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# SOME INEQUALITIES AND PROPERTIES CONCERNING CHORDAL POLYGONS

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*Abstract.* The paper deals with some inequalities and properties concerning chordal polygons. The Theorems 1–3 are proved.

## 1. Preliminaries

A polygon with vertices $A_1, A_2, \dots, A_n$ (in this order) will be denoted by $A_1 \dots A_n$. The lengths of its sides will be denoted by $|A_1A_2|, \dots, |A_nA_1|$ or $a_1, \dots, a_n$. The interior angle at the vertex $A_i$ will be denoted by $\alpha_i$ or $\sphericalangle A_i$. Thus

$$\sphericalangle A_i = \sphericalangle A_{i-1}A_iA_{i+1}, \quad i = 1, \dots, n$$

where $A_0 = A_n$ and $A_{n+1} = A_1$.

A polygon $A_1 \dots A_n$ is called *chordal polygon* if there exists a circle $\mathscr{K}$ such that $A_i \in \mathscr{K}$, $i = 1, \dots, n$.

*Remark.* We shall suppose that chordal polygon under consideration has the property that a motion from the vertex $A_i$ to the vertex $A_{i+1}$, $i = 1, \dots, n$, is always in the same sense.

DEFINITION 1. Let $\underline{A} = A_1 \dots A_n$ be a chordal polygon and let $\mathscr{C}$ be its circumcicle. By $S_{A_i}$ and $\widehat{S}_{A_i}$ we denote semicircles such that

$$S_{A_i} \cup \widehat{S}_{A_i} = \mathscr{C}, \quad A_i \in S_{A_i} \cap \widehat{S}_{A_i}.$$

The polygon $\underline{A}$ is said to be of *the first kind* if the following is fulfiled:

1) all the vertices $A_1, \dots, A_n$ do not lie on the same semicircle,
2) for every three consecutive vertices $A_i, A_{i+1}, A_{i+2}$ it holds

$$A_i \in S_{A_{i+1}} \implies A_{i+2} \in \widehat{S}_{A_{i+1}},$$

3) any two consecutive vertices $A_i, A_{i+1}$ do not lie on the same diameter.

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DEFINITION 2. Let $\underline{A} = A_1 \ldots A_n$ be a chordal polygon and let $k$ be a natural number. The polygon $\underline{A}$ is said to be *$k$-circumscribed* if it is of the first kind and if it holds

$$\sum_{i=1}^{n} \measuredangle A_i C A_{i+1} = k \cdot 360^\circ, \tag{1}$$

where $C$ is the centre of the circumscribed circle of the polygon $\underline{A}$.

Using (1) it is easy to prove that for a $k$-circumscribed polygon $A_1 \ldots A_n$ we have:

$$\sum_{i=1}^{n} \measuredangle A_i = (n - 2k)180^\circ. \tag{2}$$

DEFINITION 3. Let $\underline{A} = A_1 \ldots A_n$ be a given polygon. If there exists a polygon $\underline{B} = B_1 \ldots B_n$ and a natural number $k$ such that $\underline{B}$ is $k$-circumscribed and has sides of equal lenghts as the polygon $\underline{A}$, then $\underline{B}$ is said to be *$k$-chordal polygon determined by the polygon $\underline{A}$* and will be denoted by $\underline{A}^{(k)}$.

Of course, if $\underline{A}$ is a chordal polygon, then $A^{(1)} = \underline{A}$.

For example, if $\underline{A}$ is regular $n$-gon, then there exist polygons

$$\underline{A}^{(1)} = \underline{A},\ \underline{A}^{(2)}, \ldots, \underline{A}^{(m)}$$

where $m = \frac{n-1}{2}$ for odd $n$ and $m = \frac{n-2}{2}$ for even $n$. The case when $n = 6$ is illustrated in Fig. 1. We see that the hexagon $A^{(2)}$ is a "double triangle". Generally, if $\underline{A}$ is a regular $n$-gon and $k|n$, then $\underline{A}^{(k)}$ is $(n : k)$-gon which is $k$-fold.

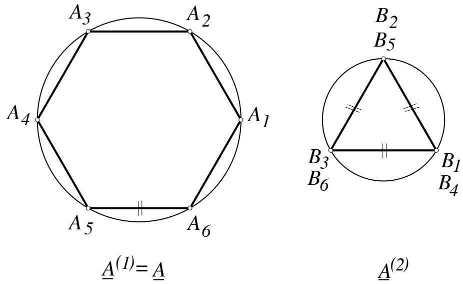


*Fig. 1.*

From (2) it follows that the integer $k$ can be at most $\frac{n-1}{2}$ for odd $n$ and $\frac{n}{2}$ for even $n$. For example, if $\underline{A}$ is a regular heptagon, then

$$\begin{aligned}
(7 - 2 \cdot 1)180^\circ &= 5 \cdot 180^\circ \\
(7 - 2 \cdot 2)180^\circ &= 3 \cdot 180^\circ \\
(7 - 2 \cdot 3)180^\circ &= 1 \cdot 180^\circ \\
(7 - 2 \cdot 4)180^\circ &= -1 \cdot 180^\circ \\
(7 - 2 \cdot 5)180^\circ &= -3 \cdot 180^\circ
\end{aligned}$$

and so one. Accordingly, for $k > 3$ there is nothing essential new.

DEFINITION 4. Let $\underline{A} = A_1 \dots A_n$ be a given polygon. If for some $k$ there exists a polygon $\underline{A}^{(k)}$, but does not exists for $k+1$, then the polygon $\underline{A}$ is said to have a *chordal degree* equal to $k$. The polygon $\underline{A}$ will be said to have maximum chordal degree if there are polygons $\underline{A}^{(1)}, \underline{A}^{(2)}, \dots, \underline{A}^{(m)}$, where $m = \frac{n-1}{2}$ for odd $n$ and $m = \frac{n-2}{2}$ for even $n$.

For example, the regular $n$-gon has the maximal chordal degree. Namely, if $\underline{A} = A_1 \dots A_n$ is a regular $n$-gon, then the polygon

$$\underline{B}^{(k)} = A_1 A_{1+k} A_{1+2k} \dots A_{1+(n-1)k} \tag{3}$$

is a chordal polygon for every $k = 1, \dots, m$ where $m = \frac{n-1}{2}$ for odd $n$ and $m = \frac{n-2}{2}$ for even $n$.

Let us state some facts by intuition. (We shall give later exact statements.) Let $\underline{A} = A_1 \dots A_7$ be a regular heptagon (Fig. 2.). We can imagine the procedure described in Fig. 2. which gives $A^{(2)}$. $A^{(3)}$ can be obtained similarly.

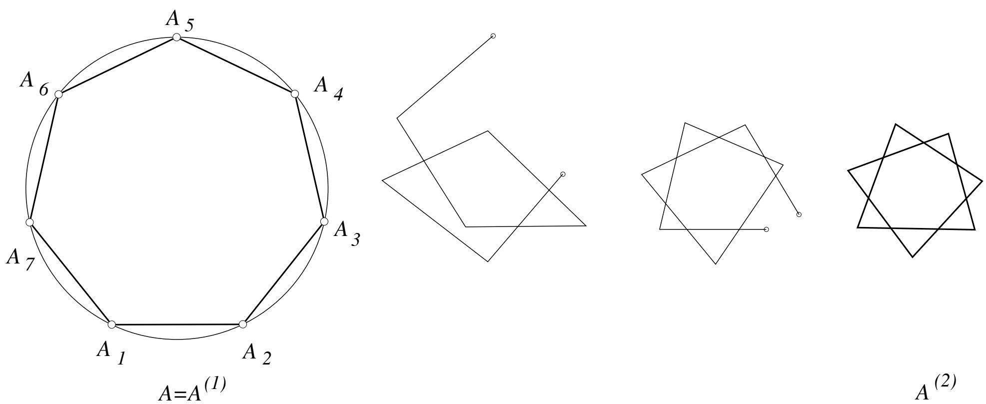


*Fig. 2.*

Every polygon has the chordal degree at least 1. Thus, every polygon can be transformed into $l$-chordal polygon so that the lengths of its sides remain unchanged.

Unfortunately, it is not always possibly to obtain $A^{(2)}$, $A^{(3)}$ and so on. Something more about this will be said later.

## 2. Some inequalities connected with $k$-chordal polygon

First we shall prove one inequality which is very important in investigation of $k$-chordal polygon and some other considerations. Therefore, in some aspect this inequality and its consequences may be regarded as the main result of this paper.

THEOREM 1. *Let $k$ and $n$ be any given positive integers such that $n - 2k > 0$, and let the angles $\beta_1, \dots, \beta_n$ satisfy*

$$\sum_{i=1}^{n} \beta_i = (n-2k)\frac{\pi}{2}, \quad 0 < \beta_i < \frac{\pi}{2}. \tag{4}$$

*Then*

$$\sum_{i=1}^{n} \cos\beta_i > 2k\cos\beta_j, \quad j=1,\ldots,n. \tag{5}$$

*Proof.* Since $\cos\pi x > 1-2x$ if $0<x<\frac{1}{2}$ (Fig. 3), putting $\alpha=\pi x$ we obtain

$$\cos\alpha > 1-\frac{2}{\pi}\alpha, \quad 0<\alpha<\frac{\pi}{2}. \tag{6}$$

Consequently,

$$\sum_{i=1}^{n} \cos\beta_i > n-\frac{2}{\pi}\sum_{i=1}^{n}\beta_i = n-\frac{2}{\pi}(n-2k)\frac{\pi}{2} = 2k > 2k\cos\beta_j.$$

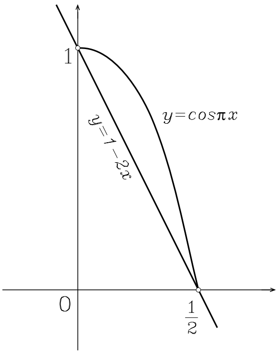


*Fig. 3.*

COROLLARY 1.1. *Let $\underline{A}=A_1\ldots A_n$ be any given $k$-chordal polygon and let*

$$\beta_i = \sphericalangle CA_iA_{i+1}, \quad i=1,\ldots,n \quad (A_{n+1}=A_1),$$

*where $C$ is the centre of the circumcircle of $\underline{A}$. Then it is valid*

$$\sum_{i=1}^{n} \cos\beta_i > 2k\cos\beta_j, \quad j=1,\ldots,n.$$

*Proof.* From Definition 2 and (2) it follows

$$\sum_{i=1}^{n} \beta_i = (n-2k)\frac{\pi}{2}, \quad 0<\beta_i<\frac{\pi}{2}.$$

COROLLARY 1.2. *If $a_1,\ldots,a_n$ are the lengths of the sides of the $k$-chordal polygon $\underline{A}$, then*

$$\sum_{i=1}^{n} a_i > 2ka_j, \quad j=1,\ldots,n. \tag{7}$$

*Proof.* It is valid $a_i = 2r\cos\beta_i$, $i=1,\ldots,n$, where $r$ is the radius of the circumcircle of the polygon $\underline{A}$.

COROLLARY 1.3. *Let* $(i_1, i_2, \dots, i_n)$ *be any permutation of the set* $\{1, \dots, n\}$. *Then*

$$a_{i_1} + \dots + a_{i_k} < a_{i_{k+1}} + \dots + a_{i_n}. \tag{8}$$

*Proof.* It can be easily seen that (7) implies (8).

Of course, instead of (8) the following can be written

$$\cos\beta_{i_1} + \dots + \cos\beta_{i_k} < \cos\beta_{i_{k+1}} + \dots + \cos\beta_{i_n}.$$

One question arises now: If $a_1, \dots, a_n$ are given lengths such that (7) holds, does always exists $k$-chordal polygon with these lengths?

The answer is negative, the counterexample is a pentagon with sides of the lengths $a_1 = 10$, $a_2 = 11$, $a_3 = 12$, $a_4 = 13$, $a_5 = 14$.

We conjecture the following: If the lengths $a_1, a_2, \dots, a_n$ satisfy

$$\sum_{i=1}^{n} a_i^{2k-1} > 2k a_j^{2k-1}, \quad j = 1, \dots, n,$$

and $k$ is maximal, then there exists $k$-chordal polygon with this lengths.

We also state as a hypothesis the following assertion:

If $\beta_1 + \dots + \beta_n = (n-2k)\dfrac{\pi}{2}$, $\beta_i > \dfrac{2k-1}{2k}(n-2k)\dfrac{\pi}{2n}$, $i = 1, \dots, n$, then

$$\cos^k\beta_1 + \dots + \cos^k\beta_n > 2k\cos^k\beta_j, \qquad j = 1, \dots, n.$$

(In the case when $k$ is maximal, we conjucture that the condition $\beta_i > \dfrac{2k-1}{2k}(n-2k)\dfrac{\pi}{2n}$ is not necessary.)

The approximation similar to (6) is too weak to prove this hypothesis in the case $k > 2$, but it strongly suggests that this assertion must be true.

## 3. Determination of chordal polygon and some relations

We shall consider now convex chordal polygon which need not be of the first kind. Its essential characteristic can be stated as follows.

Let $\underline{A} = A_1 \dots A_n$ be a convex chordal polygon and let $a_1, \dots, a_n$ be the lengths of its sides. Further, let $C$ and $r$ be the centre and the radius of the circumcircle of the polygon $\underline{A}$. Finally, let $\beta_1, \dots, \beta_n$ be the angles given by

$$\beta_i = \measuredangle CA_iA_{i+1}, \quad i = 1, \dots, n.$$

Then there are three possibilities: a) the centre $C$ is within the polygon $\underline{A}$, b) the centre $C$ is on a side of the polygon $\underline{A}$, c) the centre $C$ is outside of the polygon $\underline{A}$.

In the case a) all the angles $\beta_1, \dots, \beta_n$ are acute. In the case b) one angle has $0^\circ$ and all others are acute.

In both cases it is valid

$$\beta_1 + \ldots + \beta_n = (n-2)90^\circ.$$

It should be remarked that the equation above is not valid in the case c). One angle must be taken with the negative sign. For example, in Fig. 4 the angle $\beta_4$ must be taken with negative sign.

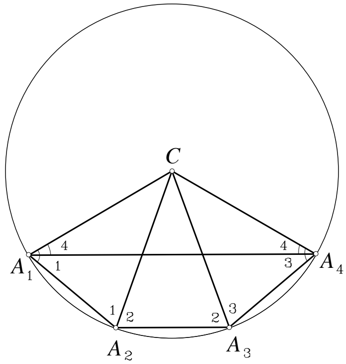


*Fig. 4.*

THEOREM 2. *Let $\underline{B} = B_1 \ldots B_n$ be any given polygon (convex or not) and let $b_1, \ldots, b_n$ be the lengths of its sides. Then there exists the unique convex chordal polygon with sides of those lengths (and in this order).*

*Proof.* It is suficient to show that there exist unique angles $\beta_1, \ldots, \beta_n$ such that

$$\frac{\cos\beta_1}{b_1} = \frac{\cos\beta_2}{b_2} = \ldots = \frac{\cos\beta_n}{b_n}, \tag{9}$$

$$\beta_1 + \beta_2 + \ldots + \beta_n = (n-2)90^\circ, \tag{10}$$

and which satisfy one of the following three statement:

1) all the angles $\beta_1, \ldots, \beta_n$ are acute,
2) one of the angles $\beta_1, \ldots, \beta_n$ has $0^\circ$, the others are acute,
3) all the angles $\beta_1, \ldots, \beta_n$ are acute and one has a negative sign.

Firstly, we conclude that there are infinitely many possibilities for the angles $\beta_1, \ldots, \beta_n$ which satisfy (9). For example, we may suppose that

$$b_1 = \max\{b_1, \ldots, b_n\}$$

and take for $\beta_1$ any acute angle. The angles $\beta_2, \ldots, \beta_n$ can be calculated from the equations

$$\cos\beta_2 = \frac{b_2}{b_1}\cos\beta_1, \ldots, \cos\beta_n = \frac{b_n}{b_1}\cos\beta_1. \tag{11}$$

If we take for $\beta_1$ an acute angle close enough to $90^\circ$, we may obtain that the angles $\beta_2, \ldots, \beta_n$ are also close to $90^\circ$. Thus, there exist acute angles $\beta_2, \ldots, \beta_n$ that satisfy (9) and the inequality

$$\beta_1 + \ldots + \beta_n > (n-2)90^\circ. \tag{12}$$

Now, let $\beta_1$ in (11) decrease to $0^\circ$. Then the angles $\beta_2, \dots, \beta_n$ will approach to the angles given by

$$\cos\beta_2 = \frac{b_2}{b_1}, \dots, \quad \cos\beta_n = \frac{b_n}{b_1}.$$

If in the decreasing we do not get (10) even in the case when $\beta_1$ is $0^\circ$, let us take $\beta_1$ with the negative sign. Now, let $\beta_1$ decrease to $-90^\circ$. Then any of the angles $\beta_2, \dots, \beta_n$ will increase to $90^\circ$. Eventualy, we must obtain the inequality

$$\beta_1 + \dots + \beta_n < (n-2)\cdot 90^\circ. \tag{13}$$

Namely, since $B_1 \dots B_n$ is a polygon, the length of any its side is less than the sum of the lengths of all other sides:

$$b_1 < b_2 + \dots + b_n, \tag{14}$$

where $b_1 = \max\{b_1, \dots, b_n\}$. This inequality, as it is easy to see from (13), implies the inequality

$$\cos\beta_1 < \cos\beta_2 + \dots + \cos\beta_n.$$

Thus, in the case when $\beta_1$ is close enough to $-90^\circ$, there exist angles $\varepsilon_1, \dots, \varepsilon_n$ such that

$$\cos(-90^\circ + \varepsilon_1) < \cos(90^\circ - \varepsilon_2) + \dots + \cos(90^\circ - \varepsilon_n)$$

or

$$\sin\varepsilon_1 < \sin\varepsilon_2 + \dots + \sin\varepsilon_n.$$

Since $\sin x \approx x$ for small $x$, we can write the inequality

$$\varepsilon_1 < \sin\varepsilon_2 + \dots + \sin\varepsilon_n < \varepsilon_2 + \dots + \varepsilon_n.$$

Hence,

$$(-90^\circ + \varepsilon_1) + (90^\circ - \varepsilon_2) + \dots + (90^\circ - \varepsilon_n) < (n-2)90^\circ$$

and (13) is proved. It is obvious now that proceeding from (12) to (13) we can obtain the equality (10), with the angles $\beta_1, \dots, \beta_n$ described in the beginning of the proof.

So, Theorem 2 is proved.

The following theorem concerns the calculation of the radius. If $k$ and $m$ are positive integers such that $k \leqslant m$, denote by $P_k^m$ the sum of $\binom{m}{k}$ products of the form

$$\cos\beta_{i_1}\cdot \ldots \cdot \cos b_{i_k}\cdot \sin\beta_{i_{k+1}}\cdot \ldots \cdot \sin\beta_{i_m}$$

where $(i_1, i_2, \dots, i_m)$ is a permutation of $\{1, 2, \dots, m\}$. For example:

$$\begin{aligned}
P_1^3 &= \cos\beta_1\sin\beta_2\sin\beta_3 + \sin\beta_1\cos\beta_2\sin\beta_3 + \sin\beta_1\sin\beta_2\cos\beta_3,\\
P_3^4 &= \cos\beta_1\cos\beta_2\cos\beta_3\sin\beta_4 + \cos\beta_1\cos\beta_2\sin\beta_3\cos\beta_4\\
&\quad + \cos\beta_1\sin\beta_2\cos\beta_3\cos\beta_4 + \sin\beta_1\cos\beta_2\cos\beta_3\cos\beta_4.
\end{aligned}$$

THEOREM 3. *If* $\beta_1+\beta_2+\ldots+\beta_{n-1}=(n-2)90^\circ-\beta_n$, *then*

$$\sum_{i=1}^{m}(-1)^{i+1}P_{2i-1}^{n-1}=\cos\beta_n \tag{15}$$

*where* $m=\frac{n-1}{2}$ *for odd* $n$ *and* $m=\frac{n}{2}$ *for even* $n$.

*Proof.* The equation

$$\cos[(n-2)90^\circ-(\beta_1+\ldots+\beta_{n-1})]=\cos\beta_n$$

can be written as

$$\begin{aligned} \sin(\beta_1+\ldots+\beta_{n-1})&=\cos\beta_n, &&\text{for}\quad n=3,7,11,\ldots\\ -\cos(\beta_1+\ldots+\beta_{n-1})&=\cos\beta_n, &&\text{for}\quad n=4,8,12,\ldots\\ -\sin(\beta_1+\ldots+\beta_{n-1})&=\cos\beta_n, &&\text{for}\quad n=5,9,13,\ldots\\ \cos(\beta_1+\ldots+\beta_{n-1})&=\cos\beta_n, &&\text{for}\quad n=6,10,14,\ldots. \end{aligned}$$

So, for $n=3,4,5,6,7$ we have

$$\begin{aligned} &P_1^2=\cos\beta_3\\ &P_1^3-P_3^3=\cos\beta_4\\ &P_1^4-P_3^4=\cos\beta_5\\ &P_1^5-P_3^5+P_5^5=\cos\beta_6\\ &P_1^6-P_3^6+P_5^6=\cos\beta_7, \end{aligned}$$

where

$$\begin{aligned} P_1^2&=\cos\beta_1\sin\beta_2+\sin\beta_1\cos\beta_2\\ P_1^3-P_3^3&=\cos\beta_1\sin\beta_2\sin\beta_3+\sin\beta_1\cos\beta_2\sin\beta_3\\ &\qquad+\sin\beta_1\sin\beta_2\cos\beta_3-\cos\beta_1\cos\beta_2\cos\beta_3,\\ P_1^4-P_3^4&=\cos\beta_1\sin\beta_2\sin\beta_3\sin\beta_4+\sin\beta_1\cos\beta_2\sin\beta_3\sin\beta_4\\ &\qquad+\sin\beta_1\sin\beta_2\cos\beta_3\sin\beta_4+\sin\beta_1\sin\beta_2\sin\beta_3\cos\beta_4\\ &\qquad-\cos\beta_1\cos\beta_2\cos\beta_3\sin\beta_4-\cos\beta_1\cos\beta_2\sin\beta_3\cos\beta_4\\ &\qquad-\cos\beta_1\sin\beta_2\cos\beta_3\cos\beta_4-\sin\beta_1\cos\beta_2\cos\beta_3\cos\beta_4, \end{aligned}$$

and so on.

Induction by $n$ we obtain (15) for all natural number $n\geqslant 3$.

COROLLARY 3.1. *If the polygon* $A^{(k)}$ *exists for some* $k\in\{1,2,\ldots,m\}$, *where* $m=\frac{n-1}{2}$ *for odd* $n$ *and* $m=\frac{n-2}{2}$ *for even* $n$, *then the angles* $\beta_1,\ldots,\beta_n$ *of this polygon satisfy*

$$\sum_{1=1}^{m}(-1)^{i+1}P_{2i-1}^{n-1}=(-1)^{k+1}\cos\beta_n. \tag{16}$$

*Proof.* Since $A^{(k)}$ is $k$-circumscribed polygon, it holds

$$\beta_1 + \ldots + \beta_n = (n-2k)90^\circ$$

and the equation

$$\cos[(n-2k)90^\circ - (\beta_1 + \ldots + \beta_{n-1})] = \cos\beta_n$$

can be written as

$$\begin{aligned}
\sin(\beta_1 + \ldots + \beta_{n-1}) &= (-1)^{k+1}\cos\beta_n, && \text{for} \quad n = 3, 7, 11, \ldots \\
-\cos(\beta_1 + \ldots + \beta_{n-1}) &= (-1)^{k+1}\cos\beta_n, && \text{for} \quad n = 4, 8, 12, \ldots \\
-\sin(\beta_1 + \ldots + \beta_{n-1}) &= (-1)^{k+1}\cos\beta_n, && \text{for} \quad n = 5, 9, 13, \ldots \\
\cos(\beta_1 + \ldots + \beta_{n-1}) &= (-1)^{k+1}\cos\beta_n, && \text{for} \quad n = 6, 10, 14, \ldots .
\end{aligned}$$

COROLLARY 3.2. *We have*

$$0 < \sum_{i=1}^{m} (-1)^{i+1} P_{2i-1}^{n-1} < 1 \quad \text{if } k \text{ is odd}, \tag{17}$$

$$-1 < \sum_{i=1}^{m} (-1)^{i+1} P_{2i-1}^{n-1} < 0 \quad \text{if } k \text{ is even}. \tag{18}$$

From (17) and (18) some other inequalities can be obtained. For example, if $n = 5$ and $k = 1$, then $P_1^4 > P_3^4$ i.e.

$$\begin{aligned}
&\cos\beta_1 \sin\beta_2 \sin\beta_3 \sin\beta_4 + \sin\beta_1 \cos\beta_2 \sin\beta_3 \sin\beta_4 \\
&\qquad + \sin\beta_1 \sin\beta_2 \cos\beta_3 \sin\beta_4 + \sin\beta_1 \sin\beta_2 \sin\beta_3 \cos\beta_4 \\
&\quad > \cos\beta_1 \cos\beta_2 \cos\beta_3 \sin\beta_4 + \cos\beta_1 \cos\beta_2 \sin\beta_3 \cos\beta_4 \\
&\qquad + \cos\beta_1 \sin\beta_2 \cos\beta_3 \cos\beta_4 + \sin\beta_1 \cos\beta_2 \cos\beta_3 \cos\beta_4 .
\end{aligned}$$

Let us denote by $r_k$ the radius of $\underline{A}^{(k)}$. From (16) we obtain the equation in $r_k$, using

$$\cos\beta_i = \frac{b_i}{2r_k}, \qquad i = 1, \ldots, n.$$

For example, if $n = 5$ and $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5$ $(= \beta)$, then (16) can be written as

$$4\cos\beta\sin^3\beta - 4\cos^3\beta\sin\beta = (-1)^{k+1}\cos\beta, \qquad k = 1, 2.$$

Since $\cos\beta = 1/2r_k$ (if $b_1 = \ldots = b_5 = 1$), we obtain equations

$$(2r_k^2 - 1)\sqrt{4r_k^2 - 1} = (-1)^{k+1} r_k^3, \qquad k = 1, 2,$$

with solutions

$$r_1 = \frac{1}{2\sin 36^\circ} = 0.85065,$$

$$r_2 = \frac{1}{2\cos 18^\circ} = 0.5257.$$

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