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AN OPERATOR INEQUALITY WHICH IMPLIES PARANORMALITY

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Abstract. Let T be a bounded linear operator on a Hilbert space. Among other things, it is shown that (1) if $|T^2| \geqslant |T|^2$, then T is paranormal, (2) if T is w-hyponormal, then $|T^2| \geqslant |T|^2$ and $|{T^*}^2| \leqslant |T^*|^2$, and (3) if T and T^* are w-hyponormal, and either $\ker T \subseteq \ker T^*$ or $\ker T^* \subseteq \ker T$, then T is normal.

Introduction

Let T be a bounded linear operator on a Hilbert space H. The polar decomposition states that the operator T can be uniquely decomposed as $T = U|T|$ where U is a partial isometry, $|T| = (T^*T)^{1/2}$ and $\ker U = \ker |T|$.

For a given operator T, consider the operators $|T^2|$ and $|T|^2$. How are they related? If the operator T is hyponormal, it is easily seen that T satisfies both the inequality

$$|T^2| \geqslant |T|^2 \tag{1}$$

and the companion inequality

$$|{T^*}^2| \leqslant |T^*|^2. \tag{2}$$

This paper studies operators T satisfying either inequality (1) or inequalities (1) and (2).

In [4], Ando obtained an operator inequality (Theorem B below) involving $|T^2|$ and $|T|^2$ which characterizes the operator T as being paranormal. Using his characterization, Ando was able to obtained a theorem [4; Theorem 2] which implies that both the p- and log-hyponormal operators are paranormal.

In this paper, we first introduce the class of w-hyponormal operators and show that p- and log-hyponormal operators are w-hyponormal. In Section 2, we show that an operator T satisfying inequality (1) satisfies Ando's characterization. Consequently, such an operator is paranormal. We then prove the main result of the paper that a w-hyponormal operator T satisfies inequalities (1) and (2). Therefore, w-hyponormal

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operators are paranormal. Applications of the main result to powers of p- and log-hyponormal operators are given in Section 3. Finally, conditions under which w-hyponormal operators become normal are given in Section 4.

1. w-hyponormality

In this section, we introduce the classes of p-, log- and w-hyponormal operators and show that p- and log-hyponormal operators are w-hyponormal. Let p be a positive real number. The operator T is said to be p-hyponormal if $(T^*T)^p \geqslant (TT^*)^p$. A p-hyponormal operator is called hyponormal if $p = 1$, semi-hyponormal if $p = 1/2$. The celebrated Löwner-Heinz inequality implies that every p-hyponormal operator is q-hyponormal for $0 < q \leqslant p$. Call an operator T log-hyponormal if T is invertible and $\log|T| \geqslant \log|T^*|$. Finally, let $T = U|T|$ be the polar decomposition of T, and following [1], let $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$. The operator T is said to be w-hyponormal if

$$|\widetilde{T}| \geqslant |T| \geqslant |\widetilde{T}^*|.$$

The class of w-hyponormal operators was introduced and its spectral properties studied in [2]. Of particular interest is the fact, shown in [2], that the class of w-hyponormal operators contains both the p- and log-hyponormal operators. Since it is central to this paper, the proof of this fact is presented here.

THEOREM 1. *If T is a p-hyponormal operator, then T is w-hyponormal.*

Proof. Without loss of generality, assume $0 < p < 1/2$. In the proof of [1; Theorem 2] it was shown that if T is a p-hyponormal operator with $0 < p < 1/2$, then the inequalities

$$|\widetilde{T}|^{1+2p} \geqslant |T|^{1+2p} \geqslant |\widetilde{T}^*|^{1+2p}$$

hold. Raise each side of the inequalities to the $1/(1+2p)$-th power, and the result follows from the Löwner-Heinz inequality.

In order to show that log-hyponormal operators are w-hyponormal, we need a theorem of Ando's ([5], [6]) which characterizes the operator chaotic order.

THEOREM A. [5] *If A and B are positive invertible operators, then $\log A \geqslant \log B$ if and only if $A^r \geqslant (A^{r/2}B^rA^{r/2})^{1/2}$ for all $r \geqslant 0$.*

THEOREM 2. *If T is a log-hyponormal operator, then T is w-hyponormal.*

Proof. Since $\log|T| \geqslant \log|T^*|$, Theorem A implies that the inequality

$$|T| \geqslant (|T|^{1/2}|T^*||T|^{1/2})^{1/2} \tag{3}$$

holds. Hence,

$$|\widetilde{T}^*| = (|T|^{1/2}U|T|U^*|T|^{1/2})^{1/2} = (|T|^{1/2}|T^*||T|^{1/2})^{1/2} \leqslant |T|.$$

Moreover, since $|T|$ and $|T^*|$ are invertible, inequality (3) is seen to be equivalent to the inequality

$$(|T^*|^{1/2}|T||T^*|^{1/2})^{1/2} \geqslant |T^*|.$$

Thus,

$$U^*(U|T|^{1/2}U^*|T|U|T|^{1/2}U^*)^{1/2}U \geqslant U^*|T^*|U,$$

and therefore,

$$(|T|^{1/2}U^*|T|U|T|^{1/2})^{1/2} \geqslant |T|.$$

Since the operator on the left side of the last inequality is equal to the operator $|\widetilde{T}|$, we have $|\widetilde{T}| \geqslant |T|$.

2. Paranormality

Recall that an operator T is paranormal [7] if T satisfies

$$\| T^2x \| \geqslant \| Tx \|^2$$

for all unit vectors $x \in H$. As an application of Ando's (Theorem B) characterization of paranormality, we show that if the operator T satisfies inequality (1), then T is paranormal.

THEOREM B. [4; Theorem 1] *An operator T is paranormal if and only if the inequality*

$$|T^2|^2 - 2\lambda|T|^2 + \lambda^2 \geqslant 0$$

holds for all $\lambda \geqslant 0$.

THEOREM 3. *If the operator T satisfies inequality (1), then T is paranormal.*

Proof. Since T satisfies $|T^2| \geqslant |T|^2$, it satisfies the inequality of Theorem B for all $\lambda \geqslant 0$.

In [4; Theorem 2], Ando obtained a sufficient condition for an operator to be paranormal. He further showed that p- and log-hyponormal operators satisfy this condition, and hence are paranormal. The next theorem shows that if an operator T is w-hyponormal, then it satisfies inequality (1) and therefore, in view of Theorem 3, is paranormal. It follows that p- and log-hyponormal operators are paranormal.

THEOREM 4. *If $T = U|T|$ is a w-hyponormal operator, then $|T^2| \geqslant |T|^2$ and $|T^{*^2}| \leqslant |T^*|^2$.*

Proof. Let $\widetilde{T} = V|\widetilde{T}|$ be the polar decomposition of $\widetilde{T}$. Then

$$V^*|\widetilde{T}^*|V = |\widetilde{T}| \geqslant |T| \tag{4}$$

Thus,

$$\begin{aligned}
|T^2| &= (T^{2^*}T^2)^{1/2} \\
&= (T^*|T|^2T)^{1/2} \\
&= (|T|U^*|T|^2U|T|)^{1/2} \\
&= (|T|^{1/2}\widetilde{T}^*\,|T|\,\widetilde{T}\,|T|^{1/2})^{1/2} \\
&\geqslant (|T|^{1/2}\widetilde{T}^*\,|\widetilde{T}^*|\,\widetilde{T}\,|T|^{1/2})^{1/2} && \text{since } |T| \geqslant |\widetilde{T}^*| \\
&= (|T|^{1/2}|\widetilde{T}|V^*|\widetilde{T}^*|V|\widetilde{T}||T|^{1/2})^{1/2} \\
&\geqslant (|T|^{1/2}|\widetilde{T}||T||\widetilde{T}||T|^{1/2})^{1/2} && \text{by (4)} \\
&= |T|^{1/2}|\widetilde{T}||T|^{1/2} \\
&\geqslant |T|^2,
\end{aligned}$$

where the last inequality follows since $|\widetilde{T}| \geqslant |T|$. On the other hand,

$$\begin{aligned}
|T^{*^2}| &= (T^2T^{*^2})^{1/2} \\
&= (T|T^*|^2T^*)^{1/2} \\
&= (U|T|U|T|^2U^*|T|U^*)^{1/2} \\
&= U(|T|^{1/2}\widetilde{T}\,|T|\,\widetilde{T}^*\,|T|^{1/2})^{1/2}U^* \\
&\leqslant U(|T|^{1/2}\widetilde{T}V^*|\widetilde{T}^*|V\widetilde{T}^*\,|T|^{1/2})^{1/2}U^* && \text{by (4)} \\
&\leqslant U(|T|^{1/2}|\widetilde{T}^*||T||\widetilde{T}^*||T|^{1/2})^{1/2}U^* && \text{since } |\widetilde{T}^*| \leqslant |T| \\
&= U|T|^{1/2}|\widetilde{T}^*||T|^{1/2}U^* \\
&\leqslant U|T|^2U^* \\
&= |T^*|^2,
\end{aligned}$$

where the last inequality follows since $|\widetilde{T}^*| \leqslant |T|$.

COROLLARY 1. *If T is w-hyponormal, then T is paranormal.*

Proof. The result follows from Theorems 3 and 4.

COROLLARY 2. *If T is w-hyponormal, then T satisfies the inequality*

$$|T^2| - 2\lambda|T| + \lambda^2 \geqslant 0$$

for all $\lambda \geqslant 0$.

3. Powers of p-hyponormal and log-hyponormal operators

It is known [7] that powers of paranormal operators are paranormal. However, this is not true for p-hyponormal operators. Indeed, Halmos [8; Problem 164] gave an example of a hyponormal operator A for which A^2 is not hyponormal. The operator A^2 of Halmos' example is seen to be semi-hyponormal as the next corollary of Theorem 4 shows.

COROLLARY 3. *If T is p-hyponormal with $0 < p \leqslant 1$, then T^{2^n} is $p/2^n$-hyponormal for any positive integer n.*

Proof. Since T is p-hyponormal, $|T|^{2p} \geqslant |T^*|^{2p}$. Theorem 4 and the Löwner-Heinz inequality imply

$$|T^2|^p \geqslant |T|^{2p} \geqslant |T^*|^{2p} \geqslant |T^{*^2}|^p = |T^{2^*}|^p,$$

and hence T^2 is $p/2$-hyponormal. Retracing the steps of the proof with T^2 in place of T and $p/2$ in place of p, we conclude that T^4 is $p/4$-hyponormal. The result follows by induction.

In fact, a result stronger than that of Corollary 3 is known. Using a different approach, it was shown in [3] that if the operator T is p-hyponormal with $0 < p \leqslant 1$, then for any positive integer n, the operator T^n is p/n-hyponormal. As for log-hyponormal operators we have the following

COROLLARY 4. *If T is log-hyponormal, then T^{2^n} is log-hyponormal for any positive integer n.*

Proof. Since $\log|T| \geqslant \log|T^*|$, it follows that

$$\log|T|^2 = 2\log|T| \geqslant 2\log|T^*| = \log|T^*|^2.$$

Hence

$$\log|T^2| \geqslant \log|T|^2 \geqslant \log|T^*|^2 \geqslant \log|T^{*^2}| = \log|T^{2^*}|$$

by Theorem 4. Again, the result follows by induction.

4. Normality

In this section we utilize inequalities (1) and (2) to give conditions under which a w-hyponormal operator becomes normal.

THEOREM 5. *If the operator T satisfies $|T^2| = |T|^2$ and $\ker T^* \subseteq \ker T$, then T is normal. Similarly, if $|T^{*^2}| = |T^*|^2$ and $\ker T \subseteq \ker T^*$, then T is normal.*

Proof. Since $|T^2| = |T|^2$, squaring both sides of the equality yields

$$T^*|T|^2T = T^*|T^*|^2T. \tag{5}$$

Denote the closure of the range of an operator A by $\operatorname{ran} A$. If $\ker T^* \subseteq \ker T$, then $\ker T^*$ is invariant under $|T|^2$. Therefore, both $\ker T^*$ and $\operatorname{ran} T = (\ker T^*)^{\perp}$ are reducing subspaces for $|T|^2$. Moreover, it is obvious that both $\ker T^*$ and $\operatorname{ran} T$ are reducing subspaces for $|T^*|^2$. Since $|T|^2 = |T^*|^2 = 0$ on $\ker T^*$ and $|T|^2 = |T^*|^2$ on $\operatorname{ran} T$ by equality (5), $|T|^2 = |T^*|^2$ on $H = \ker T^* \oplus \operatorname{ran} T$. Consequently, T is normal. The proof of the first part of the theorem is completed. With T^* in place of T, the second part of the theorem follows from the first part.

If the operator T is p-hyponormal, it follows that $\ker T \subseteq \ker T^*$. Furthermore, if both T and its adjoint T^* are p-hyponormal, then $\ker T = \ker T^*$ and T is normal. In light of these, Ando [4; Theorem 5] obtained the following generalization.

THEOREM C. *If the operator T and its adjoint T^* are paranormal, then T is normal if $\ker T = \ker T^*$.*

As an application of Theorems 4 and 5, the next corollary shows that the kernel condition in Theorem C may be relaxed if the paranormality assumption is strengthened to that of w-hyponormality.

COROLLARY 1. *If the operator T and its adjoint T^* are w-hyponormal, then T is normal if either $\ker T \subseteq \ker T^*$ or $\ker T^* \subseteq \ker T$.*

Proof. Since T and T^* are w-hyponormal, it follows from Theorem 4 that $|T^2| = |T|^2$ and $|T^{*^2}| = |T^*|^2$. The result follows from Theorem 5.

COROLLARY 2. *If the operator T is either p-hyponormal or log-hyponormal and its adjoint T^* is w-hyponormal, then T is normal.*

Proof. Since $\ker T \subseteq \ker T^*$ if T is p-hyponormal, and $\ker T = \ker T^*$ if T is log-hyponormal, it follows from the above corollary that T is normal.

Generalizing a previously known result for p-hyponormal operators, it was shown in [2] that a w-hyponormal operator T with $\ker T \subseteq \ker T^*$ is normal if the associated $\widetilde{T}$ is normal. The following corollary provides a further generalization.

COROLLARY 3. *Let the operator T be w-hyponormal. If $\widetilde{T}$ is normal and either $\ker T \subseteq \ker T^*$ or $\ker T^* \subseteq \ker T$, then T is normal and $T = \widetilde{T}$.*

Proof. Since T is w-hyponormal and $\widetilde{T}$ is normal, we have

$$|\widetilde{T}| = |T| = |\widetilde{T}^*|.$$

Thus, all the inequalities which appear in the proof of Theorem 4 become equalities. Hence, $|T^2| = |T|^2$ and $|T^{*^2}| = |T^*|^2$. Therefore, T is normal by Theorem 5. Let $T = U|T|$ be the polar decomposition of T. Since T is normal, U commutes with $|T|^{1/2}$, and hence $\widetilde{T} = |T|^{1/2}U|T|^{1/2} = U|T| = T$.

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