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INEQUALITIES FOR BRACHISTOCHRONE

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Abstract. Let $A=(0,1)$ and $B=(b,0)$ be the initial and final points, and $y=y(x)$ joins A with AB, $y''\geqslant 0$, $0\leqslant y(x)\leqslant y_0(x)$, where $y_0(x)$ is the straight line joining A and B. Denote by S the set of such $y(x)$. Let P be the polygon consisting of two segments of straight lines: $A0$ and OB, where $0=(0,0)$ is the origin. Up to a constant factor depending on the choice of units, the time needed for a particle to get from A to B along $y(x)$ in the gravitational field is $T(y)=\int_0^b \frac{\sqrt{1+y'^2}}{\sqrt{1-y}}\,dx$. Let $T_0:=T(y_0)=2\sqrt{1+b^2}$, $T_P:=T(P)=2+b$. It is conjectured that:

1) if $0<b<\frac{4}{3}$ then $T(y_{br})\leqslant T(y)<T_P$, $y\in S$,
2) if $\frac{4}{3}\leqslant b\leqslant\frac{\pi}{2}$ then $T(y_{br})\leqslant T(y)\leqslant T_0$, $y\in S$,
3) if $b>\frac{\pi}{2}$ then $T(P_{br})<T(y)\leqslant T_0$, $y\in S$

where $y_{br}=y_{br}(x)\in S$ is the classical brachistochrone curve. For $b>\frac{\pi}{2}$ this curve probably degenerates into P_{br}, the brachistochrone curve which joins A and $(\pi/2,0)$ and the straight line joining $(\pi/2,0)$ and $(b,0)$.

1. Introduction

The classical brachistochrone problem is:
Find a curve $y(x)$ which joins two points $A=(0,h)$ and $B=(b,0)$ such that

$$t(y):=\int_0^b \frac{\sqrt{1+y'^2}}{\sqrt{2g(h-y)}}\,dx=\min \tag{1.1}$$

Here $t(y)$ is the time needed for a particle moving along the curve $y=y(x)$ in the gravitational field (with the gravitational acceleration g) to go from A to B. Indeed, $dt=\frac{ds}{v}$, where $ds=\sqrt{1+y'^2}\,dx$ is the element of the length of the curve, and $v=\sqrt{2g(h-y)}$ is the velocity of the particle, which is calculated from the conservation of energy law: $\frac{mv^2}{2}+mgy=mgh$. In what follows, we take $h=1$ and choose the acceleration units in which $2g=1$. The possiblility to take $h=1$ without loss of

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generality, comes from the observation that the change of variables $\xi = \frac{x}{h}$, $\eta = \frac{y}{h}$ reduces the functional (1.1) to

$$\sqrt{h}\int_0^{\frac{b}{h}} \frac{\sqrt{1+(\eta'_\xi)^2}}{\sqrt{2g(1-\eta(\xi))}}\,d\xi.$$

So, the problem formulation, with the above remarks taken into account, is:

Find the minimum of the functional

$$T(y) := \int_0^b \frac{\sqrt{1+[y'(x)]^2}}{\sqrt{1-y(x)}}\,dx = \min \tag{1.2}$$

among all $y \in C^2(0,b)$*, such that*

$$y(0) = 1, \qquad y(b) = 0. \tag{1.3}$$

This problem was posed and solved by Johann Bernoulli in 1696 and by several other people, including G. Leibniz, Jacob Bernoulli and I. Newton at the same time. This problem is a cornerstone of the calculus of variations [1], [2]. The solution, $y = y(x)$, to (1.2)–(1.3) is called the brachistochrone curve. In the classical solution it was proved that the brachistochrone (which is understood in this discussion as an extremal of the functional (1.2)) can join any two points in the vertical plane. This was so because the Earth was assumed to be nonexistent: only the gravitational field existed. In our reformulation of the problem not any two points can be joined by an extremal of the functional (1.2) as shown in section 3 below, see Lemma 3.1. Although the brachistohrone problem is 300 years old, it is of interst to find out that it is still not quite well understood and to formulate some questions about brachistohrone the answers to which seem unknown.

In this note we reconsider problem (1.2)–(1.3) and ask the following question:

Suppose additionally to (1.3) that

$$y'' \geqslant 0, \qquad 0 \leqslant y(x) \leqslant y_0(x), \tag{1.4}$$

where $y_0(x)$ is the straight line joining A and B, that is

$$y_0(x) = 1 - \frac{x}{b}. \tag{1.5}$$

Let S be the set

$$S := \left\{ y(x) : \ (1.3) \text{ and } (1.4) \text{ hold}, \qquad y \in C^2(0,b) \right\}. \tag{1.6}$$

Is it true that

$$T(y) \leqslant T_0 := T(y_0) = 2\sqrt{1+b^2}, \qquad \forall y \in S? \tag{1.7}$$

We prove that the answer is no.

However, we conjecture that:
There exists a number $b_0 = \frac{4}{3}$ *such that:*

$$\text{if } b \geqslant b_0 \text{ then (1.7) is true,} \tag{1.8}$$

$$\text{if } b < b_0 \text{ then (1.7) is false.} \tag{1.9}$$

In addition, we give a (hopefully) new method for integration of the Euler's equation for the classical functional (1.2) and show, using our solution, that for $b > \frac{\pi}{2}$ there is no solution to Euler's equation for the functional (1.2) which satisfies the boundary conditions (1.3).

Proofs are given in section 2, in section 3 some auxiliary results are obtained, and in section 4 the summary of the results is given.

We conjecture that for $b > \frac{\pi}{2}$ the brachistochrone is the curve P_{br} consisting of two parts: one is the classical brachistochrone curve which joins A with $(\pi/2, 0)$ and the second is the segment of the straight line which joins $(\pi/2, 0)$ with $(b, 0)$.

2. Proofs and discussions

2.1. Let us first prove (1.9). Let P_a be a polygon joining A and B and consisting of two segments: one joins A with the point $(a, 0)$, $0 < a < b$, and the second joins $(a, 0)$ and $(b, 0)$. This curve is not a C^2 curve, but can be smoothed to become C^2 with an arbitrary small change of the functional $T_a := T(P_a)$. The smoothed curve belongs to S (see (1.6)). It is easy to check that

$$T(P_a) = 2\sqrt{1 + a^2} + b - a. \tag{2.1}$$

We claim that if $0 < b < \frac{4}{3}$, then, for sufficiently small $a > 0$, one has

$$T(P_a) > T_0. \tag{2.2}$$

Indeed, if a is sufficiently small, then

$$T(P_a) = 2 + b + O(a), \qquad a \to 0, \tag{2.3}$$

and the inequality

$$2 + b > 2\sqrt{1 + b^2} \quad \text{if} \quad 0 < b < \frac{4}{3} \tag{2.4}$$

is checked easily. Thus, claim (1.9) is proved. □

2.2. Let us now discuss claim (1.8). The functional (1.2) has not more than one critical point, (see section 3), and if it has critical point, this point is a point of minimum of $T(y)$, the brachistochrone, which is a $C^2(0, b) \cap H^1[0, b]$ function, where H^1 is the Sobolev space. We postpone a verification of this statement, which can be found in section 3. There is at most one critical point of the functional (1.2) in S, and if a critical point exists, it is an element of S (see section 3). Therefore, maximum of the functional (1.2) can be attained only at the boundary points of the set S. The boundary points of the set S are the points which are not interior. One such a point is $y_0(x)$. The other should have been the polygon P consisting of two straight line segments: one joins S and the origin $(0,0)$, and the second joins $(0,0)$ and $(b,0)$. This polygon does not belong to S, but can be considered as a limiting point for S since there is a sequence $y_n(x)$ of the elements of S which converges to P in the following sense: $\rho(y_n,P)\to 0$ as $n\to\infty$. Here $\rho(y_1,y_2)=\inf_{\xi_1\in C_1}\sup_{\xi_2\in C_2}|\xi_1-\xi_2|$, where C_j is the set of points which lie on the j-th curve C_j, $j=1,2$. In our case $C_1=\{x,y_n(x)\}_{0\leqslant x\leqslant b}$, and $C_2=P$.

One can easily check that

$$T(P)=2+b, \tag{2.5}$$

where, by definition,

$$T(P):=\lim_{a\downarrow 0}T(P_a), \tag{2.6}$$

(see equation (2.3)). It is also easy to check that

$$2+b\leqslant 2\sqrt{1+b^2}\quad \text{if}\quad b\geqslant \frac{4}{3}. \tag{2.7}$$

For $b>\frac{4}{3}$, we would like to conclude that

$$T_0=\max\{T_0,T(P)\}\geqslant T(y)\qquad \forall y\in S. \tag{2.8}$$

One can also check, using formula (3.14) of section 3, that if $b>\pi/2$ then $T(P_{br})<T(P)<T_0$. $\square$

3. Integration of the Euler's equation

The derivation here differs from the usual one and is included by this reason. It also allows one to see for what b there is no solution to the Euler's equation which satisfies the boundary conditions (1.3). The Euler's equation for (1.2) is

$$\frac{1}{2}(1+y'^2)^{\frac12}(1-y)^{-\frac32}-\frac{d}{dx}\left[(1-y)^{-\frac12}(1+y'^2)^{-\frac12}y'\right]=0. \tag{3.1}$$

Let us use the standard change of variables: $p=p(y)=y'(x)$. Then (3.1) becomes

$$\frac{1}{2}(1+p^2)^{\frac12}(1-y)^{-\frac32}-p\frac{d}{dy}\left[(1-y)^{-\frac12}(1+p^2)^{-\frac12}p\right]=0. \tag{3.2}$$

Denote

$$z:=z(y):=(1-y)^{-\frac12}(1+p^2)^{-\frac12}p. \tag{3.3}$$

Then (3.2) becomes

$$2z\frac{dz}{dy}=(1-y)^{-2},\qquad z^2(y)=c_0(1-y)^{-1}, \tag{3.4}$$

or

$$(1-y)^{-1}\frac{p^2}{1+p^2}=c_0+(1-y)^{-1},\quad \frac{p^2}{1+p^2}=c_0(1-y)+1, \tag{3.5}$$

where c_0 is an integration constant. From (3.5) it follows that:

$$-\frac{1}{1+p^2} = c_0(1-y), \qquad p^2 = \frac{-1}{c_0(1-y)} - 1. \tag{3.6}$$

Since $0 < y < 1$, it is clear from (3.6) that $c_0 < 0$. Thus we denote $-\frac{1}{c_0} = c_1 > 0$ and write (3.6), taking into account that $y'(x) \leqslant 0$, as

$$\frac{dy}{dx} = -\sqrt{\frac{c_1 - 1 + y}{1-y}} := -\sqrt{\frac{c+y}{1-y}}, \qquad c := c_1 - 1 > 0. \tag{3.7}$$

The boundary condition $y(0) = 1$ and equation (3.7) imply:

$$x = \int_y^1 \sqrt{\frac{1-y}{c+y}}\, dy. \tag{3.8}$$

The constant c in (3.8) is to be determined from the condition $y(b) = 0$:

$$b = \int_0^1 \sqrt{\frac{1-y}{c+y}}\, dy := I. \tag{3.9}$$

The integral in (3.9) is calculated by the standard substitution: $\frac{1-y}{c+y} = t^2$, which yields

$$y = \frac{1-ct^2}{1+t^2}, \quad dy = -2(c+1)\frac{t\,dt}{(1+t^2)^2}, \quad I = 2(c+1)\int_0^{c^{-\frac{1}{2}}} \frac{t^2\,dt}{(1+t^2)^2}. \tag{3.10}$$

Let $t = \tan s$. Then

$$\int \frac{t^2\,dt}{(1+t^2)^2} = \int \frac{\sin^2 s \cos^4 s}{\cos^2 s}\frac{ds}{\cos^2 s} = \int \sin^2 s\, ds$$
$$= \frac{s}{2} - \frac{1}{2}\sin s \cos s = \frac{s}{2} - \frac{\tan s}{2[1+(\tan s)^2]}.$$

Therefore

$$I = (c+1)\arctan\frac{1}{\sqrt{c}} - \sqrt{c}, \tag{3.11}$$

and (3.9) becomes

$$b = (c+1)\arctan\frac{1}{\sqrt{c}} - \sqrt{c} := g(c), \tag{3.12}$$

where $c \geqslant 0$ since in (3.9) $c + y > 0$ for all $y \in [0,1]$. The function $g(c)$ decays on $[0,\infty)$ since $g' < 0$ for $c > 0$. Therefore

$$\max_{c \geqslant 0} g(c) = g(0) = \frac{\pi}{2}. \tag{3.13}$$

We have proved the following:

LEMMA 3.1. *Equation (3.12) is not solvable if* $b > \frac{\pi}{2}$.

Therefore, *if* $b > \frac{\pi}{2}$ *the Euler's equation (3.1) has no solution satisfying the boundary condition (1.3).*

This means that for $b > \frac{\pi}{2}$ *the functional* $T(y)$ *defined in (1.2) does not have critical points.*

Therefore if $b > \frac{\pi}{2}$ *then* $T(y)$ *attains its minimum at a boundary point, maybe at* P_{br}.

Note that (3.8) can be written as

$$x = (c+1)\left[\arctan\sqrt{\frac{1-y}{c+y}} - \sqrt{\frac{1-y}{c+y}}\ \frac{c+y}{c+1}\right], \tag{3.14}$$

and using the condition $y(b) = 0$, one obtains from (3.14) equation (3.12) once again.

Using (3.14), one calculates $T(P_{br}) = b + \frac{\pi}{2} < 2\sqrt{1+b^2} = T_0$ provided that $b > \frac{\pi}{2}$, as was claimed in section 2.

It follows from (3.14) that the functions $y = y(x)$ and $x = x(y)$ defined by (3.14) are not in C^2 in the closed intervals $[0, b]$ and $[0, 1]$ respectively. Indeed, in a neighborhood of $y = 1$, the function $x = x(y)$ behaves like $\text{const} \cdot (1-y)^{\frac{3}{2}}$ and in a neighborhood of $x = 0$ the function $y = y(x)$ is a smooth function of $x^{\frac{2}{3}}$ such that $y'(x) \in L^2(0, b)$, $y \notin C^2[0, b]$, but $y \in C^2(0, b) \cap H^1[0, b]$.

Finally, one can check that $x'_y = -\frac{1}{2(c+1)}\sqrt{\frac{1-y}{c+y}} < 0$, and $y''_{xx} = -\frac{x''}{x'^3} > 0$, and therefore $y \in S$, as was mentioned in section 2.

4. Conclusions

Let us summarize:

1) if $b < \frac{4}{3}$ then there exists $y \in S$ such that $T(y) > T(y_0)$;
   if y_{br} is given by equation (3.14) in which $c > 0$ is the unique solution to (3.12), then:
   $T(y_{br}) = \min_{y\in S} T(y) = \min_{y\in C^2} T(y)$ and
   $T(P) > T(y) \geqslant T(y_{br})$, $y \in S$;
2) if $\frac{4}{3} \leqslant b \leqslant \frac{\pi}{2}$, then $T(y_{br}) \leqslant T(y) \leqslant T_0$, $y \in S$;
   $T(y_{br}) = \min_{y\in S} T(y)$ if y_{br} is given by (3.14) with $c > 0$ by (3.12);
3) if $b > \frac{\pi}{2}$, then $T(P_{br}) \leqslant T(y) \leqslant T_0$, $y \in S$; equation (3.12) has no solution for $b > \frac{\pi}{2}$.

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