Mathematical
Inequalities
& Applications
Volume 2, Number 1 (1999), 121–134

STUDY OF GENERALIZED QUASI–VARIATIONAL INEQUALITIES FOR LOWER AND UPPER HEMI–CONTINUOUS OPERATORS ON NON–COMPACT SETS

MOHAMMAD S. R. CHOWDHURY AND KOK-KEONG TAN *

(*communicated by J. Pečarić*)

Abstract. Results are obtained on existence theorems of generalized quasi-variational inequalities with monotone and lower hemi-continuous operators, or semi-monotone and upper hemi-continuous operators on paracompact sets.

1. Preliminaries

If X is a non-empty set, we shall denote by 2^X the family of all non-empty subsets of X.

Let X be a non-empty subset of a topological vector space E. Given the maps $S : X \to 2^X$ and $T : X \to E^*$, the quasi-variational inequality problem (QVI) is to find a point $\hat{y} \in S(\hat{y})$ such that $Re\langle T(\hat{y}), \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$. The QVI was introduced by Bensoussan and Lions in 1973 (see, e.g., [3]) in connection with impulse control. Again, if we consider a set-valued map $T : X \to 2^{E^*}$, then the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$. The GQVI was introduced by Chan and Pang [4] in 1982 if $E = \mathbb{R}^n$ and by Shih and Tan [8] in 1985 if E is infinite dimensional.

If X is a topological space and $\{U_\alpha : \alpha \in \mathcal{A}\}$ is an open cover for X, then a partition of unity subordinated to the open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ is a family $\{\beta_\alpha : \alpha \in \mathcal{A}\}$ of continuous real-valued functions $\beta_\alpha : X \to [0, 1]$ such that

(1) $\beta_\alpha(y) = 0$ for all $y \in X \setminus U_\alpha$,

(2) $\{$ support $\beta_\alpha : \alpha \in \mathcal{A}\}$ is locally finite and

(3) $\Sigma_{\alpha\in\mathcal{A}}\beta_\alpha(y) = 1$ for each $y \in X$.

The following result is Lemma 1 of Shih and Tan in [8, pp.334-335]:

Mathematics subject classification (1991): 47H04, 47H05, 47H09, 47H10, 49J35, 49J40, 54C60.

Key words and phrases: Generalized quasi-variational inequality, locally convex space, lower semi-continuous, upper semi-continuous, lower hemi-continuous, upper hemi-continuous, monotone and semi-monotone operators.

* The author was partially supported by NSERC of Canada under Grant A-8096.

Paper MIA-02-10

LEMMA A. *Let X be a non-empty subset of a Hausdorff topological vector space E and $S : X \to 2^E$ be an upper semicontinuous map such that $S(x)$ is a bounded subset of E for each $x \in X$. Then for each continuous linear functional p on E, the map $f_p : X \to \mathbb{R}$ defined by $f_p(y) = \sup_{x \in S(y)} Re\langle p, x\rangle$ is upper semicontinuous; i.e., for each $\lambda \in \mathbb{R}$, the set $\{y \in X : f_p(y) = \sup_{x\in S(y)} Re\langle p,x\rangle < \lambda\}$ is open in X.*

The following result is Lemma 3 of Takahashi in [10, p.177] (see also Lemma 3 in [9, pp.71-72]:

LEMMA B. *Let X and Y be topological spaces, $f : X \to \mathbb{R}$ be non-negative and continuous and $g : Y \to \mathbb{R}$ be lower semicontinuous. Then the map $F : X \times Y \to \mathbb{R}$, defined by $F(x,y) = f(x)g(y)$ for all $(x,y) \in X \times Y$, is lower semicontinuous.*

The following result is essentially Theorem 1 Bae-Kim-Tan in [2, p.231]:

THEOREM A. *Let E be a topological vector space, X be a non-empty convex subset of E and $f, g : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(a) $g(x,x) \leqslant 0$ for all $x \in X$ and $f(x,y) \leqslant g(x,y)$ for all $x, y \in X$;

(b) for each fixed $x \in X$, $y \longmapsto f(x,y)$ is lower semicontinuous on non-empty compact subsets of X;

(c) for each fixed $y \in X$, the set $\{x \in X : g(x,y) > 0\}$ is convex;

(d) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $f(x,y) > 0$.

Then there exists $\hat{y} \in K$ such that $f(x,\hat{y}) \leqslant 0$ for all $x \in X$.

We shall need the following Kneser's minimax theorem in [7, pp.2418-2420] (see also Aubin [1, pp.40-41]):

THEOREM B. *Let X be a non-empty convex subset of a vector space and Y be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \mapsto f(x,y)$ is lower semicontinuous and convex on Y and for each fixed $y \in Y$, the map $x \mapsto f(x,y)$ is concave on X. Then*

$$\min_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \min_{y\in Y} f(x,y).$$

2. Generalized quasi-variational inequalities for lower hemi-continuous operators on non-compact sets.

In this section we shall obtain some existence theorems for generalized quasi-variational inequalities for monotone and lower hemi-continuous operators on para-compact sets.

The following definition is Definition 2.1(a) in [5, pp.28-29]:

DEFINITION 1. Let E be a topological vector space, X be a non-empty subset of E and $T : X \to 2^{E^*}$. Then T is said to be *lower hemi-continuous* on X if and only if for each $p \in E$, the function $f_p : X \to \mathbb{R} \cup \{+\infty\}$, defined by

$$f_p(z) = \sup_{u \in T(z)} Re\langle u, p\rangle \text{ for each } z \in X,$$

is lower semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \to \mathbb{R} \cup \{-\infty\}$, defined by

$$g_p(z) = \inf_{u \in T(z)} Re\langle u, p\rangle \text{ for each } z \in X,$$

is upper semicontinuous on X);

The following proposition is Proposition 2.2 in [5, p.29]:

PROPOSITION 1. *Let E be a topological vector space and X be a non-empty subset of E. Let $T : X \to 2^{E^*}$ be lower semicontinuous from relative topology on X to the weak* topology $\sigma\langle E^*, E\rangle$ on E^*. Then T is lower hemi-continuous on X.*

The following result is Lemma 4.1 in [5, pp.37-38]:

LEMMA 1. *Let E be a topological vector space, X be a non-empty convex subset of E, $h : X \to \mathbb{R}$ be convex and $T : X \to 2^{E^*}$ be lower hemi-continuous along line segments in X. Suppose $\hat{y} \in X$ is such that $\sup_{u \in T(x)} Re\langle u, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y}) \textit{ for all } x \in X.$$

We shall first establish the following result:

THEOREM 1. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex subset of E. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be monotone and be lower hemi-continuous along line segments in X to the weak*-topology on E^*. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} \sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0\}$$

is open in X. Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in co(X_0 \cup \{y\}) \cap S(y)$ with $\sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. We shall prove this theorem in two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\sup_{u \in T(x)} Re\langle u, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leqslant 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exist $x \in S(y)$ and $u \in T(x)$ such that $Re\langle u, y-x\rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $Re\langle p, y\rangle - \sup_{x\in S(y)} Re\langle p, x\rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x\in S(y)} [\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : Re\langle p, y\rangle - \sup_{x\in S(y)} Re\langle p, x\rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p\in E^*} V_p$. Since each V_p is open in X by Lemma A and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see, e.g., Theorem VIII.4.2 of Dugundji in [6]); that is for each $p \in E^*$, $\beta_p : X \to [0,1]$ and $\beta_0 : X \to [0,1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and $\{$ support β_0, support $\beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \Sigma_{p\in E^*}\beta_p(y) = 1$ for each $y \in X$. Define $\phi, \psi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \beta_0(y)[\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] + \Sigma_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle,$$

and

$$\psi(x,y) = \beta_0(y)[\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] + \Sigma_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle,$$

for each $x, y \in X$. Then we have the following.

(1) For each $x, y \in X$, since T is monotone, $\phi(x,y) \leqslant \psi(x,y)$ and $\psi(x,x) = 0$ for all $x \in X$.

(2) For each fixed $x \in X$ and each fixed $u \in T(x)$, the map

$$y \longmapsto Re\langle u, y-x\rangle + h(y) - h(x)$$

is continuous on X and therefore the map

$$y \longmapsto \beta_0(y)[\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)]$$

is lower semicontinuous on X by Lemma B. Also for each fixed $x \in X$,

$$y \mapsto \Sigma_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle$$

is continuous on X. Hence, for each fixed $x \in X$, the map $y \mapsto \phi(x,y)$ is lower semicontinuous on X.

(3) Clearly, for each $y \in X$, the set $\{x \in X : \psi(x,y) > 0\}$ is convex.

(4) By hypothesis, there exists a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in co(X_0 \cup \{y\}) \cap S(y)$ such that $\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x) > 0$.

Thus $\beta_0(y)[\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y)-h(x)] > 0$ whenever $\beta_0(y) > 0$. Also $Re\langle p, y-x\rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, $\phi(x,y) = \beta_0(y)[\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y)-h(x)] + \Sigma_{p\in E^*}\beta_p(y)Re\ \langle p,\ y-x\rangle > 0$.

Then ϕ and ψ satisfy all the hypotheses of Theorem A. Thus by Theorem A, there exists $\hat{y}\in K$ such that $\phi(x,\hat{y}) \leqslant 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y})[\sup_{u\in T(x)} Re\langle u, \hat{y}-x\rangle + h(\hat{y})-h(x)] + \Sigma_{p\in E^*}\beta_p(\hat{y})Re\langle p, \hat{y}-x\rangle \leqslant 0 \tag{2.1}$$

for all $x\in X$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y}\in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$. Choose $\hat{x}\in S(\hat{y}) \subset X$ such that

$$\sup_{u\in T(\hat{x})} Re\langle u, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x}) \geqslant \frac{\gamma(\hat{y})}{2} > 0;$$

it follows that

$$\beta_0(\hat{y})[\sup_{u\in T(\hat{x})} Re\langle u, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x})] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p\in E^*$, then $\hat{y}\in V_p$ and hence

$$Re\langle p, \hat{y}\rangle > \sup_{x\in S(\hat{y})} Re\langle p, x\rangle \geqslant Re\langle p, \hat{x}\rangle$$

so that $Re\langle p, \hat{y}-\hat{x}\rangle > 0$. Then note that

$$\beta_p(\hat{y})Re\langle p, \hat{y}-\hat{x}\rangle > 0 \quad \text{whenever} \quad \beta_p(\hat{y}) > 0 \text{ for } p\in E^*.$$

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p\in E^*$, it follows that

$$\phi(\hat{x},\hat{y}) = \beta_0(\hat{y})[\sup_{u\in T(\hat{x})} Re\langle u, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x})] + \Sigma_{p\in E^*}\beta_p(\hat{y})Re\langle p, \hat{y}-\hat{x}\rangle > 0,$$

which contradicts (2.1). This contradiction proves Step 1.

Step 2.

$$\sup_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \text{ for all } x\in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y}\in S(\hat{y})$ which is a convex subset of X, and

$$\sup_{u\in T(x)} Re\langle u, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all} \quad x\in S(\hat{y}).$$

Hence by Lemma 1, we have

$$\sup_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all} \quad x\in S(\hat{y}). \quad \Box$$

If X is compact, Theorem 1 reduces to the following:

THEOREM 2. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be monotone and be lower hemi-continuous along line segments in X to the weak*-topology on E^*. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} \sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0\}$$

is open in X. Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

REMARK 1. Theorem 1 and Theorem 2 generalize Theorem 1 of Shih-Tan in [8, p.335].

Note that if X is also bounded in Theorem 1 and the map $S : X \to 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)}[\sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0\}$, T is lower semicontinuous at some point x in $S(y)$ with $\sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$, then the set Σ in Theorem 1 is always open in X. Thus we obtain the following result:

THEOREM 3. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex and bounded subset of E. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be monotone and be lower hemi-continuous along line segments in X to the weak*-topology on E^*. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)}[\sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0\}$, T is lower semicontinuous at some point x in $S(y)$ with $\sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$. Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in co(X_0 \cup \{y\}) \cap S(y)$ with $\sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in K$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. Note that Theorem 3 follows from Theorem 1 if we show that the set

$$\Sigma := \{y \in X : \sup_{x \in S(y)} [\sup_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0\}$$

is open in X. Indeed, let $y_0 \in \Sigma$; then by hypothesis, T is lower semicontinuous at some point x_0 in $S(y_0)$ with $\sup_{u \in T(x_0)} Re\langle u, y_0 - x_0\rangle + h(y_0) - h(x_0) > 0$. Hence there exists $u_0 \in T(x_0)$ such that $Re\langle u_0, y_0 - x_0\rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := Re\langle u_0, y_0 - x_0\rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$U_1 := \{u \in E^* : \sup_{z_1, z_2 \in X} |\langle u - u_0, z_1 - z_2\rangle| < \frac{\alpha}{6}\}.$$

Then U_1 is a strongly open neighborhood of u_0 in E^*. Since T is lower semicontinuous at x_0 and $U_1 \cap T(x_0) \neq \emptyset$, there exists an open neighborhood V_1 of x_0 in X such that $T(x) \cap U_1 \neq \emptyset$ for all $x \in V_1$.

As the map $x \longmapsto Re\langle u_0, x_0 - x\rangle + h(x_0) - h(x)$ is continuous at x_0, there exists an open neighborhood V_2 of x_0 in X such that

$$|Re\langle u_0, x_0 - x\rangle + h(x_0) - h(x)| < \frac{\alpha}{6} \quad \text{for all} \quad x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X. Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0, there exists an open neighborhood N_1 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_1$.

Since the map $y \longmapsto Re\langle u_0, y - y_0\rangle + h(y) - h(y_0)$ is continuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that

$$|Re\langle u_0, y - y_0\rangle + h(y) - h(y_0)| < \frac{\alpha}{6} \quad \text{for all} \quad y \in N_2.$$

Let $N_0 := N_1 \cap N_2$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

(i) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_1$; so we can choose any $x_1 \in S(y_1) \cap V_0$;

(ii) $|Re\langle u_0, y_1 - y_0\rangle + h(y_1) - h(y_0)| < \frac{\alpha}{6}$ as $y_1 \in N_2$;

(iii) $T(x_1) \cap U_1 \neq \emptyset$ as $x_1 \in V_1$; choose any $u_1 \in T(x_1) \cap U_1$ so that

$$\sup_{z_1, z_2 \in X} |\langle u_1 - u_0, z_1 - z_2\rangle| < \frac{\alpha}{6};$$

(iv) $|Re\langle u_0, x_0 - x_1\rangle + h(x_0) - h(x_1)| < \frac{\alpha}{6}$ as $x_1 \in V_2$.

It follows that

$$\begin{aligned}
&Re\langle u_1, y_1 - x_1\rangle + h(y_1) - h(x_1) \\
&= Re\langle u_1 - u_0, y_1 - x_1\rangle + Re\langle u_0, y_1 - x_1\rangle + h(y_1) - h(x_1) \\
&\geqslant -\frac{\alpha}{6} + Re\langle u_0, y_1 - y_0\rangle + h(y_1) - h(y_0) \\
&+ Re\langle u_0, y_0 - x_0\rangle + h(y_0) - h(x_0) \\
&+ Re\langle u_0, x_0 - x_1\rangle + h(x_0) - h(x_1) (\text{ by } (iii)), \\
&\geqslant -\frac{\alpha}{6} - \frac{\alpha}{6} + \alpha - \frac{\alpha}{6} = \frac{\alpha}{2} > 0 \;\; (\text{ by } (ii) \text{ and } (iv));
\end{aligned}$$

therefore

$$\sup_{x \in S(y_1)} [\sup_{u \in T(x)} Re\langle u, y_1 - x\rangle + h(y_1) - h(x)] > 0$$

as $x_1 \in S(y_1)$ and $u_1 \in T(x_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X. This completes the proof. $\square$

If X is compact, Theorem 3 reduces to the following:

THEOREM 4. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be monotone and be lower hemi-continuous along line segments in X to the weak*-topology on E^*. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x\in S(y)}[\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$, T is lower semicontinuous at some point x in $S(y)$ with $\sup_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ *and*

(ii) $\sup_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y})$ *for all* $x \in S(\hat{y})$.

REMARK 2. Theorem 3 and Theorem 4 generalize Theorem 2 of Shih-Tan in [8, p.338].

3. Generalized quasi-variational inequalities for upper hemi-continuous operators on non-compact sets.

In this section we shall obtain some existence theorems for generalized quasi-variational inequalities for semi-monotone and upper hemi-continuous operators on paracompact sets.

The following definition is Definition 2.1(b) in [5, pp.28-29]:

DEFINITION 2. Let E be a topological vector space, X be a non-empty subset of E and $T : X \to 2^{E^*}$. Then T is said to be *upper hemi-continuous* on X if and only if for each $p \in E$, the function $f_p : X \to \mathbb{R} \cup \{+\infty\}$, defined by

$$f_p(z) = \sup_{u\in T(z)} Re\langle u, p\rangle \text{ for each } z \in X,$$

is upper semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \to \mathbb{R} \cup \{-\infty\}$, defined by

$$g_p(z) = \inf_{u\in T(z)} Re\langle u, p\rangle \text{ for each } z \in X,$$

is lower semicontinuous on X).

The following proposition is Proposition 2.4 in [5, p.30]:

PROPOSITION 2. *Let E be a topological vector space and X be a non-empty subset of E. Let $T : X \to 2^{E^*}$ be upper semicontinuous from relative topology on X to the weak* topology $\sigma\langle E^*, E\rangle$ on E^*. Then T is upper hemi-continuous on X.*

Note that there is a typo in Proposition 2.4 in [5, p.30]. The set X is not required to be convex.

The following simple result is Lemma 2.1.6 in [11]:

LEMMA 2. *Let E be a topological vector space and A be a non-empty bounded subset of E. Let C be a non-empty strongly compact subset of E^*. Define $f : A \to \mathbb{R}$ by $f(x) = \min_{u \in C} Re\langle u, x\rangle$ for all $x \in A$. Then f is weakly continuous on A.*

The following result is Lemma 4.2 in [5, p.38]:

LEMMA 3. *Let E be a topological vector space, X be a non-empty convex subset of E, $h : X \to \mathbb{R}$ be convex and $T : X \to 2^{E^*}$ be upper hemi-continuous along line segments in X. Suppose $\hat{y} \in X$ is such that $\inf_{u \in T(x)} Re\langle u, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y}) \text{ for all } x \in X.$$

We shall now establish the following result:

THEOREM 5. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex and bounded subset of E. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be semi-monotone and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0\}$$

is open in X. Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in co(X_0 \cup \{y\}) \cap S(y)$ with $\inf_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. We shall prove this theorem in three steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{u \in T(x)} Re\langle u, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leqslant 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x \in S(y)} [\inf_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p\in E^*} V_p$. Since each V_p is open in X by Lemma A and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$. Define $\phi, \psi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \beta_0(y)[\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] + \Sigma_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle ,$$

and

$$\psi(x,y) = \beta_0(y)[\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] + \Sigma_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle$$

for each $x, y \in X$. Then we have the following.

(1) For each $x, y \in X$, since T is semi-monotone, $\phi(x,y) \leqslant \psi(x,y)$ and $\psi(x,x) = 0$ for all $x \in X$.

(2) For each fixed $x \in X$, the map

$$y \longmapsto \inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)$$

is weakly lower semicontinuous (and therefore lower semi-continuous) on X by Lemma 2 and the fact that h is continuous; therefore the map

$$y \longmapsto \beta_0(y)[\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)]$$

is lower semicontinuous on X by Lemma B. Also for each fixed $x \in X$,

$$y \mapsto \Sigma_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle$$

is continuous on X. Hence, for each fixed $x \in X$, the map $y \mapsto \phi(x,y)$ is lower semicontinuous on X.

(3) Clearly, for each $y \in X$, the set $\{x \in X : \psi(x,y) > 0\}$ is convex.

(4) By hypothesis, there exists a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in co(X_0 \cup \{y\}) \cap S(y)$ such that $\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x) > 0$. Thus $\beta_0(y)[\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0$ whenever $\beta_0(y) > 0$. Also $Re\langle p, y-x\rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, $\phi(x,y) = \beta_0(y)[\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] + \Sigma_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle > 0$.

Then ϕ and ψ satisfy all the hypotheses of Theorem A. Thus by Theorem A, there exists $\hat{y} \in K$ such that $\phi(x,\hat{y}) \leqslant 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y})[\inf_{u\in T(x)} Re\langle u, \hat{y}-x\rangle + h(\hat{y}) - h(x)] + \Sigma_{p\in E^*}\beta_p(\hat{y}) Re\langle p, \hat{y}-x\rangle \leqslant 0 \tag{3.1}$$

for all $x \in X$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$. Choose $\hat{x} \in S(\hat{y}) \subset X$ such that

$$\inf_{u\in T(\hat{x})} Re\langle u, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x}) \geqslant \frac{\gamma(\hat{y})}{2} > 0;$$

it follows that

$$\beta_0(\hat{y})[\inf_{u\in T(\hat{x})} Re\langle u, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x})] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$Re\langle p, \hat{y}\rangle > \sup_{x\in S(\hat{y})} Re\langle p, x\rangle \geqslant Re\langle p, \hat{x}\rangle$$

so that $Re\langle p, \hat{y}-\hat{x}\rangle > 0$. Then note that

$$\beta_p(\hat{y}) Re\langle p, \hat{y}-\hat{x}\rangle > 0 \quad \text{whenever} \quad \beta_p(\hat{y}) > 0 \text{ for } p \in E^*.$$

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x},\hat{y}) = \beta_0(\hat{y})[\inf_{u\in T(\hat{x})} Re\langle u, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x})] + \Sigma_{p\in E^*}\beta_p(\hat{y}) Re\langle p, \hat{y}-\hat{x}\rangle > 0,$$

which contradicts (3.1). This contradiction proves Step 1.

Step 2.

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y} \in S(\hat{y})$ which is a convex subset of X, and

$$\inf_{u\in T(x)} Re\langle u, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}).$$

Hence by Lemma 3, we have

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}). \tag{3.2}$$

Step 3. There exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Indeed, from Step 2 we have

$$\sup_{x\in S(\hat{y})} [\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)] \leqslant 0, \tag{3.3}$$

where $T(\hat{y})$ is a strongly compact convex subset of the Hausdorff topological vector space E^* and $S(\hat{y})$ is a convex subset of X.

Now, define $f : S(\hat{y}) \times T(\hat{y}) \to \mathbb{R}$ by $f(x,w) = Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)$ for each $x \in S(\hat{y})$ and each $w \in T(\hat{y})$. Note that for each fixed $x \in S(\hat{y})$, the map $w \longmapsto f(x,w)$ is convex and continuous on $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, the map $x \longmapsto f(x,w)$ is concave on $S(\hat{y})$. Thus by Theorem B, we have

$$\min_{w\in T(\hat{y})} \sup_{x\in S(\hat{y})} [Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)] = \sup_{x\in S(\hat{y})} \min_{w\in T(\hat{y})} [Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)].$$

Hence

$$\min_{w\in T(\hat{y})} \sup_{x\in S(\hat{y})} [Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)] \leqslant 0, \quad \text{by (3.3)}.$$

Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}). \quad \square$$

If X is compact, Theorem 5 reduces to the following:

THEOREM 6. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be semi-monotone and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x\in S(y)} [\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$$

is open in X. Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x)-h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if the map $S : X \to 2^X$ is, in addition lower semicontinuous, and for each $y \in \Sigma$, T is upper semicontinuous at some point x in $S(y)$ with $\inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0$, then the set Σ in Theorem 5 is always open in X. Thus we obtain the following result:

THEOREM 7. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex and bounded subset of E. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \to 2^F$ be semi-monotone and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x\in S(y)}[\inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0\}$, T is upper semi-continuous at some point x in $S(y)$ with $\inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$. Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in co(X_0 \cup \{y\}) \cap S(y)$ with $\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x) > 0$. Then there exists $\hat{y} \in K$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x)-h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. Note that Theorem 7 follows from Theorem 5 if we show that the set

$$\Sigma = \{y \in X : \sup_{x\in S(y)} [\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$$

is open in X. Indeed, let $y_0 \in \Sigma$; then by hypothesis, T is upper semicontinuous at some point x_0 in $S(y_0)$ with $\inf_{u\in T(x_0)} Re\langle u, y_0 - x_0\rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := \inf_{u\in T(x_0)} Re\langle u, y_0 - x_0\rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$W := \{w \in E^* : \sup_{z_1,z_2\in X} |\langle w, z_1 - z_2\rangle| < \alpha/6\}.$$

Then W is a strongly open neighborhood of 0 in E^* so that $U_1 := T(x_0) + W$ is an open neighborhood of $T(x_0)$ in E^*. Since T is upper semicontinuous at x_0, there exists an open neighborhood V_1 of x_0 in X such that $T(x) \subset U_1$ for all $x \in V_1$.

As the map $x \longmapsto \inf_{u\in T(x_0)} Re\langle u, x_0 - x\rangle + h(x_0) - h(x)$ is continuous at x_0, there exists an open neighborhood V_2 of x_0 in X such that

$$|\inf_{u\in T(x_0)} Re\langle u, x_0 - x\rangle + h(x_0) - h(x)| < \alpha/6 \quad \text{for all} \quad x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X. Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0, there exists an open neighborhood N_1 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_1$.

Since the map $y \longmapsto \inf_{u\in T(x_0)} Re\langle u, y - y_0\rangle + h(y) - h(y_0)$ is continuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that

$$|\inf_{u\in T(x_0)} Re\langle u, y - y_0\rangle + h(y) - h(y_0)| < \alpha/6 \quad \text{for all} \quad y \in N_2.$$

Let $N_0 := N_1 \cap N_2$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

(i) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_1$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
(ii) $|\inf_{u\in T(x_0)} Re\langle u, y_1 - y_0\rangle + h(y_1) - h(y_0)| < \alpha/6$ as $y_1 \in N_2$;
(iii) $T(x_1) \subset U_1 = T(x_0) + W$ as $x_1 \in V_1$;
(iv) $|\inf_{u\in T(x_0)} Re\langle u, x_0 - x_1\rangle + h(x_0) - h(x_1)| < \alpha/6$ as $x_1 \in V_2$.

It follows that

$$\begin{aligned}
&\inf_{u\in T(x_1)} Re\langle u, y_1 - x_1\rangle + h(y_1) - h(x_1) \\
&\geqslant \inf_{[u\in T(x_0)+W]} Re\langle u, y_1 - x_1\rangle + h(y_1) - h(x_1) \qquad (\text{ by } (iii)), \\
&\geqslant \inf_{u\in T(x_0)} Re\langle u, y_1 - x_1\rangle + h(y_1) - h(x_1) + \inf_{u\in W} Re\langle u, y_1 - x_1\rangle \\
&\geqslant \inf_{u\in T(x_0)} Re\langle u, y_1 - y_0\rangle + h(y_1) - h(y_0) \\
&+ \inf_{u\in T(x_0)} Re\langle u, y_0 - x_0\rangle + h(y_0) - h(x_0) \\
&+ \inf_{u\in T(x_0)} Re\langle u, x_0 - x_1\rangle + h(x_0) - h(x_1) + \inf_{u\in W} Re\langle u, y_1 - x_1\rangle \\
&\geqslant -\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{2} > 0 \qquad (\text{ by } (ii) \text{ and } (iv));
\end{aligned}$$

therefore

$$\sup_{x\in S(y_1)} [\inf_{u\in T(x)} Re\langle u, y_1 - x\rangle + h(y_1) - h(x)] > 0$$

as $x_1 \in S(y_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X. This completes the proof. □

If X is compact, Theorem 7 reduces to the following:

THEOREM 8. *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be semi-monotone and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \to \mathbb{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)}[\inf_{u \in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$, T is upper semi-continuous at some point x in $S(y)$ with $\inf_{u \in T(x)} Re\langle u, y-x\rangle + h(y) - h(x) > 0$. Then there exists $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

REFERENCES

[1] J. P. AUBIN, *Applied Functional Analysis*, Wiley-Interscience, New York, 1979.

[2] J. S. BAE, W. K. KIM AND K.-K. TAN, *Another generalization of Ky Fan's minimax inequality and its applications*, Bull. Inst. Math. Academia Sinica **21** (1993), 229-244.

[3] A. BENSOUSSAN AND J. L. LIONS, *Nouvelle formulation des problèmes de contrôle impulsionnel et applications*, C. R. Acad. Sci. **29** (1973), 1189-1192.

[4] D. CHAN AND J. S. PANG, *The generalized quasi-variational inequality problem*, Math. Oper. Res. **7** (1982), 211-222.

[5] M. S. R. CHOWDHURY AND K.-K. TAN, *Generalized variational inequalities for quasi-monotone operators and applications*, Bulletin of the Polish Academy of Sciences **45, No. 1** (1997), 25-54.

[6] J. DUGUNDJI, *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[7] H. KNESER, *Sur un thèorème fondamental de la thèorie des jeux*, C. R. Acad. Sci. Paris **234** (1952), 2418-2420.

[8] M.-H. SHIH AND K.-K. TAN, *Generalized quasi-variational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl. **108** (1985), 333-343.

[9] M.-H. SHIH AND K.-K. TAN, *Generalized bi-quasi-variational inequalities*, J. Math. Anal. Appl. **143** (1989), 66-85.

[10] W. TAKAHASHI, *Nonlinear variational inequalities and fixed point theorems*, J. Math. Soc. Japan **28** (1976), 166-181.

[11] K.-K. TAN, *Lecture Notes on Topics in Topology and Functional Analysis*, unpublished, 1985, 1991 and 1994.

(Received June 26, 1998)

Mohammad S. R. Chowdhury
Kok-Keong Tan

Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5