpha}$. It gives the conditions (I) or (II).

COROLLARY 20. *Let A_j be positive operator on a Hilbert space H satisfying $mI\leqslant A_j\leqslant MI$ $(j=1,2,\ldots,k)$, where $0<m<M$. Let $x_1,x_2,\ldots,x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k}\|x_j\|^2=1$. Then for any real numbers $\alpha>0$ and $\beta>0$ or $\alpha<0$ and $\beta<0$ the following inequality holds*

$$\sum_{j=1}^{k}\left(e^{\beta A_j}x_j,x_j\right)\leqslant \lambda+\exp\left(\alpha\sum_{j=1}^{k}(A_jx_j,x_j)\right), \tag{70}$$

where λ is defined by (69) for $\bar{\mu}=\left(e^{\beta M}-e^{\beta m}\right)/(M-m)$.

Proof. We obtain the inequality (70) if we put $f(t)=e^{\beta t}$ for a real number $\beta\neq 0$ in Corollary 19.

COROLLARY 21. *Assume that the conditions of Corollary 10 hold. Then for any real number $\alpha\neq 0$ the following inequality holds*

$$\sum_{j=1}^{k}\left(e^{\alpha A_j}x_j,x_j\right)-\exp\left(\alpha\sum_{j=1}^{k}(A_jx_j,x_j)\right)$$
$$\leqslant \frac{Me^{\alpha m}-me^{\alpha M}}{M-m}+\frac{e^{\alpha M}-e^{\alpha m}}{\alpha(M-m)}\ln\left(\frac{e^{\alpha M}-e^{\alpha m}}{\alpha(M-m)}\right) \tag{71}$$

Proof. Put $\alpha=\beta\neq 0$ in Corollary 20. As $f(t)=e^{\alpha t}$ is a real valued continuous convex function for any real number $\alpha\neq 0$, then the first condition in the inequality (69) holds.

REMARK 17. We have the inequalities

$$\left(e^{-A}x,x\right)-e^{-(Ax,x)}\leqslant \frac{Me^{-m}-me^{-M}}{M-m}+\frac{e^{-m}-e^{-M}}{M-m}\ln\left(\frac{e^{-m}-e^{-M}}{M-m}\right)$$

if we put $k = 1, \alpha = -1$ in (71) and

$$\left(e^{A}x,x\right) - e^{(Ax,x)} \leqslant \frac{Me^{m} - me^{M}}{M-m} + \frac{e^{M} - e^{m}}{M-m} \ln\left(\frac{e^{M} - e^{m}}{M-m}\right)$$

for $k = 1, \alpha = 1$.

7. Multiple positive definite matrix case

If instead of the positive operators we observe the positive definite Hermitian matrices, then the results of all the above analogons are valid with the necessary modifications. For example, we have Theorems 10 and 12 instead of Theorem 2. We recall that $\mu = (f(M) - f(m))/(M-m)$ for a real valued function f on the interval $[m,M]$.

7.1. Extension of the Ky Fan inequality.

THEOREM 9. *Let A_j, $(j = 1,2,\ldots,k)$ be positive definite Hermitian matrices of order n with eigenvalues in the interval $[m,M]$, where $0 < m < M$, and also let $X_j, j = 1,2,\ldots,k$, be any finite number of vectors in the unitary n-space such that $\sum_{j=1}^{k}(X_j,X_j) = 1$. If $f(t)$ is a real valued continuous convex function on $[m,M]$, then the following inequality holds*

$$\sum_{j=1}^{k}\left(f\,(A_j)\,X_j,X_j\right) \leqslant f(m) + \mu\left(\sum_{j=1}^{k}(A_jX_j,X_j) - m\right). \tag{72}$$

Proof. This inequality was given in [13, Theorem 1].

THEOREM 10. *Assume that the conditions of Theorem 9 hold and let $g(t)$ be a real valued continuous function on $[m,M]$. Let U and V be two intervals such that $U \supset f[m,M]$ and $V \supset g[m,M]$. If $F(u,v)$ is a real valued function defined on $U \times V$, non-decreasing in u, then the following inequality holds*

$$\begin{aligned} &F\Big[\sum_{j=1}^{k}\left(f\,(A_j)\,X_j,X_j\right), g\Big(\sum_{j=1}^{k}(A_jX_j,X_j)\Big)\Big] \\ &\leqslant \max_{m\leqslant t\leqslant M} F\left[f(m) + \mu(t-m), g(t)\right] \\ &= \max_{0\leqslant\theta\leqslant1} F\left[\theta f(m) + (1-\theta)f(M), g\left(\theta m + (1-\theta)M\right)\right]. \end{aligned} \tag{73}$$

Proof. Proof is the almost same as one in the Theorem 2.

7.2. Matrix version of the Ky Fan inequality.

THEOREM 11. *Let A_j, $(j=1,2,\ldots,k)$ be positive definite Hermitian matrices of order n with eigenvalues in the interval $[m,M]$, where $0<m<M$, and also let $U_j, j=1,2,\ldots,k$, be $r\times n$ matrices such that $\sum_{j=1}^k U_jU_j^*=I$. If $f(t)$ is a real valued continuous convex function on $[m,M]$, then the following inequality holds*

$$\sum_{j=1}^{k} U_j f\,(A_j)\,U_j^* \leqslant f(m)I + \mu\Bigg(\sum_{j=1}^{k} U_jA_jU_j^* - mI\Bigg). \tag{74}$$

Proof. This inequality was given in [14, Theorem 1].

THEOREM 12. *Assume that the conditions of Theorem 9 hold and let $g(t)$ be a real valued continuous function on $[m,M]$. If U and V are two intervals such that $U\supset f[m,M]$ and $V\supset g[m,M]$ and if $F(u,v)$ is a real valued function defined on $U\times V$, matrix non-decreasing in u, then the following inequality holds*

$$\begin{aligned} &F\Big[\sum_{j=1}^{k} U_j f\,(A_j)\,U_j^*, g\Big(\sum_{j=1}^{k} U_jA_jU_j^*\Big)\Big] \\ &\leqslant \Big\{\max_{m\leqslant t\leqslant M} F\,[f\,(m)+\mu(t-m), g(t)]\Big\}\, I \\ &= \Big\{\max_{0\leqslant\theta\leqslant 1} F\,[\theta f\,(m)+(1-\theta)f\,(M), g\,(\theta m+(1-\theta)M)]\Big\}\, I. \end{aligned} \tag{75}$$

Proof. By (74) and the matrix non-decreasing character of $F(\cdot,v)$, we have

$$\begin{aligned} &F\Big[\sum_{j=1}^{k} U_j f\,(A_j)\,U_j^*, g\Big(\sum_{j=1}^{k} U_jA_jU_j^*\Big)\Big] \\ &\leqslant F\left[f\,(m)I+\mu\left(\tilde{A}-mI\right), g(\tilde{A})\right], \end{aligned} \tag{76}$$

where $\tilde{A}=\sum_{j=1}^k U_jA_jU_j^*$. Now, again consider the inequality for a real valued function

$$F\,[f\,(m)+\mu\,(t-m)\,, g(t)] \leqslant \lambda$$

where

$$\lambda = \max_{m\leqslant t\leqslant M} F\,[f\,(m)+\mu\,(t-m)\,, g(t)]\,.$$

As in [14, Theorem 1], we can get the matrix inequality

$$F\left[f\,(m)I+\mu\left(\tilde{A}-m\right), g(\tilde{A})\right] \leqslant \lambda I \tag{77}$$

for matrices $\tilde{A}$ such that $mI\leqslant\tilde{A}\leqslant MI$. Now (76) and (77) give (75).

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